Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 121–142.
DOI:10.3934/math.2022007
Received: 02 June 2021
Accepted: 08 September 2021
Published: 08 October 2021

Research article

Optimal control for a phase field model of melting arising from inductive heating

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Abstract: Due to its unique performance of high efficiency, fast heating speed and low power consumption, induction heating is widely and commonly used in many applications. In this paper, we study an optimal control problem arising from a metal melting process by using a induction heating method. Metal melting phenomena can be modeled by phase field equations. The aim of optimization is to approximate a desired temperature evolution and melting process. The controlled system is obtained by coupling Maxwell's equations, heat equation and phase field equation. The control variable of the system is the external electric field on the local boundary. The existence and uniqueness of the solution of the controlled system are showed by using Galerkin's method and Leray-Schauder's fixed point theorem. By proving that the control-to-state operator P is weakly sequentially continuous and Fréchet differentiable, we establish an existence result of optimal control and derive the first-order necessary optimality conditions. This work improves the limitation of the previous control system which only contains heat equation and phase field equation.

Keywords: induction heating; optimal control; phase field equation; Maxwell's equations; existence; necessary conditions
Mathematics Subject Classification: 35K55, 35Q60, 49J20, 49K20

1. Introduction

As a fairly new processing technology, induction heating is widely and commonly used in many applications, for example, induction melting, induction heat treatment, and so on (see [1]). The similar mathematical model is described by partial differential equations (see [2]). The metal materials is heated by the eddy current which is produced by electromagnetic induction. The solid phase of the material begins to melt when the temperature achieves its melting point.

In this paper, we use bold letter represents a vector or vector function in three space dimensions. For convenience, a product space B^n is often simply written as B, such as, $\mathbf{L}^2(\Omega) := [L^2(\Omega)]^3$. Suppose that a fixed time $T > 0$ is given and a metal material occupies a bounded C^2-domain $\Omega \subset \mathbf{R}^3$. The electric filed $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in Ω satisfy the Maxwell's equations (see [3,4]):

$$\begin{cases} \varepsilon\mathbf{E}_t + \sigma\mathbf{E}(x,t) = \nabla\times\mathbf{H}, (x,t)\in Q_T, \\ \mu\mathbf{H}_t + \nabla\times\mathbf{E} = 0, (x,t)\in Q_T, \end{cases}$$

where $Q_T = \Omega\times(0,T]$, ε, μ and σ are the electric permittivity, magnetic permeability, and electric conductivity, respectively. Since the induction material is highly conductive, the displacement current $\mathbf{J}_d = \varepsilon\mathbf{E}$ is very weak in comparison with the eddy currents $\mathbf{J}_e = \sigma\mathbf{E}$ and is negligible ($\varepsilon = 0$) [5]. Hence, Maxwell's equations become a single system with respect to $\mathbf{H}(x,t)$:

$$\mu\mathbf{H}_t + \nabla\times[\rho\nabla\times\mathbf{H}] = \mathbf{0}, (x,t)\in Q_T,$$

where $\rho = \frac{1}{\sigma}$ represents the electric resistivity, the permeability $\mu = \mu_1 - i\mu_2$, $\mu_1 > 0, \mu_2 > 0$ is a complex constant.

During the heating process, the local density of Joule's heat is described as (see [5,6])

$$Q(x,t) = \mathbf{E}\cdot\mathbf{J}^* = \rho|\nabla\times\mathbf{H}|^2, (x,t)\in Q_T,$$

where $\mathbf{J}^*$ is the complex conjugate of $\mathbf{J}$.

When the metal material is heated, it will melt. Therefore, solid and liquid coexist in the heated object during induction heating. This process can be modeled as system of phase field equations. A solid-liquid coexistence region which is usually described by using the interface with finite thickness ξ (See [2]). Let function u be the temperature and phase function ϕ describe the degree of melting. Then, in the process of induction heating, the phase change phenomena can be modeled by a strong coupling system as follows (a similar model can refer to the literature [2]):

$$\begin{cases} u_t + \frac{1}{2}l\phi_t - \nabla\cdot(k(x)\nabla u) = \rho|\nabla\times\mathbf{H}|^2, \quad (x,t)\in Q_T, \\ \tau\phi_t - \xi^2\triangle\phi - \frac{1}{2}(\phi - \phi^3) = 2u, \quad (x,t)\in Q_T, \end{cases}$$

where l, τ and $k(x)$ are the latent heat (per unit mass), relaxation time and the thermal conductivity, respectively.

Since magnetic field is more difficult to measure and control than electric field in engineering practice, we consider the boundary control problem of electric field. In addition, some parts of the boundary are insulated for ease of operation in the specific induction heating and melting process. Therefore, a part of the boundary is adiabatic, the other part is used to control magnetic field by electric field. After imposing the initial boundary value and normalizing certain physical parameters, we present the following system of phase field arising from inductive heating:

$$
\begin{cases}
\mathbf{H}_t + \nabla\times\nabla\times\mathbf{H} = \mathbf{0}, \quad (x,t)\in Q_T, & (1.1)\\
u_t - \nabla\cdot(k(x)\nabla u) = |\nabla\times\mathbf{H}|^2 - \frac{1}{2}l\phi_t, \quad (x,t)\in Q_T, & (1.2)\\
\phi_t - \triangle\phi - \frac{1}{2}(\phi-\phi^3) = 2u, \quad (x,t)\in Q_T, & (1.3)\\
\mathbf{n}\times\mathbf{H} = \mathbf{0}, \quad (x,t)\in S_{\Gamma_1} = \Gamma_1\times(0,T], & (1.4)\\
\mathbf{n}\times[\nabla\times\mathbf{H}(x,t)] = \mathbf{n}\times\mathbf{G}(x,t), (x,t)\in S_{\Gamma_2} = \Gamma_2\times(0,T], & (1.5)\\
\mathbf{H}(x,0) = \mathbf{H}_0(x), \quad x\in\Omega, & (1.6)\\
(u_{\mathbf{n}}, \phi_{\mathbf{n}}) = (0,0), \quad (x,t)\in S_\Gamma = \partial\Omega\times(0,T], & (1.7)\\
(u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad x\in\Omega, & (1.8)
\end{cases}
$$

where the boundary $\partial\Omega = \Gamma_1\cup\Gamma_2$ is split into two disjoint measurable subsets Γ_1 and Γ_2, both of which are nonempty, $\mathbf{n}$ is the outward unit normal on $\partial\Omega, u_{\mathbf{n}} = \nabla u\cdot\mathbf{n}$ is the normal derivative on $\partial\Omega$ and $\mathbf{G}$ is the electric field generated by external optoelectronic devices which will be considered as a control variable.

In order to obtain optimal strategy for the electric field action such that the temperature profile at the final stage has a relative uniform distribution and minimum energy consumption, we state our optimal control problem as follows.

Optimal control problem (P): Assume that the temperature $u_T(\cdot)$ and degree of melting $\phi_T(\cdot)$ are given in $L^2(\Omega)$, we would like to find an optimal control $\mathbf{G}^*\in U_{ad}$ such that the cost functional

$$
\begin{aligned}
J(\mathbf{G};\mathbf{H},\phi,u) = {} & \frac{1}{2}\int_\Omega |u(x,T)-u_T(x)|^2dx + \frac{1}{2}\int_\Omega |\phi(x,T)-\phi_T(x)|^2dx\\
& + \frac{\lambda}{2}\int_0^T\int_{\Gamma_2}|\mathbf{G}(x,t)|^2dsdt
\end{aligned} \tag{1.9}
$$

achieves its minimum at $(\mathbf{H}^*,\phi^*,u^*)$ under the state equations by coupled systems (1.1)–(1.8), where $\lambda>0$ is a typical regularization parameter, $\mathbf{G}(\cdot,\cdot)$ belongs to the following admissible control set:

$$
U_{ad} = \Big\{\mathbf{G}\in L^2(0,T;L^2(\Gamma_2)) : \|\mathbf{G}\|_{L^2(0,T;L^2(\Gamma_2))} \le A_0 < +\infty\Big\},
$$

where A_0 is a known constant.

For optimal control problems of the system coupled by Maxwell's equations with nonlinear heat equation in microwave heating, there are several papers dealing with in recent years. Wei and Yin [7] discussed the existence and necessary conditions of the optimal control on the boundary electric field control. Further, they proved the regularity of weak solutions of coupled nonlinear systems and gave necessary conditions for uniform microwave heating when microwave acts in three directions in the paper [8]. The problem of realizing uniform microwave heating through frequency control was considered (see, for examples, Yin [9] and Liao [10]). The Bang-Bang properties for time optimal controls on microwave heating was also discussed in [11]. But the controlled system in researches above does not contain the phase field process.

