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Research article

Pinning-controlled synchronization of partially coupled dynamical networks via impulsive control

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Abstract: In this paper, global exponential outer synchronization of coupled nonlinear systems with general coupling matrices are investigated via pinning impulsive control. More realistic and more general partially coupled drive-response systems are established, where the completely communication channel matrix between coupled nodes may not be a permutation matrix. By using pinning impulsive strategy involving pinning ratio and our generalised lower average impulsive interval method, a number of novel and less restrictive synchronization criteria are proposed. In the end, a numerical example is constructed to indicate the effectiveness of our theoretical results.

Keywords: complex dynamical networks; discontinuous control; pinning synchronization; impulsive control; outer synchronization
Mathematics Subject Classification: 34K20, 34K45, 35R12

1. Introduction

Complex network usually consists of interconnected nodes, where every node is a dynamical network. Due to complicated links and interactions between nodes, it displays complicated dynamics which may be completely differ from those of alone node. As far as we know, research on complex networks originated in 1960 [1], where P. Erdȯs and A. Rénti advanced random graphical mathematical model and utilized it to characterize topological feature of complex networks. Up to now, complex networks have provoked extensive interest because of their widely existing in natural world and universal applications in multidisciplinary field such as neural networks [2, 3], parallel image processing [4], pattern recognition [5], social network [6], the world wide web [7] and so on. Many results on delayed complex networks also have been reported [8–10].

As everyone knows, synchronization is one of very typical collective behaviors of complex networks, which means that all the individual nodes with diverse initial values in the network approach to a common state as time goes on. Synchronization is widely applied to multitudinous fields, such as information science, secure communication, flocking of birds, agreement of opinions, biological systems, see [11–13] and so on. These years, people get a lot of significant research on synchronization problem. For example, [14] investigated globally exponential synchronization of coupled complex systems. [15] studied synchronization of coupled complex networks with hybrid impulses. Note that time-delay is inevitable, thus, a great deal of interesting works consider effects of time-delay to achieve synchronization of coupled complex networks [16, 17]. Recently, some results about the synchronization were addressed for nonlinear discontinuous coupled dynamical networks, see [18–20]. The bulk of the above mentioned research of complex networks concentrated around inner synchronization, which considers that collective behavior amongst complete nodes in the same network. In addition, outer synchronization among multiple complex networks is widespread in daily life. Therefore, it is also very necessary to research how to achieve outer synchronization among several complex network. Numerous meaningful results concerning outer synchronization have been proposed. [21] investigated outer synchronization of both coupled complex networks with identical topology. Thereafter, [22] studied outer synchronization about two coupled complex networks with nonidentical topologies. [23] discussed generalized outer synchronization and [24] investigated adaptive outer synchronization about two differ coupled complex networks.

Controller design is the core issue to realize synchronization of complex network. Until now, there are a large amount of effective control methods including impulsive control [25–27], switching control [28], feedback control [29], sampled-data control [30], intermittent control [31, 32], pinning control [33, 34], sliding mode control [35], event-triggered control [36], fuzzy control [37], etc. Among them, impulsive control strategy is widely used since it has a very straight forward structure and only needs to be controlled discretely [38–40]. There are considerable number of valuable results on synchronization of complex networks via impulsive control, see [41, 42] and so on. Furthermore, controlling the whole nodes is difficult usually particularly when the system is composed of many high-dimensional nodes. Because of these considerations, pinning control is proposed, which means that nothing but a few part of nodes are controlled [43, 44]. To combine good points of pinning strategy and impulsive control, pinning impulsive control method has been conceived. For instance, [45] realized outer synchronization of partially coupled complex networks via pinning impulsive control. In [45], the completely communication channel matrix explored between coupled nodes is a unit matrix. Subsequently, based on regrouping method, some more realistic partially coupled networks were investigated in [46], where the completely communication channel matrix between coupled nodes is a permutation matrix. Such restriction also leads to more conservativeness. In addition, [47] presented a more flexible method involving pinning ratio to achieve synchronization of coupled complex networks via impulsive control.