Without Maxwell's equations, the research on the qualitative theory and optimal control of phase field equations is a very active topic from the 1980s. On phase field model, one can refer to Temam's related references [12] and Caginalp [2]. In order to more accurately express the subtle physical characteristics, some authors studied the phase field equations with corresponding parameters [13–16]. However, it is worth mentioning that, based on Caginalp's phase field model [2], Hoffman and Jiang [15] generalized the model and obtained the well-posedness of the solution for controlled system and necessary conditions for related optimal problem. In addition, the solidification or melting models of some metal alloys with multiple crystals were described. For the two kinds of crystallization [16], the well-posedness of the phase field equation was studied, and the maximum principle is deduced by using Dubovitskii-Milyutin method [17]. Limited by mathematical techniques, the well-posedness of the phase field equations of three crystals was studied only in one dimension [18]. Later, Colli studied the distributed optimal control problem of this kind of system to supplement this analysis [19]. There are abundant articles on several numerical simulation aspects of phase field model, such as [20–25]. However, the previous work did not involve the phase field coupling system (1.1)–(1.8) in the induction heating process. There exist some difficulties in this study. Firstly, the boundary conditions (1.4) and (1.5) are mixed. Secondly, the heat source $|\nabla \times \mathbf{H}|^2$ is required in $L^2(Q_T)$. Finally, the Fréchet differentiability of state variables $\mathbf{H}, u$ and ϕ on control variable $\mathbf{G}$ is also a complicated problem.

The rest of this article is organized as follows. In Section 2, some symbols and assumptions used in this paper are given, and we showed that there is unique solution for the controlled system. In Section 3, we prove several important properties of the control-to-state operator. Section 4 is devoted to the existence and necessary condition for optimal control problem. Some concluding remarks are given in Section 5.

2. Existence and uniqueness of solution for the underlying system

2.1. Existence of a weak solution for Maxwell's equations

For the sake of convenience, we recall some function spaces associated with Maxwell's equations. Let

$$V(curl,\Omega) = \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega)\}, X = \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega), \mathbf{n} \times \mathbf{M} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Then, $V(curl,\Omega)$ and X are Hilbert spaces equipped with inner product

$$(\mathbf{M},\mathbf{N}) = \int_\Omega [(\nabla \times \mathbf{M}) \cdot (\nabla \times \mathbf{N}^*) + \mathbf{M} \cdot \mathbf{N}^*],$$

where $\mathbf{N}^*$ represents the complex conjugate of $\mathbf{N}$. Obviously, X is a linear subspace of $V(curl,\Omega)$. A norm on $V(curl,\Omega)$ and X is given by $\|\cdot\|_{V(curl,\Omega)} = \sqrt{(\cdot,\cdot)}$.

To take account of the boundary conditions, we introduce two trace mappings $\Upsilon_t : V(curl,\Omega) \to Y(\partial\Omega)$ and $\Upsilon_T : V(curl,\Omega) \to Y'(\partial\Omega)$ (the dual space of $Y(\partial\Omega)$) defined by $\Upsilon_t(\mathbf{M}) = \mathbf{n} \times \mathbf{M}|_{\partial\Omega}$ and $\Upsilon_T(\mathbf{M}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{M}|_{\partial\Omega})$, respectively, where $\mathbf{n}$ is the above description and $Y(\partial\Omega)$ is a Hilbert space (see [26]) as follow

$$Y(\partial\Omega) = \{\mathbf{f} \in H^{-\frac{1}{2}}(\Omega) : \text{there exists } \mathbf{M} \in V(curl,\Omega) \text{ with } \Upsilon_t(\mathbf{M}) = \mathbf{f}\},$$

with norm

$$\|\mathbf{f}\|_{Y(\partial\Omega)} = \inf_{\mathbf{M}\in V(curl,\Omega),\Upsilon_t(\mathbf{M})=\mathbf{f}} \|\mathbf{M}\|_{V(curl,\Omega)}.$$

We begin with some basic assumptions.

A(2.1) The vector function $\mathbf{H}_0(\cdot) \in L^2(\Omega)$ is given and nonnegative.

A(2.2) For almost every $t \in (0, T)$, the function $G(\cdot, t)$ is defined on Γ_2 with an extension such that $\mathbf{G}(\cdot, t) \in V(curl, \Omega)$ with extended function $\bar{\mathbf{G}}(\cdot, t)$ satisfies

$$||\bar{\mathbf{G}}(\cdot, t)||_{V(curl,\Omega)} \leq C_0||\mathbf{G}(\cdot, t)||_{L^2(\Gamma_2)},$$

where C_0 is a constant that depends only on Ω.

For convenience, we shall denote the extended function $\bar{\mathbf{G}}(\cdot, t)$ by $\mathbf{G}(\cdot, t)$ with $\mathbf{G}(\cdot, t) \in V(curl, \Omega)$.

Lemma 1. *Under the assumptions **A(2.1)** and **A(2.2)**, there is a unique weak solution* $\mathbf{H} \in L^2(0, T; X)$ *for the parabolic problem* (1.1) *and* (1.4)–(1.6)*. The weak solution* $\mathbf{H} \in L^2(0, T; X)$ *is defined by*

$$\begin{aligned} &-\int_0^T \int_\Omega \mathbf{H} \cdot \mathbf{\Phi}_t dxdt + \int_0^T \int_\Omega (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{\Phi}) dxdt \\ &= \int_\Omega \mathbf{H}_0 \cdot \mathbf{\Phi}(x, 0) dx - \int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{\Phi}) \rangle_{\Gamma_2} dt, \end{aligned}$$

for any function $\mathbf{\Phi} \in H^1(0, T; X)$ *with* $\mathbf{\Phi}(x, T) = 0$ *a.e.* $x \in \Omega$*. Moreover, there are constants* $C_1 \geq 0$ *and* $C_2 \geq 0$*, which depend only on known data, such that*

$$\sup_{t\in[0,T]} ||\mathbf{H}(\cdot, t)||_{L^2(\Omega)} + ||\mathbf{H}||_{L^2(0,T;V(curl,\Omega))} \leq C_1(||\mathbf{G}||_{L^2(0,T;L^2(\Gamma_2))} + ||\mathbf{H}_0||_{L^2(\Omega)}), \tag{2.1}$$

$$||\nabla \times \mathbf{H}(\cdot, t)||^6_{L^6(\Omega)} \leq C_2||\mathbf{G}(\cdot, t)||^2_{L^2(\Omega)}, \ a.e.\ t \in [0, T]. \tag{2.2}$$

Proof. Since (1.1) is a linear equations with conditions (1.4)–(1.6), the existence and uniqueness of the solution can be proved by Galerkin's method (see [27]). Meanwhile, estimates (2.1) are also derived (see [27], Chapter III, Theorem 2.2). According to $\mathbf{E} = \nabla \times \mathbf{H}$ in Section 1 and Lemma 3 in [7], we find that $\nabla \times \mathbf{H} = \mathbf{E} \in H^1(\Omega)$ and $||\nabla \times \mathbf{H}||_{H^1(\Omega)} \leq C_2||\mathbf{G}||^2_{L^2(\Omega)}$. Finally, by virtue of Sobolev's embedding theorem with $N = 3$, we can derive the desired estimate (2.2). □

Remark 1. *Note that the boundary condition* (1.4) *gives no information about* $\mathbf{n} \times [\nabla \times \mathbf{H}]$ *on* S_{Γ_1}*. We eliminate this portion of the integral by taking test function* $\mathbf{\Phi}$ *such that* $\Upsilon_T(\mathbf{\Phi}) = 0$ *on* S_{Γ_1}*. Hence, the variational problem of Lemma 1 leaves only the integral term on* S_{Γ_2}*.*

Next, a stability result is established which will be useful for deriving the first-order necessary optimality conditions for Problem **(P)** in Section 1.

Lemma 2. *In addition to the assumptions **A(2.1)** and **A(2.2)**, assume that* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are two weak solutions of the system* (1.1) *and* (1.4)–(1.6) *corresponding to* $\mathbf{G}_1$ *and* $\mathbf{G}_2$*, respectively. Then the following stability estimate holds:*

$$||\mathbf{H}_1 - \mathbf{H}_2||_{L^2(0,T;V(curl,\Omega))} \leq C||\mathbf{G}_1 - \mathbf{G}_2||_{L^2(0,T;L^2(\Gamma_2))},$$

*where the constant C depends only on known data in **A(2.2)**.*

Proof. Define $\mathbf{G} = \mathbf{G}_1 - \mathbf{G}_2$ and $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$. Then, in the sense of weak solution, $\mathbf{H} \in L^2(0,T;X)$ solves the system

$$\begin{cases} \mathbf{H}_t + \nabla\times\nabla\times\mathbf{H} = \mathbf{0}, \quad (x,t)\in Q_T, \\ \mathbf{n}\times\mathbf{H} = \mathbf{0}, \quad (x,t)\in S_{\Gamma_1}, \\ \mathbf{n}\times[\nabla\times\mathbf{H}(x,t)] = \mathbf{n}\times\mathbf{G}(x,t), (x,t)\in S_{\Gamma_2}, \\ \mathbf{H}(x,0) = \mathbf{0}, \quad x\in\Omega. \end{cases}$$

By means of similar estimates in (2.1), it can then be shown that the required estimate holds. □

2.2. Existence and uniqueness of solution for phase field equations

With the weak solution $\mathbf{H} \in L^2(0,T;X)$ of problem (1.1) and (1.4)–(1.6), we now turn to study the problem (1.2), (1.3) and (1.6)–(1.8) (i.e., the phase field problem). Let us list some Banach spaces:

$$W(0,T) = \left\{u\in L^2(0,T;H^1(\Omega)) : u_t\in L^2(0,T;H^{-1}(\Omega))\right\},$$
$$W_2^{2,1}(Q_T) = \{u\in L^2(Q_T) : u_{x_i}, u_{x_ix_j}, u_t \in L^2(Q_T), i=1,2,3\},$$
$$W_4^2(\Omega) = \{u\in L^4(\Omega) : u_{x_i}, u_{x_ix_j} \in L^4(\Omega), i=1,2,3\}.$$

We impose some basic assumptions which ensure the well-posedness of controlled system:
A(2.3) (a) Let $u_0(\cdot), u_T(\cdot)\in L^2(\Omega)$ be nonnegative.
(b) The function $k:\Omega\to R$ is given with $0<k_1\le k(x)\le k_2$ for constants k_1 and k_2.
A(2.4) Function $\phi_0(\cdot)\in W_4^2(\Omega)$ satisfying $\frac{\partial\phi_0}{\partial\mathbf{n}}|_{\partial\Omega}=0$ and $\phi_T(\cdot)\in L^2(\Omega)$ is nonnegative.

In order to study the phase field model, we state some lemmas.

Lemma 3. *(Lions-Peetre's embedding theorem [28])The embedding $W_2^{2,1}(Q_T)\hookrightarrow L^{10}(Q_T)$ is continuous. Moreover, whenever $2\le\tilde{p}<10$, the embedding $W_2^{2,1}(Q_T)\hookrightarrow L^{\tilde{p}}(Q_T)$ is compact.*

Lemma 4. *Assume that $v\in L^2(Q_T)$ and $\psi_0\in W_4^2(\Omega)$ with $\frac{\partial\psi_0}{\partial\mathbf{n}}|_{\partial\Omega}=0$. Then the following problem*

$$\begin{cases} \psi_t - \triangle\psi - \frac{1}{2}(\psi-\psi^3) = v(x,t), \quad (x,t)\in Q_T, \\ \psi_{\mathbf{n}}(x,t) = 0, \quad (x,t)\in S_\Gamma, \\ \psi(x,0) = \psi_0(x), \quad x\in\Omega, \end{cases} \tag{2.3}$$

has a unique strong solution $\psi\in W_2^{2,1}(Q_T)$ satisfying

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le C(\|\psi_0\|_{W_4^2(\Omega)} + \|v\|_{L^2(\Omega)} + C_0), \tag{2.4}$$

where $C_0=\max_{y\in\mathbf{R}}(\frac{3}{2}y^2-\frac{1}{4}y^4)$ and C is a constant only depends on Ω and T.

Proof. The proof is given by applying Leray-Schauder's fixed point theorem (see [29]). To this end, let $B=L^6(Q_T)$ and consider the mapping $T_\sigma : B\to B$, which is given by

$$T_\sigma(\eta) = \psi,$$

where $\sigma \in [0,1]$ and ψ is the unique solution of the following linear system:

$$\begin{cases} \psi_t - \triangle\psi = \sigma[\frac{1}{2}(\eta - \eta^3) + 2v(x,t)], \quad (x,t) \in Q_T, \\ \psi_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_\Gamma, \\ \psi(x,0) = \psi_0(x), \quad x \in \Omega. \end{cases} \tag{2.5}$$

It is not hard to know that $\sigma[\frac{1}{2}(\eta - \eta^3) + 2v] \in L^2(Q_T)$. By using the L^p-theory of parabolic equation, the system (2.5) has a unique solution $\psi \in W_2^{2,1}(Q_T)$. From Lemma 3, mapping T_σ is compact on B.

Next, let $\psi \in B$ be a fixed point of T_σ for some $\sigma \in [0,1]$. Thus, ψ solves the following problem

$$\begin{cases} \psi_t - \triangle\psi = \sigma[\frac{1}{2}(\psi - \psi^3) + 2v(x,t)], \quad (x,t) \in Q_T, \\ \psi_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_\Gamma, \\ \psi(x,0) = \psi_0(x), \quad x \in \Omega. \end{cases} \tag{2.6}$$

Multiplying Eq (2.6) by ψ, integrating on $(0,t) \times \Omega$ with $t \in [0,T]$, integrating by parts and applying Young's inequality, it follows that

$$\begin{aligned} &\int_\Omega \psi^2(t)dx + \int_0^t \int_\Omega |\nabla\psi|^2 dxdt + \int_0^t \int_\Omega |\psi|^4 dxdt \\ \le &C_1\Big[\int_0^T \int_\Omega v^2 dxdt + \int_\Omega \psi_0^2(x)dx + \max_{y\in\mathbf{R}}(\frac{3}{2}y^2 - \frac{1}{4}y^4)\Big], \end{aligned} \tag{2.7}$$

where C_1 only depends on Ω and T.

Multiplying Eq (2.6) by ψ_t and making the similar operation as before, it is not hard to get that

$$\begin{aligned} &\int_\Omega \psi^4(t)dx + \int_\Omega |\nabla\psi|^2 dxdt + \int_0^t \int_\Omega \psi_t^2 dxdt \\ \le &C_2\Big(\int_\Omega |\nabla\psi_0|^2 dxdt + \int_0^t \int_\Omega v^2 dxdt + \int_\Omega \psi_0^4 dxdt + \int_\Omega \psi^2(t)dx\Big), \end{aligned} \tag{2.8}$$

where C_2 only depends on T.

Multiplying Eq (2.6) by $-\triangle\psi$ and performing the similar calculation, it yields that

$$\begin{aligned} &\int_\Omega |\nabla\psi(t)|^2 dx + \int_0^t \int_\Omega |\triangle\psi|^2 dxdt + \int_0^t \int_\Omega \psi^2|\nabla\psi|^2 dxdt \\ \le &C_3\Big(\int_\Omega |\nabla\psi_0|^2 dxdt + \int_0^t \int_\Omega v^2 dxdt + \int_0^t \int_\Omega |\nabla\psi|^2 dxdt\Big), \end{aligned} \tag{2.9}$$

where C_3 only depends on T.

According to (2.7)–(2.9), we take $C = \max(C_1, C_2, C_2) \ge 0$ which depends on Ω and T, such that

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le C(\|\psi_0\|_{W_4^2(\Omega)} + \|v\|_{L^2(\Omega)} + C_0), \tag{2.10}$$

where $C_0 = \max_{y\in\mathbf{R}} (\frac{3}{2}y^2 - \frac{1}{4}y^4)$.

By the compact embedding $W_2^{2,1}(Q_T) \hookrightarrow L^6(Q_T)$, we get

$$\|\psi\|_B \le C\|\psi\|_{W_2^{2,1}(Q_T)} \le C'.$$

where the positive constant C' does not depend on σ.

By Leray-Schauder's fixed point theorem, there exists a fixed point $\psi \in L^6(Q_T) \cap W_2^{2,1}(Q_T)$ for the operator T_1, i.e., $\psi = T_1\psi$.

The rest of the work is to prove the uniqueness of the solution. Let $v \in L^2(Q_T)$ and $\psi_1, \psi_2 (i = 1, 2)$ be two solutions of problem (2.3). Then $\Psi := \psi_1 - \psi_2$ satisfies the following problem

$$\begin{cases} \Psi_t - \triangle\Psi = D(\psi_1, \psi_2)\Psi, \quad (x,t) \in Q_T, \\ \Psi_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_\Gamma, \\ \Psi(x,0) = 0, \quad x \in \Omega, \end{cases} \tag{2.11}$$

where $D(\psi_1, \psi_2) = \frac{1}{2}[1 - (\psi_1^2 + \psi_1\psi_2 + \psi_2^2)] \in L^5(Q_T)$ by applying Lemma 3 and $D(\psi_1, \psi_2) \le \frac{1}{2}$.

Testing the equation in (2.11) by $e^{-t}\Psi(x,t)$, integrating by parts, it holds that

$$\int_\Omega \Psi^2(t)e^{-t}dx + \int_0^t \int_\Omega |\nabla\Psi|^2 e^{-t}dxdt = \int_0^t \int_\Omega \Big(-\frac{1}{2} + D\Big)\Psi^2 e^{-t}dxdt \le 0.$$

Therefore, one can obtain that

$$\int_\Omega \Psi^2(t)e^{-t}dx = \int_0^t \int_\Omega |\nabla\Psi|^2 e^{-t}dxdt = 0,$$

which implies that $\Psi = 0$ a.e. $x \in \Omega$ for every $t \in [0, T]$. □

Lemma 5. *[30, p175–177] Under assumption **A(2.3)**, for any given function $f \in L^2(Q_T)$, the following problem:*

$$\begin{cases} u_t - \nabla \cdot (k(x)\nabla u) = f(x,t), \quad (x,t) \in Q_T, \\ u_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_\Gamma, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

has a unique weak solution $u \in W(0,T)$. Moreover, there exists a positive constant C which depends on known data, such that

$$\|u\|_{W(0,T)} \le C(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}). \tag{2.12}$$

It follows from Lemmas 4 and 5 that there is a unique solution for the problem (1.2)–(1.3) and (1.7)–(1.8).