Motivated by our previous discussion, in this paper, we investigate the outer synchronization problem of coupled nonlinear systems with more general inner coupling matrices than in [45, 46] via pinning impulsive control. Based on the impulse strategy involving pinning ratio and our generalised lower average impulsive interval concept, a number of more flexible and less restrictive novel outer synchronization criterion are proposed. The rest of this paper is structured as below: In Section 2, we give fundamental model formulation, a few necessary definitions and assumptions. In Section 3, some more widely applicable and less conservative criteria are proposed to obtain outer synchronization of coupled systems. In Section 4, we cite a numerical example to show the effectiveness of derived results. At the final part, a brief conclusion is given in Section 5.

Notations. Let $\mathbb{N}$ represents the nonnegative integer set, I_n denotes the identity matrix of dimension n. Let $Y^{\frac{1}{2}}(Y^{-\frac{1}{2}})$ represents the arithmetic roots of positive matrix $Y(Y^{-1})$, which means that $Y^{\frac{1}{2}}Y^{\frac{1}{2}} = Y(Y^{-\frac{1}{2}}Y^{-\frac{1}{2}} = Y^{-1})$. For a matrix A, let $\lambda_{max}(A)$ denotes the maximum eigenvalues of matrix A, and $A > 0(< 0)$ indicates A is a positive (negative) definite matrix. $\mathbb{R}^n$ denotes the Euclidean space of dimension n, $\mathbb{R}^{N\times N}$ represents the linear space composed of real square matrices of order n . This symbol T expresses the transpose of a matrix. $\sharp\mathfrak{D}$ is the number of elements obtained in finite set $\mathfrak{D}$. $max\mathfrak{D}$ denotes the maximum in finite set $\mathfrak{D}$.

2. Model description and preliminaries

Consider the complex network below consists of N partially coupled nodes:

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + c\sum_{j=1,j\neq i}^{N} g_{ij}R_{ij}(x_j(t) - x_i(t)), \tag{2.1}$$

where $x_i(t) = (x_{i1}(t), \cdots, x_{in}(t))^\top \in \mathbb{R}^n$ represents the state variable of the ith node; $A = diag\{a_1, a_2, \cdots, a_n\} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}; f(x_i(t)) = (f_1(x_i(t)), \cdots, f_n(x_i(t)))^\top$ is one non-linear function of the ith node at time t satisfying $f(\mathbf{0}) = \mathbf{0}$; $c > 0$ is the coupling strength of this complex network. $G = (g_{ij}) \in \mathbb{R}^{N\times N}$ denotes the outer coupling matrix which be assigned values according to the following rules: if there exists one connection between node j and node $i(j \neq i)$, $g_{ij} > 0$; or else $g_{ij} = 0$. $R_{ij} = (r_{ij}^{kl}) \in \mathbb{R}^{n\times n}$ represents the channel matrix. In detail, r_{ij}^{kl} is assigned value as below: if the klth channel of the connection from the lth component of node j to the kth component of node i is active, then $r_{ij}^{kl} > 0$; otherwise, $r_{ij}^{kl} = 0$. In view of the configuration of drive-response system, we take the network (2.1) as drive system. Then, the corresponding response system is formulated by:

$$\dot{y}_i(t) = Ay_i(t) + Bf(y_i(t)) + c\sum_{j=1,j\neq i}^{N} g_{ij}R_{ij}(y_j(t) - y_i(t)) + u_i(t), \tag{2.2}$$

where $y_i(t) = (y_{i1}(t), \cdots, y_{in}(t))^\top \in \mathbb{R}^n$ is the state variable of the ith node for response system, and $u_i(t)$ is the designed impulsive controller of the ith node.

Let $C_{ij} = g_{ij}R_{ij} := (c_{ij}^{kl}) \in \mathbb{R}^{n\times n}(j \neq i)$ and $C_{ii} = -\sum_{j=1,j\neq i}^{N} C_{ij}$, where $c_{ij}^{kl} = g_{ij}r_{ij}^{kl}$.

Let $e_i(t) = y_i(t) - x_i(t)$, then we get the next error system:

$$\dot{e}_i(t) = Ae_i(t) + B\tilde{f}(e_i(t)) + c\sum_{j=1}^{N} C_{ij}e_j(t) + u_i(t),$$

where $\tilde{f}(e_i(t)) = f(y_i(t)) - f(x_i(t))$.