Lemma 6. *Under the assumptions **A(2.1)–A(2.4)**, the coupled system* (1.2)–(1.3) *with the initial boundary condition* (1.7)–(1.8) *has at least one solution $(u, \phi) \in W(0,T) \times W_2^{2,1}(Q_T)$ for any given* $\mathbf{H} \in L^2(0, T; X)$. *Moreover, the following estimates hold:*

$$\|u\|_{W(0,T)} + \|\phi\|_{W_2^{2,1}(Q_T)} \le C(\|\phi_0\|_{W_4^2(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))} + C_0'), \tag{2.13}$$

where the positive constants C and C_0' depend on known data.

Proof. Let $B = L^2(Q_T)$. Define a mapping $T_\sigma : B \to B$ by

$$T_\sigma v = u,$$

where $\sigma \in [0,1]$ and u is the unique solution of the following problem:

$$\begin{cases} u_t - \nabla\cdot(k(x)\nabla u) = |\nabla\times\mathbf{H}|^2 - \frac{1}{2}l\phi_t, \quad (x,t)\in Q_T,\\ \phi_t - \triangle\phi - \frac{1}{2}(\phi-\phi^3) = 2\sigma v, \quad (x,t)\in Q_T,\\ (u_{\mathbf{n}}(x,t), \phi_{\mathbf{n}}(x,t)) = (0,0), \quad (x,t)\in S_\Gamma,\\ (u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad x\in\Omega. \end{cases} \tag{2.14}$$

The first equation of (2.14) has a unique solution by Lemma 4. Combining Lemma 1, we can get $|\nabla\times\mathbf{H}(x,t)|^2 - \frac{1}{2}l\phi_t(x,t) \in L^2(Q_T)$. It follows from Lemma 5 that the second equation of (2.14) has a unique solution $u \in W(0,T)$. Note that $W(0,T) \hookrightarrow L^2(Q_T)$ are compact by Aubin's Lemma ([30], p148). Therefore, the mapping T_σ is well defined and compact from B into B.

It remains to estimate all fixed points of T_σ. Assume that $u\in B$ is a fixed point, i.e., $T_\sigma u = u$. Then,

$$\begin{cases} u_t - \nabla\cdot(k(x)\nabla u) = |\nabla\times\mathbf{H}|^2 - \frac{1}{2}l\phi_t, \quad (x,t)\in Q_T,\\ \phi_t - \triangle\phi - \frac{1}{2}(\phi-\phi^3) = 2\sigma u, \quad (x,t)\in Q_T,\\ (u_{\mathbf{n}}(x,t), \phi_{\mathbf{n}}(x,t)) = (0,0), \quad (x,t)\in S_\Gamma,\\ (u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad x\in\Omega. \end{cases} \tag{2.15}$$

Multiplying the first equation of (2.15) by $u + \frac{1}{2}l\sigma\phi$, integrating over $\Omega\times(0,t)$ with $0\le t\le T$, integrating by parts, using Young's inequality, we know that

$$\begin{aligned}
&\frac{1}{2}(1-\sigma l\varepsilon')\int_\Omega u^2(t)dx + \frac{1}{2}\Big(\frac{l^2\sigma^2}{4} - \sigma l C(\varepsilon')\Big)\int_\Omega \phi^2(t)dx\\
&+ k_1\Big(1-\frac{l\sigma}{2}\varepsilon''\Big)\int_0^t\int_\Omega |\nabla u|^2dxdt - \frac{lk_1\sigma}{2}C(\varepsilon'')\int_0^t\int_\Omega|\nabla\phi|^2dxdt\\
\le& C\Big(\int_0^t\int_\Omega u^2dxdt + \int_0^t\int_\Omega\phi^2dxdt + \|u_0\|^2_{L^2(\Omega)} + \|\phi_0\|^2_{L^2(\Omega)} + \|\mathbf{G}\|^4_{L^2(0,T;L^2(\Gamma_2))}\Big),
\end{aligned} \tag{2.16}$$

where ε' and ε'' are two sufficiently small parameters, $C(\varepsilon')$ and $C(\varepsilon'')$ are sufficiently large constants corresponding to ε' and ε'', respectively.

Multiplying the first equation of (2.15) by ϕ and acting the similar process as before, it yields that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\phi^2(t)dx + \int_0^t\int_\Omega|\nabla\phi|^2dxdt + \frac{1}{4}\int_0^t\int_\Omega\phi^4dxdt\\
\le& C\Big[\max_{\substack{y\in\mathbf{R}\\(x,t)\in Q_T}}\Big(\frac{1}{2}y^2 - \frac{1}{4}y^4\Big) + \|\phi_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega u^2dxdt + \int_0^t\int_\Omega\phi^2dxdt\Big].
\end{aligned} \tag{2.17}$$

Multiplying (2.17) by $A>0$ and adding it to (2.16). It can be easily deduced that

$$\begin{aligned}
&\frac{1}{2}(1-\sigma l\varepsilon')\int_\Omega u^2(t)dx + \frac{1}{2}\Big(A + \frac{l^2\sigma^2}{4} - \sigma lC(\varepsilon')\Big)\int_\Omega\phi^2(t)dx + k_1\Big(1-\frac{l\sigma}{2}\varepsilon''\Big)\int_0^t\int_\Omega|\nabla u|^2dxdt\\
&+\Big(A - \frac{lk_1\sigma}{2}C(\varepsilon'')\Big)\int_0^t\int_\Omega|\nabla\phi|^2dxdt + \frac{A}{4}\int_0^t\int_\Omega\phi^4dxdt
\end{aligned}$$

$$\leq C\Big[\int_0^t\int_\Omega (u^2+\phi^2)dxdt + \|u_0\|^2_{L^2(\Omega)} + \|\phi_0\|^2_{L^2(\Omega)} + \|\mathbf{G}\|^4_{L^2(0,T;L^2(\Gamma_2))} + C_0'\Big], \tag{2.18}$$

where $C_0' = \max_{y\in\mathbf{R}}(\frac{1}{2}y^2 - \frac{1}{4}y^4)$ and $A \in \mathbf{R}$ is an arbitrary parameter. Choosing the appropriate A, ε' and ε'', it can be ensured that all the integrals on the left side of (2.18) are positive. This means that there exists a constant $C > 0$ such that

$$\begin{aligned}\int_\Omega [u^2(t)+\phi^2(t)]dx \leq & \int_\Omega [u^2(t)+\phi^2(t)]dx + \int_0^t\int_\Omega (|\nabla u|^2 + |\nabla\phi|^2 + \phi^4)dxdt \\ \leq & C\Big[\int_0^t\int_\Omega (u^2+\phi^2)dxdt + \|u_0\|^2_{L^2(\Omega)} + \|\phi_0\|^2_{L^2(\Omega)} + \|\mathbf{G}\|^4_{L^2(0,T;L^2(\Gamma_2))} + C_0'\Big].\end{aligned} \tag{2.19}$$

By Gronwall's inequality, it implies that

$$\int_\Omega [u^2(t)+\phi^2(t)]dx \leq C\Big(\|u_0\|^2_{L^2(\Omega)} + \|\phi_0\|^2_{L^2(\Omega)} + \|\mathbf{G}\|^4_{L^2(0,T;L^2(\Gamma_2))} + C_0'\Big),$$

Substituting the previous inequlity into (2.19), it produces that

$$\begin{aligned}&\int_\Omega [u^2(t)+\phi^2(t)]dx + \int_0^t\int_\Omega (|\nabla u|^2 + |\nabla\phi|^2 + \phi^4)dxdt \\ \leq & 2C\Big(\|u_0\|^2_{L^2(\Omega)} + \|\phi_0\|^2_{L^2(\Omega)} + \|\mathbf{G}\|^4_{L^2(0,T;L^2(\Gamma_2))} + C_0'\Big)\end{aligned} \tag{2.20}$$

for all $t \in [0, T]$.
Multiplying the second equation of (2.14) respectively by ϕ_t and $-\triangle\phi$ and performing the similar calculation, one can get

$$\int_\Omega [\phi^4(t) + |\nabla\phi(t)|^2]dx + \int_0^t\int_\Omega \phi_t^2 dxdt \leq C(\|\nabla\phi_0\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} + \|\phi_0\|^4_{L^4(\Omega)} + \|u\|^2_{L^2(Q_T)}) \tag{2.21}$$

and

$$\int_\Omega |\nabla\phi(t)|^2 dx + \int_0^t\int_\Omega (|\triangle\phi|^2 + \phi^2|\nabla\phi|^2)dxdt \leq C(\|\nabla\phi_0\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(Q_T)} + \|u\|^2_{L^2(Q_T)}). \tag{2.22}$$

Combining (2.20)–(2.22), one can obtain

$$\|\phi\|_{W_2^{2,1}(Q_T)} \leq C(\|\phi_0\|_{W_4^2(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))} + C_0'). \tag{2.23}$$

Invoking (2.12) in Lemma 5, one have the following estimate

$$\int_0^t \|u_t\|_{H^{-1}(\Omega)}dt \leq C(\|u_0\|_{L^2(\Omega)} + \|\phi_t\|_{L^2(Q_T)} + \|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))}). \tag{2.24}$$

Estimates (2.20) and (2.24) yield to

$$\|u\|_{W(0,T)} \leq C(\|u_0\|_{L^2(\Omega)} + \|\phi_0\|_{L^2(Q_T)} + \|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))} + C_0'). \tag{2.25}$$

It follows from (2.25) and the embedding $W(0,T) \hookrightarrow L^2(Q_T)$ that all fixed points of T_σ are uniformly bounded in B. By Leray–Schauder's fixed point theorem, there exists a fixed point $u \in L^2(Q_T) \cap W(0,T)$ of the operator T_1, i.e., $u = T_1 u$. □

Next, we will establish a stability result.