The following is the designed pinning impulsive controller:

$$u_i(t) = \begin{cases} \sum\limits_{k=1}^{+\infty} q_k e_i(t)\delta(t - t_k), & i \in \mathfrak{D}_k, \sharp\mathfrak{D}_k = s_k \\ 0, & i \notin \mathfrak{D}_k \end{cases} \tag{2.3}$$

where the impulsive time sequence $\zeta = \{t_k, k \in \mathbb{N}\}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim\limits_{k\to+\infty} t_k = +\infty$. $\delta(\cdot)$ is the Dirac function. Let s_k represents the number of pinned nodes, $\mathfrak{D}_k = \{i_1, i_2, \cdots, i_{s_k}\} \subseteq$

$\{1, 2, \cdots, N\}$ is the set of pinned nodes and q_k is the impulsive control gain at impulsive instant t_k respectively.

With the designed impulsive controller (2.3), the dynamical error system is expressed by:

$$\begin{cases} \dot{e}_i(t) = Ae_i(t) + B\tilde{f}(e_i(t)) + c\sum_{j=1}^{N} C_{ij}e_j(t), & t \neq t_k, \\ e_i(t_k) = (1 + q_k)e_i(t_k^-), & i \in \mathfrak{D}_k, \sharp\mathfrak{D}_k = s_k. \end{cases} \tag{2.4}$$

We assume that the solution of (2.4) satisfies $e_i(t_k^+) = e_i(t_k), k \in \mathbb{N}$.

We give the next definitions and assumption to derive main synchronization criteria.

Definition 1. *We define the lower average impulsive interval* $\underline{T_a}$ *of impulsive sequence* $\zeta = \{t_k, k \in \mathbb{N}\}$ *as below*

$$\underline{T_a} = \varliminf_{t\to+\infty} \frac{t - t_0}{N_\zeta(t_0, t)},$$

where $N_\zeta(t_0, t)$ *represents the number of impulse occurrences of* ζ *in interval* (t_0, t).

Remark 1. *This definition evolved from the concept of average impulsive interval [15]. Our the lower limit of average impulsive interval* $\underline{T_a}$ *always exists for any impulsive sequence. Therefore, we present a general less conservative concept which be able to characterize more broader impulsive sequences.*

Definition 2. *([47]).*

$$\eta_k := \frac{\sum_{i\in\mathfrak{D}(t_k)} e_i^T(t_k)e_i(t_k)}{\sum_{j=1}^{N} e_j^T(t_k)e_j(t_k)}, \tag{2.5}$$

is called the pinning ratio of the system (2.4) at impulsive instant $t = t_k$.

Definition 3. *Response system (2.2) be called globally exponentially outer synchronized with drive system (2.1), in other words error system (2.4) converges exponentially to zero, i.e.* $\exists$ *scalars* $\vartheta > 0, M > 0$ *and* $T > 0$*, s.t. for arbitrarily* $e_i(0)(i = 1, 2, \cdots, N)$*, hold*

$$\| e_i(t) \| \leq Me^{-\vartheta t}, t > T, i = 1, 2, \cdots, N.$$

Assumption 1. *About non-linear function* $f(\cdot)$*, suppose that* $\exists$ *real numbers* $l_{ik} > 0(i, k = 1, 2, \cdots, n)$ *satisfy* $|f(x_1) - f(x_2)| \leq \sum_{k=1}^{n} l_{ik}|x_{1k} - x_{2k}|$ *for any* $x_1, x_2 \in \mathbb{R}^n$.

For the convenience of later use, let

$$L = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}.$$

3. Main results

In this section, a few pinning-controlled synchronization criterion of dynamical networks via impulsive control will be deduced.

Theorem 1. *Under Assumption 1, and assume that* $\underline{T_a}$ *of the sequence* ζ *is a finite real number. Then response system (2.2) can be globally exponentially outer synchronized with drive system (2.1) if*

$$\vartheta = \frac{ln\xi}{\underline{T_a}} + \lambda < 0, \tag{3.1}$$

where $\xi = \overline{\lim_{k\to+\infty}} \frac{\xi_1+\xi_2+\cdots+\xi_k}{k} > 0, \xi_k = q_k^2\eta_k + 2q_k\eta_k + 1, \lambda = \lambda_{max}(A + A^T) + \lambda_{max}(BB^T) + \lambda_{max}(L^T L) + cN(1 + \max\{\lambda_{max}(C_{ij}^\top C_{ij})|i, j = 1, \cdots, N\})$.