Lemma 7. *Suppose that* $\mathbf{H}_i (i = 1, 2)$ *are two weak solutions of the system* (1.1) *and* (1.4)–(1.6) *corresponding to* $\mathbf{G}_i (i = 1, 2)$ *and* $(u_i, \phi_i)(i = 1, 2)$ *are two solutions of the phase equations* (1.2), (1.3), (1.7) *and* (1.8). *Then the following stability estimate holds.*

$$\|u_1 - u_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{W_2^{2,1}(Q_T)} \leq C\|\mathbf{G}_1 - \mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}, \tag{2.26}$$

where the constant C depends only on known data in ***A(2.2)***.

Proof. Define $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2, u = u_1 - u_2$ and $\phi = \phi_1 - \phi_2$. Then $(u, \phi) \in W(0, T) \times W_2^{2,1}(Q_T)$ is a solution of the following problem.

$$\begin{cases} u_t - \nabla \cdot [k(x)\nabla u] = \nabla \times (\mathbf{H}_1 + \mathbf{H}_2) \cdot \nabla \times \mathbf{H} - \frac{1}{2} l\phi_t, \quad (x, t) \in Q_T, \\ \phi_t - \triangle\phi = D(\phi_1, \phi_2)\phi + 2u, \quad (x, t) \in Q_T, \\ (u_{\mathbf{n}}(x, t), \phi_{\mathbf{n}}(x, t)) = (0, 0), \quad (x, t) \in S_\Gamma, \\ (u(x, 0), \phi(x, 0)) = (0, 0), \quad x \in \Omega, \end{cases} \tag{2.27}$$

where $D(\phi_1, \phi_2) = \frac{1}{2}[1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)] \leq \frac{1}{2}$ and $D(\phi_1, \phi_2) \in L^5(Q_T)$.

Multiplying the second equation of (2.27) by $e^{-t}\phi$, integrating on $\Omega \times (0, t)$, one can obtain that

$$\int_\Omega \phi^2(t)dx + \int_0^t \int_\Omega |\nabla\phi|^2 dxdt \leq C\Big(\int_0^t \int_\Omega u^2 dxdt + \int_0^t \int_\Omega \phi^2 dxdt\Big). \tag{2.28}$$

Gronwall's inequality implies that

$$\int_\Omega \phi^2(t)dx + \int_0^t \int_\Omega |\nabla\phi|^2 dxdt \leq C \int_0^T \int_\Omega u^2 dxdt. \tag{2.29}$$

Testing the first equation of (2.27) by $u + \frac{l}{2}\phi$, integrating by parts and using Young's, Hölder's and Cauchy's inequalities, we get

$$\begin{aligned} &\frac{1 - l\varepsilon'}{2} \int_\Omega u^2(t)dx + \Big(\frac{l}{8} - \frac{lC(\varepsilon')}{2}\Big) \int_\Omega \phi^2(t)dx + \Big(k_1 - \frac{l}{2}k_2\varepsilon''\Big) \int_0^t \int_\Omega |\nabla u|^2 dxdt \\ &- \frac{lk_2C(\varepsilon'')}{2} \int_0^t \int_\Omega |\nabla\phi|^2 dxdt \\ \leq &\frac{1}{2}\Big(\int_0^t \int_\Omega |\nabla \times (\mathbf{H}_1 + \mathbf{H}_2)|^4 dxdt\Big)^{\frac{1}{4}} \Big[\Big(\int_0^t \int_\Omega |\nabla \times \mathbf{H}|^4 dxdt\Big)^{\frac{1}{2}} + \int_0^t \int_\Omega |u|^2 dxdt\Big]. \end{aligned} \tag{2.30}$$

Mltiplying (2.28) by a parameter $A > 0$ and adding it to (2.30), next, choosing suitable parameters $A, \varepsilon', \varepsilon''$ such that the coefficients of $\int_\Omega u^2 dx, \int_\Omega \phi^2 dx, \int_0^t \int_\Omega |\nabla u|^2 dxdt$ and $\int_0^t \int_\Omega |\nabla\phi|^2 dxdt$ are all nonnegative, then there exists a constant C such that

$$\begin{aligned} &\int_\Omega [u^2(t) + \phi^2(t)]dx + \int_0^t \int_\Omega (|\nabla u|^2 + |\nabla\phi|^2) dxdt \\ \leq\ &C\Big[\Big(\int_0^t \int_\Omega |\nabla \times \mathbf{H}|^4 dxdt\Big)^{\frac{1}{2}} + \int_0^t \int_\Omega (|u|^2 + |\phi|^2) dxdt\Big]. \end{aligned}$$

Using the Gronwall's inequality, one obtains

$$\int_\Omega [u^2(t) + \phi^2(t)]dx + \int_0^T \int_\Omega (|\nabla u|^2 + |\nabla\phi|^2)dxdt \le C\Big(\int_0^T \int_\Omega |\nabla \times \mathbf{H}|^4 dxdt\Big)^{\frac{1}{2}}. \tag{2.31}$$

Similarly, testing the second equation of (2.27) by ϕ_t, integrating by parts and using Young's, Hölder's and Cauchy's inequalities, we get

$$\int_0^t \int_\Omega \phi_t^2 dxdt + \int_\Omega |\nabla\phi(t)|^2 dx \le C'\Big[\int_0^t \int_\Omega u^2 dxdt + \Big(\int_0^t \int_\Omega \phi^{\frac{10}{3}} dxdt\Big)^{\frac{3}{5}}\Big], \tag{2.32}$$

where C' depends on $\|D\|_{L^5(Q_T)}$.
Finally, testing the second equation of (2.27) by $-\triangle\phi$, integrating by parts and using Young's, Hölder's and Cauchy's inequality, we get

$$\int_0^t \int_\Omega |\triangle\phi|^2 dxdt + \int_\Omega |\nabla\phi(t)|^2 dx \le C''\Big[\int_0^t \int_\Omega |u|^2 dxdt + \Big(\int_0^t \int_\Omega \phi^{\frac{10}{3}} dxdt\Big)^{\frac{3}{5}}\Big], \tag{2.33}$$

where C'' depends on $\|D\|_{L^5(Q_T)}$.

Combining (2.31)–(2.33), we find that

$$\|\phi\|_{W_2^{2,1}(Q_T)} \le C\big(\|\phi\|_{L^{\frac{10}{3}}(Q_T)} + \|\nabla \times \mathbf{H}\|_{L^4(Q_T)}\big). \tag{2.34}$$

Since $W_2^{2,1}(Q_T) \hookrightarrow L^{\frac{10}{3}}(Q_T)$, the following interpolation inequality holds

$$\|\phi\|_{L^{\frac{10}{3}}(Q_T)} \le \varepsilon\|\phi\|_{W_2^{2,1}(Q_T)} + C(\varepsilon)\|\phi\|_{L^2(Q_T)}. \tag{2.35}$$

Substituting (2.35) in (2.34), choosing the appropriate ε and invoking (2.33), it follows that

$$\|\phi\|_{W_2^{2,1}(Q_T)} \le C\|\nabla \times \mathbf{H}\|_{L^4(Q_T)} \le C\|\mathbf{G}_1 - \mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}. \tag{2.36}$$

By Lemma 5, we also have

$$\|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le C\|\mathbf{G}_1 - \mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}. \tag{2.37}$$

Finally, estimates (2.31) and (2.37) yield

$$\|u\|_{W(0,T)} \le C\|\mathbf{G}_1 - \mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}.$$

□

Corollary 1. *Under the assumptions* ***A(2.1)–A(2.4)****, the solution* $(u, \phi) \in W(0,T)\times W_2^{2,1}(Q_T)$ *in Lemma 6 is unique.*

With Lemmas 1, 6 and 7, we can obtain the following theorems.

Theorem 1. *Under the assumptions* ***A(2.1)–A(2.4)****, the coupled system* (1.1)–(1.8) *has a unique solution* $(\mathbf{H}, u, \phi) \in L^2(0,T;X) \times W(0,T) \times W_2^{2,1}(Q_T)$*. Meanwhile, there exists a constant* K_1 *such that*

$$\sup_{t\in[0,T]} \|\mathbf{H}(\cdot,t)\|_{L^2(\Omega)} + \|\mathbf{H}\|_{L^2(0,T;V(curl,\Omega))} + \|u\|_{W(0,T)} + \|\phi\|_{W_2^{2,1}(Q_T)} \le K_1. \tag{2.38}$$

3. The control-to-state operator

Definition 1. *The mapping*

$$P : L^2(0,T;L^2(\Gamma_2)) \to L^2(0,T;X) \times W(0,T) \times W_2^{2,1}(Q_T), \mathbf{G} \mapsto P(\mathbf{G}) = (\mathbf{H}, u, \phi),$$

defined by Theorem 1 is called the control-to-state operator.