Proof. We select Lyapunov function as below:

$$V(t) = \sum_{i=1}^{N} e_i^\top(t)e_i(t).$$

For any $k \in \mathbb{N}, t \in [t_k, t_{k+1})$, we calculate the derivative of $V(t)$ about t along the trajectory of error system (2.4):

$$\begin{aligned}
\dot{V}(t) &= 2\sum_{i=1}^{N} e_i^\top(t)\dot{e}_i(t) \\
&= 2\sum_{i=1}^{N} e_i^\top(t)[Ae_i(t) + B\tilde{f}(e_i(t)) + c\sum_{j=1}^{N} C_{ij}e_j(t)] \\
&= 2\sum_{i=1}^{N} e_i^\top(t)Ae_i(t) + 2\sum_{i=1}^{N} e_i^\top(t)B\tilde{f}(e_i(t) + 2c\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^\top(t)C_{ij}e_j(t).
\end{aligned} \tag{3.2}$$

Note that

$$2\sum_{i=1}^{N} e_i^\top(t)Ae_i(t) = \sum_{i=1}^{N} e_i^\top(t)(A + A^\top)e_i(t) \le \lambda_{max}(A + A^T)\textstyle\sum_{i=1}^{N} e_i^\top(t)e_i(t). \tag{3.3}$$

Under the premise of Assumption 1, one can see that

$$\begin{aligned}
&2\sum_{i=1}^{N} e_i^\top(t)B\tilde{f}(e_i(t)) \le \sum_{i=1}^{N} e_i^\top(t)BB^\top e_i(t) + \sum_{i=1}^{N} \tilde{f}(e_i(t))^\top \tilde{f}(e_i(t)) \\
&\le \sum_{i=1}^{N} e_i^\top(t)BB^\top e_i(t) + \sum_{i=1}^{N} e_i(t)^\top L^\top Le_i(t) \\
&\le (\lambda_{max}(BB^T) + \lambda_{max}(L^T L))\sum_{i=1}^{N} e_i^\top(t)e_i(t).
\end{aligned} \tag{3.4}$$

In addition

$$2c\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^{\top}(t)C_{ij}e_j(t) \le c\sum_{i=1}^{N}\sum_{j=1}^{N}[e_i^{\top}(t)e_i(t)+e_j^{\top}(t)C_{ij}^{\top}C_{ij}e_j(t)]$$

$$\le cN(1+\max\{\lambda_{max}(C_{ij}^{\top}C_{ij})|i,j=1,\cdots,N\})\times\sum_{j=1}^{N}e_j^{\top}(t)e_j(t). \tag{3.5}$$

Using inequalities (3.3)–(3.5) in (3.2), we obtain the following inequality $\dot{V}(t)\le\lambda V(t)$.

Hence for $t\in[t_k,t_{k+1})$, we have

$$V(t)\le V(t_k)e^{\lambda(t-t_k)}. \tag{3.6}$$

It follows from (2.4) and (2.5), we get

$$\begin{aligned} V(t_k) &= \sum_{i\in\mathfrak{D}_k}e_i^{\top}(t_k)e_i(t_k)+\sum_{i\notin\mathfrak{D}_k}e_i^{\top}(t_k)e_i(t_k) \\ &= (1+q_k)^2\sum_{i\in\mathfrak{D}_k}e_i^{\top}(t_k^-)e_i(t_k^-)+\sum_{i\notin\mathfrak{D}_k}e_i^{\top}(t_k^-)e_i(t_k^-) \\ &= [(1+q_k)^2\eta_k+(1-\eta_k)]\sum_{i=1}^{N}e_i^{\top}(t_k^-)e_i(t_k^-) \\ &= \xi_kV(t_k^-). \end{aligned} \tag{3.7}$$