In this section, we give and show several important properties of P, which will be very useful in proving the existence of optimal control and deriving optimality conditions.

Theorem 2. *Under the assumptions* ***A(2.1)–A(2.4)****, the operator P is weakly sequentially continuous.*

Proof. Taking $\{\mathbf{G}_m\}_{m=1}^{\infty} \subset L^2(0,T;L^2(\Gamma_2))$ such that

$$\mathbf{G}_m \to \mathbf{G}^* \text{ weakly in } L^2(0,T;L^2(\Gamma_2)), \text{ whenever } m \to +\infty. \tag{3.1}$$

Assuming that $(\mathbf{H}_m, u_m, \phi_m)$ is the solution of (1.1)–(1.8) associated with $\mathbf{G}_m$ for $m = 1, 2, \cdots$, i.e.,

$$P(\mathbf{G_m}) = (\mathbf{H}_m, u_m, \phi_m).$$

We will show that there exists $(\mathbf{H}^*, u^*, \phi^*) \in L^2(0,T;X) \times W(0,T) \times W_2^{2,1}(Q_T)$ such that

$$P(\mathbf{G}_m) \to P(\mathbf{G}^*) = (\mathbf{H}^*, u^*, \phi^*)$$

weakly in $L^2(0,T;X) \times W(0,T) \times W_2^{2,1}(Q_T)$, as $m \to \infty$.

According to the definition of P, it follows that

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbf{H}_m \cdot \mathbf{\Phi}_t dxdt + \int_0^T \int_\Omega \nabla \times \mathbf{H}_m \cdot \nabla \times \mathbf{\Phi} dxdt \\
=\ &\int_\Omega \mathbf{H}_0(x) \cdot \mathbf{\Phi}(x,0)dx - \int_0^T \langle \mathbf{n} \times \mathbf{G}_m, \Upsilon_T(\mathbf{\Phi})\rangle_{\Gamma_2} dt, &(3.2)\\
&\int_0^T \int_\Omega u_m v_t dxdt + \int_0^T \int_\Omega k\nabla u_m \cdot \nabla v dxdt \\
=\ &\int_\Omega u_0(x)v(0,x)dx + \int_0^T \int_\Omega |\nabla \times \mathbf{H}_m|^2 v dxdt - \frac{l}{2}\int_0^T \int_\Omega (\phi_m)_t v dxdt, &(3.3)\\
&\int_0^T \int_\Omega \phi_m \eta_t dxdt + \int_0^T \int_\Omega \nabla\phi_m \cdot \nabla\eta dxdt + \frac{1}{2}\int_0^T \int_\Omega (\phi_m - \phi_m^3)\eta dxdt \\
=\ &\int_\Omega \phi_0(x)\eta(x,0)dx - 2\int_0^T \int_\Omega u_m \eta dxdt, &(3.4)
\end{aligned}$$

for any $v \in H^1(Q_T)$ with $v(T,x) = 0$, $\mathbf{\Phi} \in H^1(0,T;X)$ with $\mathbf{\Phi}(x,T) = 0$ and $\eta \in W_2^{2,1}(Q_T)$ with $\eta(x,T) = 0$. By Theorem 1, the sequence $\{(\mathbf{H}_m, u_m, \phi_m)\}$ is bounded in reflexive space $L^2(0,T;V(curl,\Omega)) \times W(0,T) \times W_2^{2,1}(Q_T)$. Thus, there is a subsequence of $\{(\mathbf{H}_m, u_m, \phi_m)\}$, again denoted by $\{(\mathbf{H}_m, u_m, \phi_m)\}$, such that

$$\mathbf{H}_m \ \to\ \mathbf{H}^* \text{ weakly in } L^2(0,T;V(curl,\Omega)), \tag{3.5}$$

$$\begin{aligned} u_m &\rightarrow u^* \text{ weakly in } L^2(Q_T), && (3.6)\\ \nabla u_m &\rightarrow \nabla u^* \text{ weakly in } L^2(0,T;L^2(\Omega)), && (3.7)\\ \nabla\times\mathbf{H}_m &\rightarrow \nabla\times\mathbf{H}^* \text{ weakly in } L^2(0,T;L^2(\Omega)), && (3.8)\\ \phi_m &\rightarrow \phi^* \text{ weakly in } W_2^{2,1}(Q_T). && (3.9)\end{aligned}$$

It follows from Eq (3.2) and (3.5), (3.8) that

$$\int_0^T \langle \mathbf{n}\times\mathbf{G}_m, \Upsilon_T(\mathbf{\Phi})\rangle_{\Gamma_2} dt \rightarrow \int_\Omega \mathbf{H}_0(x)\cdot\mathbf{\Phi}(x,0)dx - \int_0^T\int_\Omega \mathbf{H}^*\cdot\mathbf{\Phi}_t dxdt - \int_0^T\int_\Omega \nabla\times\mathbf{H}^*\cdot\nabla\times\mathbf{\Phi}dxdt$$

as $m\rightarrow\infty$. It can be verified from (3.1) that

$$\begin{aligned} &\int_0^T\int_\Omega \mathbf{H}^*\cdot\mathbf{\Phi}_t dxdt + \int_0^T\int_\Omega \nabla\times\mathbf{H}^*\cdot\nabla\times\mathbf{\Phi}dxdt\\ = &\int_\Omega \mathbf{H}_0(x)\cdot\mathbf{\Phi}(x,0)dx - \int_0^T \langle \mathbf{n}\times\mathbf{G}^*, \Upsilon_T(\mathbf{\Phi})\rangle_{\Gamma_2} dt \qquad (3.10)\end{aligned}$$

From the compact embedding $W_2^{2,1}(Q_T)\hookrightarrow L^6(\Omega)$, we obtain

$$\phi_m \rightarrow \phi^* \text{strongly in } L^6(Q_T). \qquad (3.11)$$

It follows from Eqs (3.4) and (3.6), (3.9), (3.11) that

$$\begin{aligned} &\int_0^T\int_\Omega \phi^*\eta_t dxdt + \int_0^T\int_\Omega \nabla\phi^*\cdot\nabla\eta dxdt + \frac{1}{2}\int_0^T\int_\Omega (\phi^* - (\phi^*)^3)\eta dxdt\\ = &\int_\Omega \phi_0(x)\eta(x,0)dx - 2\int_0^T\int_\Omega u^*\eta dxdt. \qquad (3.12)\end{aligned}$$

Since $\nabla\times\mathbf{H}_m(\cdot,t)\in L^6(\Omega)$ and $\|\nabla\times\mathbf{H}\|^6_{L^6(\Omega)} \le \|\mathbf{G}\|^2_{L^2(\Gamma_2)}$, we have

$$\nabla\times\mathbf{H}_m(\cdot,t)\rightarrow\nabla\times\mathbf{H}^*(\cdot,t) \text{ weakly in } L^6(\Omega). \qquad (3.13)$$

It follows from Eqs (3.4) and (3.6), (3.7), (3.13) that

$$\begin{aligned} &\int_0^T\int_\Omega u^* v_t dx + \int_0^T\int_\Omega k(x)\nabla u^*\cdot\nabla v dx\\ = &\int_\Omega u_0(x)v(0,x)dx + \int_0^T\int_\Omega |\nabla\times\mathbf{H}^*|^2 v dx - \frac{l}{2}\int_0^T\int_\Omega \phi^*_t v dxdt, \qquad (3.14)\end{aligned}$$

Equations (3.10), (3.12) and (3.14) imply that $(\mathbf{H}^*, u^*, \phi^*)$ is a weakly solution of problem (2.1) corresponding to $\mathbf{G}^*$. That is $P(\mathbf{G}^*) = (\mathbf{H}^*, u^*, \phi^*)$. The above statement shows that every subsequence has a subsequence converging to the same $(\mathbf{H}^*, u^*, \phi^*)$. Hence, the entire sequence $(\mathbf{H}_m, u_m, \phi_m)$ converges weakly to $(\mathbf{H}^*, u^*, \phi^*)$. □

With Lemmas 2 and 7, one can easily see that P is Lipschitz continuous.

Theorem 3. *Under the assumptions **A(2.1)–A(2.4)**, the operator P is Lipschitz continuous, that is, there exists a constant $L > 0$ such that*

$$\|\mathbf{H}_1 - \mathbf{H}_2\|_{L^2(0,T;V(curl,\Omega))} + \|u_1 - u_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{W_2^{2,1}(Q_T)} \le L\|\mathbf{G}_1 - \mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))},$$

whenever $\mathbf{G}_i \in L^2(0,T;L^2(\Gamma_2))$ and $(\mathbf{H}_i, u_i, \phi_i) = P(\mathbf{G}_i), i = 1, 2$.