For $t\in[t_k,t_{k+1})$, from inequalities (3.6)–(3.7), we have

$$\begin{aligned} V(t) &\le V(t_k)e^{\lambda(t-t_k)} \\ &= \xi_kV(t_k^-)e^{\lambda(t-t_k)} \\ &\le \xi_kV(t_{k-1})e^{\lambda(t-t_{k-1})} \\ &= \xi_k\xi_{k-1}V(t_{k-1}^-)e^{\lambda(t-t_{k-1})} \\ &\cdots \\ &\le (\prod_{i=1}^{k}\xi_i)V(0)e^{\lambda(t-t_0)} \\ &\le (\frac{\xi_1+\xi_2+\cdots+\xi_k}{k})^ke^{\lambda(t-t_0)}V(0) \\ &= e^{kln\frac{\xi_1+\xi_2+\cdots+\xi_k}{k}}e^{\lambda(t-t_0)}V(0). \end{aligned}$$

Using the definition of $\underline{T_a}$, for $\forall\vartheta_1,\vartheta<\vartheta_1<0,\exists\, T>0$, s.t. for $\forall t>T$, we conclude

$$\begin{aligned} V(t) &\le e^{(\frac{ln\xi}{\underline{T_a}}+\vartheta_1-\vartheta)(t-t_0)}e^{\lambda(t-t_0)}V(0) \\ &\le e^{\vartheta_1(t-t_0)}V(0). \end{aligned}$$

Note that $\vartheta_1<0$. Consequently, response system (2.2) is exponentially outer synchronized with drive system (2.1). Theorem 1 is proved. □

Remark 2. *[46] designed impulsive controller based on a fixed weighted-norm to pinning those nodes which have biggest norm values. Different from this result, the pinning impulsive strategy of Theorem 1 above based on pinning ratio η_k. In this case, it is possible that a node with smaller norm value can be selected to save cost. Thus, the pinning impulsive strategy in Theorem 1 is more flexible and effective.*

Especially, if we select the completely communication channel matrix R to be a permutation matrix, and note that permutation matrices are all orthogonal matrices, we could obtain the next corollary immediately.

Corollary 1. *Suppose that $R_{ij} = diag(r_{ij}^{1l_1}, r_{ij}^{2l_2}, \cdots, r_{ij}^{nl_n})R$, where $(l_1, l_2, \cdots, l_n)$ is a permutation of $(1, 2, \cdots, n)$, R is a permutation matrix. Under Assumption 1, response system (2.2) is globally exponentially outer synchronized with drive system (2.1) if*

$$\vartheta = \frac{ln\xi}{\underline{T_a}} + \lambda < 0, \tag{3.8}$$

where $\xi = \overline{\lim_{k\to+\infty}} \frac{\xi_1+\xi_2+\cdots+\xi_k}{k} > 0, \xi_k = q_k^2\eta_k + 2q_k\eta_k + 1$, $\lambda = \lambda_{max}(A + A^T) + \lambda_{max}(BB^T) + \lambda_{max}(L^T L) + cN(1 + \max\{(g_{ij} r_{ij}^{kl_k})^2 | i, j, k = 1, \cdots, N\})$.

Remark 3. *Note that in [45] and [46], their completely channel matrix $R = (r^{kl})$ is assumed to be a unit matrix or permutation matrix. However, these simplifications do not conform to the characteristics of many real networks. In fact, there may be more than one communication from the components of node j to the kth component of node i, and R may be different for a distinct pair of nodes i, j in many real networks. In this paper, these restrictions are removed, thus our system are broader and in line with the real circumstances.*

In addition, if the impulsive control gain is an invariant constant, then we have the following more specific version criterion.

Corollary 2. *Suppose that $q_k \equiv q, \forall k \in \mathbb{N}, k \geq 1$. Under Assumption 1, response system (2.2) is globally exponentially outer synchronized with drive system (2.1) if*

$$\eta = \underline{\lim_{k\to+\infty}} \frac{\eta_1 + \eta_2 + \cdots + \eta_k}{k} > \frac{e^{-\lambda \underline{T_a}} - 1}{q^2 + 2q}, \tag{3.9}$$

where $\lambda = \lambda_{max}(A+A^T)+\lambda_{max}(BB^T)+\lambda_{max}(L^T L)+cN(1+\max\{\lambda_{max}(C_{ij}^{\top} C_{ij}) | i, j = 1, \cdots, N\}), q^2+2q < 0$.