Theorem 4. *Under the assumptions **A(2.1)–A(2.4)**, the operator P is Fréchet differentiable. Its directional derivative at $\mathbf{G} \in U_{ad}$ in the direction $\bar{\mathbf{G}}$ is given by*

$$P'(\mathbf{G})\bar{\mathbf{G}} = (\bar{\mathbf{H}}, \bar{u}, \bar{\phi}),$$

where the triple of functions $(\bar{\mathbf{H}}, \bar{u}, \bar{\phi})$ denotes the weakly solution of the following linear system at $(\mathbf{H}, u, \phi) = P(\mathbf{G})$:

$$\begin{cases} \bar{\mathbf{H}}_t + \nabla\times\nabla\times\bar{\mathbf{H}} = \mathbf{0}, (x,t) \in Q_T, \\ \bar{u}_t - \nabla\cdot[k(x)\nabla\bar{u}] = 2\nabla\times\mathbf{H}\cdot\nabla\times\bar{\mathbf{H}} - \frac{1}{2}l\bar{\phi}_t, (x,t) \in Q_T, \\ \bar{\phi}_t - \triangle\bar{\phi} - \frac{1}{2}\bar{\phi}(1-3\phi^2) = 2\bar{u}, \quad (x,t) \in Q_T, \\ \mathbf{n}\times\bar{\mathbf{H}} = \mathbf{0}, (x,t) \in S_{\Gamma_1}, \\ \mathbf{n}\times(\nabla\times\bar{\mathbf{H}}) = \mathbf{n}\times\bar{\mathbf{G}}, (x,t) \in S_{\Gamma_2}, \\ \bar{\mathbf{H}}(x,0) = \mathbf{0}, x \in \Omega. \\ (\bar{u}_{\mathbf{n}}(x,t), \bar{\phi}_{\mathbf{n}}(x,t)) = (0,0), (x,t) \in S_\Gamma, \\ (\bar{u}(x,0), \bar{\phi}(x,0)) = (0,0), x \in \Omega. \end{cases} \tag{3.15}$$

Proof. We present the proof in the following steps.

Step 1: We show that the operator P is Gâteaux differentiable for each $\mathbf{G} \in U_{ad}$.

Set $\mathbf{G}_\varepsilon = \mathbf{G} + \varepsilon\bar{\mathbf{G}}$ with a sufficiently small parameter $\varepsilon > 0$ such that $\mathbf{G} + \varepsilon\bar{\mathbf{G}} \in U_{ad}$. Moreover, assume that $(\mathbf{H}_\varepsilon, u_\varepsilon, \phi_\varepsilon)$ is the solution of (1.1)–(1.8) corresponding to $\mathbf{G}_\varepsilon$. Define

$$\bar{\mathbf{H}}_\varepsilon = \frac{\mathbf{H}_\varepsilon - \mathbf{H}}{\varepsilon}, \bar{u}_\varepsilon = \frac{u_\varepsilon - u}{\varepsilon}, \bar{\phi}_\varepsilon = \frac{\phi_\varepsilon - \phi}{\varepsilon}.$$

It is easily shown that $(\bar{\mathbf{H}}_\varepsilon, \bar{u}_\varepsilon, \bar{\phi}_\varepsilon)$ satisfies

$$\begin{cases} (\bar{\mathbf{H}}_\varepsilon)_t + \nabla\times\nabla\times\bar{\mathbf{H}}_\varepsilon = 0, (x,t) \in Q_T, \\ (\bar{u}_\varepsilon)_t - \nabla\cdot[k(x)\nabla\bar{u}_\varepsilon] = [\nabla\times(\mathbf{H}_\varepsilon + \mathbf{H})]\cdot\nabla\times\bar{\mathbf{H}}_\varepsilon - \frac{1}{2}l(\bar{\phi}_\varepsilon)_t, (x,t) \in Q_T, \\ (\bar{\phi}_\varepsilon)_t - \triangle\bar{\phi}_\varepsilon - \frac{1}{2}\bar{\phi}_\varepsilon[1 - (\phi_\varepsilon^2 + \phi_\varepsilon\phi + \phi^2)] = 2\bar{u}_\varepsilon, \quad (x,t) \in Q_T, \\ \mathbf{n}\times\bar{\mathbf{H}}_\varepsilon = 0, (x,t) \in S_{\Gamma_1}, \\ \mathbf{n}\times(\nabla\times\bar{\mathbf{H}}_\varepsilon) = \mathbf{n}\times\bar{\mathbf{G}}, (x,t) \in S_{\Gamma_2}, \\ \bar{\mathbf{H}}_\varepsilon(x,0) = 0, x \in \Omega. \\ ((\bar{u}_\varepsilon)_{\mathbf{n}}(x,t), (\bar{\phi}_\varepsilon)_{\mathbf{n}}(x,t)) = (0,0), (x,t) \in S_\Gamma, \\ (\bar{u}_\varepsilon(x,0), \bar{\phi}_\varepsilon(x,0)) = (0,0), x \in \Omega. \end{cases} \tag{3.16}$$

Under assumptions **A(2.2)** and **A(2.3)**, by using similar estimates as Theorem 3, we can derive the following estimates:

$$\|\bar{\mathbf{H}}_\varepsilon\|_{L^2(0,T;V(curl,\Omega))} + \|\bar{u}_\varepsilon\|_{W(0,T)} + \|\bar{\phi}_\varepsilon\|_{W_2^{2,1}(Q_T)} \quad \le C\|\bar{\mathbf{G}}\|_{L^2(0,T;L^2(\Gamma_2))}, \tag{3.17}$$

$$\|\mathbf{H}_\varepsilon-\mathbf{H}\|_{L^2(0,T;V(curl,\Omega))}+\|u_\varepsilon-u\|_{W(0,T)}+\|\phi_\varepsilon-\phi\|_{W_2^{2,1}(Q_T)}\quad\le C'\|\varepsilon\bar{\mathbf{G}}\|_{L^2(0,T;L^2(\Gamma_2))},\tag{3.18}$$

where C and C' depend only on known data. It follows from (3.17) that there exists a subsequence of $(\bar{\mathbf{H}}_\varepsilon,\bar{u}_\varepsilon,\bar{\phi}_\varepsilon)$ (still denoted by $(\bar{\mathbf{H}}_\varepsilon,\bar{u}_\varepsilon,\bar{\phi}_\varepsilon)$) and $(\bar{\mathbf{H}},\bar{u},\bar{\phi})\in L^2(0,T;V(curl,\Omega))\times W(0,T)\times W_2^{2,1}(Q_T)$ such that

$$\bar{\mathbf{H}}_\varepsilon\to\bar{\mathbf{H}}\text{ weakly in }L^2(0,T;V(curl,\Omega)),\tag{3.19}$$
$$\bar{u}_\varepsilon\to\bar{u}\text{ weakly in }L^2(Q_T),\tag{3.20}$$
$$\nabla\bar{u}_\varepsilon\to\nabla\bar{u}\text{ weakly in }L^2(0,T;L^2(\Omega)),\tag{3.21}$$
$$\nabla\times\bar{\mathbf{H}}_\varepsilon\to\nabla\times\bar{\mathbf{H}}\text{ weakly in }L^2(0,T;L^2(\Omega)),\tag{3.22}$$
$$\bar{\phi}_\varepsilon\to\bar{\phi}\text{ weakly in }W_2^{2,1}(Q_T),\tag{3.23}$$

as $\varepsilon\to0$.

It is easy to show that $\nabla\times\bar{\mathbf{H}}_\varepsilon(\cdot,t)\in L^6(\Omega)$ and $\|\nabla\times\bar{\mathbf{H}}_\varepsilon\|^6_{L^6(\Omega)}\le\|\bar{\mathbf{G}}\|^2_{L^2(\Gamma_2)}$. Therefore,

$$\nabla\times\bar{\mathbf{H}}_\varepsilon(\cdot,t)\to\nabla\times\bar{\mathbf{H}}(\cdot,t)\text{ weakly in }L^6(\Omega).\tag{3.24}$$

By the compact embedding $W_2^{2,1}(Q_T)\hookrightarrow L^i(Q_T)(i=2,3,\cdots,9)$ and (3.18)–(3.24), taking the limits of (3.16) as $\varepsilon\to0$, we obtain that $(\bar{\mathbf{H}},\bar{u},\bar{\phi})$ satisfies the problem (3.15).

The uniqueness of solution for the system (3.15) is easy to prove because it is linear. The calculations above means that the control-to-state operator P is Gâteaux differentiable at $\mathbf{G}$, that is,

$$DP(\mathbf{G};\bar{\mathbf{G}})=(\bar{\mathbf{H}},\bar{u},\bar{\phi})(\mathbf{G};\bar{\mathbf{G}}).$$

Step 2: We show that $DP(\mathbf{G};\bar{\mathbf{G}})$ is a linear bounded operator with respect to $\bar{\mathbf{G}}$.