Proof. Note that $\xi_k = (q^2 + 2q)\eta_k + 1$, we have $\xi = 1 + (q^2 + 2q) \underline{\lim_{k\to+\infty}} \frac{\eta_1+\eta_2+\cdots+\eta_k}{k}$. Then we can derive Corollary 2 from Theorem 1 immediately. □

4. Numerical simulations

In this section, we propose one numerical simulation to reveal the effectiveness of theoretical results obtained above.

Example 1. *We discuss a drive-response system with* $n = 3$ *and* $N = 3$*. The coupling parameters of this model are taken as below:*

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, B = \begin{bmatrix} -0.08 & 0 & 0.1 \\ 0 & -0.02 & 0 \\ 0.05 & 0 & 0.1 \end{bmatrix}, G = \begin{bmatrix} -1.2 & 1 & 0.2 \\ 0.1 & -1.1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, c = 1.$$

We take the channel matrices as follows:

$$R_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.3 \\ 0 & 0.2 & 0 \end{bmatrix}, R_{13} = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}, R_{21} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix},$$

$$R_{23} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0.3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R_{31} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}.$$

Clearly, our channel matrices do not satisfy the conditions in [45] and [46], so results in [45] and [46] cannot guarantee synchronization of this drive-response system in Example 1. Set

$$f(x_i) = (tanh(x_{i1}), tanh(x_{i2}), tanh(x_{i3}))^T, L = diag\{1, 1, 1\}.$$

The initial values $e_1(0), e_2(0), e_3(0)$ *of this drive-response system are selected uniformly and randomly in the interval* $[-100, 100]$.

It follows from the simulation that error system is not scynchronized if there is no control input, see Figure 1 (a).

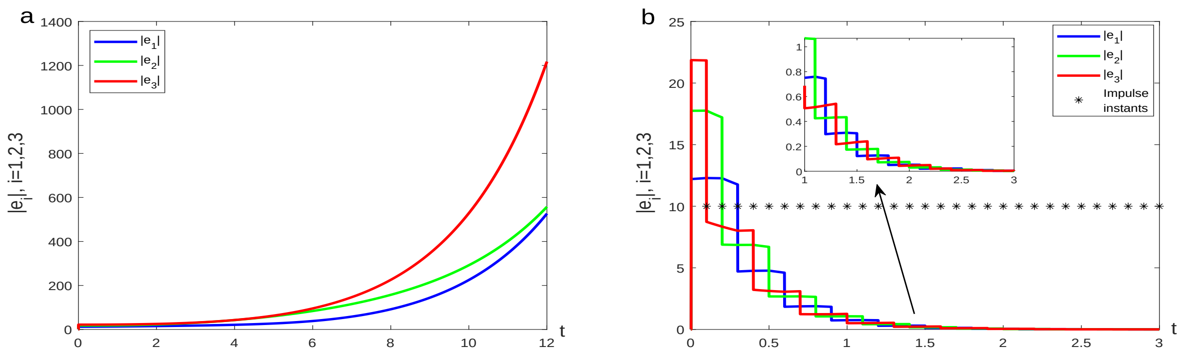

Figure 1. (a) The trajectories of synchronization errors $|e_i(t)|$ without control; (b) The trajectories of synchronization errors $|e_i(t)|$ with the pinning impulsive control.

By calculation, we get $\lambda = 0.5869$*. Set* $q_k \equiv -1.4, t_k = 0.1k, k \in \mathbb{N}$*, then* $T_a = 0.1$*. The pinning ratio needs to meet* $\eta_k > 0.4917$*, we set* $\eta_k = 0.5$*. It can be verified soon that they meet the whole requirements of Theorem 1. As indicated in Figure 1 (b), one can see that response system (2) becomes synchronized with drive system (1) via such a sort of pinning impulsive controller. In addition, Figure 2 (a)–(c) show that the trajectories for the components of error state of the drive-response system under the pinning impulsive control. We can also see from Figure 2 that the synchronization is obtained* *based on our pinning control strategy. The numerical simulations have testified the effectiveness of our results.*

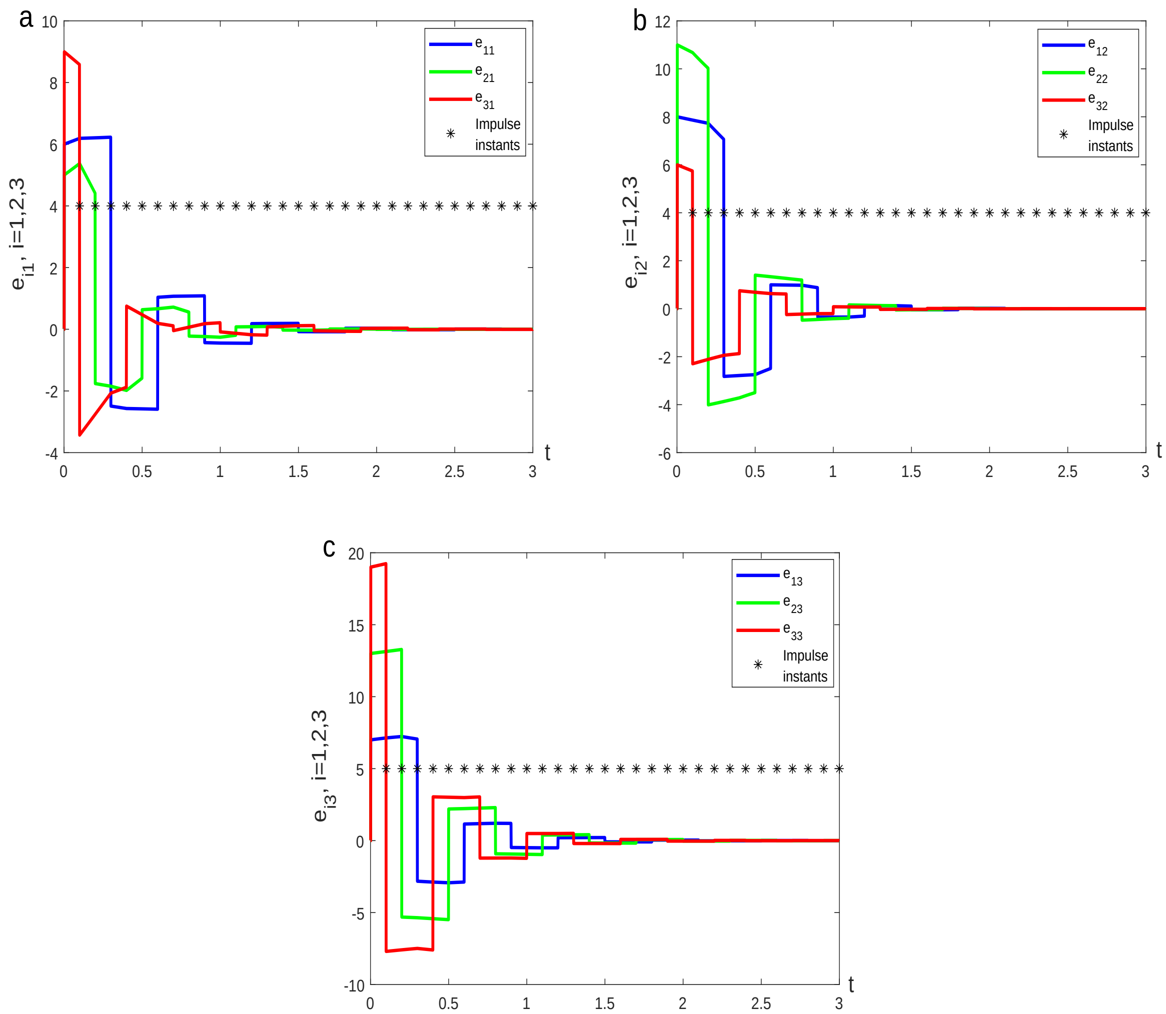

Figure 2. The trajectories for the components of error state with the pinning impulsive control.

5. Conclusions

In this paper, we mainly investigate outer synchronization problem of coupled dynamical systems with general coupling matrices. By using the pinning impulsive strategy involving pinning ratio and our generalised lower average impulsive interval concept, a few novel and less restrictive synchronization criteria are proposed to obtain the globally exponential outer synchronization of complex networks. A numerical example has been proposed to indicate the effectiveness of these theoretical results. Later, we will research synchronization problem for fractional-order coupled dynamical networks and the finite time synchronization of partially coupled complex networks with time-varying delay.

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Conflict of interest

We declare that we have no conflict of interest.

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