Obviously, $DP(\mathbf{G};\bar{\mathbf{G}})$ is linear. Similar to Lemmas 2 and 7, it can be concluded from (3.15) that

$$\|\bar{\mathbf{H}}\|_{L^2(0,T;V(curl,\Omega))}+\|\bar{u}\|_{W(0,T)}+\|\bar{\phi}\|_{W_2^{2,1}(Q_T)}\le C\|\mathbf{K}\|_{L^2(0,T;L^2(\Gamma_2))},$$

which means that $DP(\mathbf{G};\bar{\mathbf{G}})$ is bounded with respect to $\bar{\mathbf{G}}$. Hence, one has

$$DP(\mathbf{G};\bar{\mathbf{G}})=(\bar{\mathbf{H}},\bar{u},\bar{\phi})(\mathbf{G};\bar{\mathbf{G}})=P'(\mathbf{G})\bar{\mathbf{G}},$$

where $P'(\mathbf{G})$ is a bounded and linear operator.

Step 3: We verify that $P'(\mathbf{G})$ is continuous with $\mathbf{G}$.

Choosing $\mathbf{G}_1,\mathbf{G}_2\in U_{ad}$, for any $\mathbf{K}\in U_{ad}$ with $\|\mathbf{K}\|_{L^2(0,T;L^2(\Gamma_2))}=1$, define

$$(\widetilde{\mathbf{H}},\widetilde{u},\widetilde{\phi}):=(\bar{\mathbf{H}},\bar{u},\bar{\phi})(\mathbf{G}_1;\mathbf{K})-(\bar{\mathbf{H}},\bar{u},\bar{\phi})(\mathbf{G}_2;\mathbf{K}).$$

It is easy to verify that $(\widetilde{\mathbf{H}},\widetilde{u},\widetilde{\phi})\in L^2(0,T;X)\times W(0,T)\times W_2^{2,1}(Q_T)$ satisfies the following system in weakly sense.

$$\begin{cases}\widetilde{\mathbf{H}}_t+\nabla\times\nabla\times\widetilde{\mathbf{H}}=\mathbf{0},\ \text{in }Q_T,\\ \widetilde{u}_t-\nabla\cdot[k(x)\nabla\widetilde{u}]=f_1-\frac{1}{2}l\widetilde{\phi}_t,\ \text{in }Q_T,\\ \widetilde{\phi}_t-\triangle\widetilde{\phi}=(\frac{1}{2}-\frac{1}{3}\phi^2(\mathbf{G}_2))\widetilde{\phi}+f_2+2\widetilde{u},\ \text{in }Q_T,\\ \mathbf{n}\times\widetilde{\mathbf{H}}=\mathbf{0},\ \text{in }S_{\Gamma_1},\\ \mathbf{n}\times(\nabla\times\widetilde{\mathbf{H}})=\mathbf{0},\ \text{in }S_{\Gamma_2},\\ \widetilde{\mathbf{H}}(x,0)=\mathbf{0},\ \text{in }\Omega,\\ (\widetilde{u}_{\mathbf{n}}(x,t),\widetilde{\phi}_{\mathbf{n}}(x,t))=(0,0),\ \text{in }S_\Gamma,\\ (\widetilde{u}(x,0),\widetilde{\phi}(x,0))=(0,0),\ \text{in }\Omega,\end{cases}\tag{3.25}$$

where

$$f_1 = 2[\nabla\times\mathbf{H}(\mathbf{G_1})\cdot\nabla\times\bar{\mathbf{H}}(\mathbf{G}_1;\mathbf{K}) - \nabla\times\mathbf{H}(\mathbf{G}_2)\cdot\nabla\times\bar{\mathbf{H}}(\mathbf{G}_2;\mathbf{K})],$$
$$f_2 = -\frac{1}{3}\bar{\phi}(\mathbf{G}_1;\mathbf{K})(\phi(\mathbf{G}_1)+\phi(\mathbf{G}_2))(\phi(\mathbf{G}_1)-\phi(\mathbf{G}_2)).$$

Obviously, (3.25) implies $\widetilde{\mathbf{H}} = \mathbf{0}$ which directly means $\bar{\mathbf{H}}(\mathbf{G}_1;\mathbf{K}) = \bar{\mathbf{H}}(\mathbf{G}_2;\mathbf{K})$. Therefore, (3.25) can be simplified as the following system

$$\begin{cases} \widetilde{u}_t - \nabla\cdot[k(x)\nabla\widetilde{u}] = 2\nabla\times\mathbf{H}(\mathbf{G}_1,\mathbf{K})\cdot\nabla\times(\mathbf{H}(\mathbf{G}_1)-\mathbf{H}(\mathbf{G}_2)) - \frac{1}{2}l\widetilde{\phi}_t, \text{ in } Q_T,\\ \widetilde{\phi}_t - \triangle\widetilde{\phi} = (\frac{1}{2}-\frac{1}{3}\phi^2(\mathbf{G}_2))\widetilde{\phi} + f_2 + 2\widetilde{u}, \text{ in } Q_T,\\ (\widetilde{u}_{\mathbf{n}}(x,t),\widetilde{\phi}_{\mathbf{n}}(x,t)) = (0,0), \text{ in } S_\Gamma,\\ (\widetilde{u}(x,0),\widetilde{\phi}(x,0)) = (0,0), \text{ in } \Omega. \end{cases}$$

Similar to Lemma 7, the following estimates can be derived

$$\begin{aligned} \|\widetilde{u}\|_{W(0,T)} + \|\widetilde{\phi}\|_{W_2^{2,1}(Q_T)} &\leq C(\|f_2\|_{L^2(Q_T)} + \|\nabla\times(\mathbf{H}(\mathbf{G}_1)-\mathbf{H}(\mathbf{G}_2))\|_{L^4(Q_T)})\\ &\leq C\|\mathbf{G}_1-\mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}\|\mathbf{K}\|_{L^2(0,T;L^2(\Gamma_2))}. \end{aligned}$$

In addition, we also have estimates:

$$\|\widetilde{\mathbf{H}}\|_{L^2(0,T;V(curl,\Omega))} + \|\widetilde{u}\|_{W(0,T)} + \|\widetilde{\phi}\|_{W_2^{2,1}(Q_T)} \leq C\|\mathbf{G}_1-\mathbf{G}_2\|_{L^2(0,T;L^2(\Gamma_2))}\|\mathbf{K}\|_{L^2(0,T;L^2(\Gamma_2))}.$$

This completes the proof. □

4. Existence and necessary conditions for optimal control problem

4.1. Existence of an optimal control

In this section, we will show the existence of an optimal control for problem **(P)**.

Theorem 5. *Under the assumptions **A(2.1)–A(2.4)**, there exists an optimal control for the problem **(P)**.*

Proof. Owing to $P(\mathbf{G}) = (\mathbf{H},u,\phi)$, we can eliminate $\mathbf{H}, u$ and ϕ from J to obtain the reduced cost functional

$$J(\mathbf{G};\mathbf{H},u,\phi) = J(\mathbf{G};P(\mathbf{G})) =: f(\mathbf{G}).$$

Since $f(\mathbf{G})\geq 0$, there exists the infimum

$$j := \inf_{\mathbf{G}\in U_{ad}} f(\mathbf{G}) \in \mathbf{R},$$

and there is a minimizing sequence $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ such that $\lim_{m\to\infty} f(\mathbf{G}_m) = j$.

It follows from $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ and $\|\mathbf{G}_m\|_{L^2(0,T;L^2(\Gamma_2))} < \infty$ that there exists a subsequence of $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, again denoted by $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, such that

$$\mathbf{G}_m \to \mathbf{G}^* \text{ weakly in } L^2(0,T;L^2(\Gamma_2)).$$

Moreover, the closeness of set U_{ad} implies that $\mathbf{G}^*\in U_{ad}$.

Observe that f is weakly sequentially lower semi-continuous and the control-to-state operator P is weakly sequentially continuous by Theorem 2. Consequently,

$$j = \lim_{m\to\infty} f(\mathbf{G}_m) \geq f(\mathbf{G}^*) \geq j.$$

Therefore, $\mathbf{G}^*$ is an optimal control. □

4.2. Existence of solution for the adjoint equation

Using the Lagrange technique, we can deduce the adjoint system of (1.1)–(1.8) as follows:

$$\begin{cases} \mathbf{N}_t - \nabla \times \nabla \times \mathbf{N} = -\nabla \times [2p\nabla \times \mathbf{H}^0], (x,t) \in Q_T, \\ p_t + \nabla \cdot [k(x)\nabla p] = -2\psi, (x,t) \in Q_T, \\ \psi_t + \triangle\psi + \frac{1}{2}\psi[1 - 3(\phi^0)^2] = -\frac{1}{2}lp_t, (x,t) \in Q_T, \\ \mathbf{n} \times \mathbf{N} = \mathbf{0}, (x,t) \in S_{\Gamma_1}, \\ \mathbf{n} \times (\nabla \times \mathbf{N}) = \mathbf{n} \times [2p\nabla \times \mathbf{H}^0], (x,t) \in S_{\Gamma_2}, \\ \mathbf{N}(x,T) = \mathbf{0}, x \in \Omega, \\ (p_{\mathbf{n}}(x,t), \psi_{\mathbf{n}}(x,t)) = (0,0), (x,t) \in S_\Gamma, \\ (p(x,T), \psi(x,T)) = (u^0(T) - u_T, \phi^0(x,T) - \phi_T(x) - \frac{1}{2}lp(x,T)), x \in \Omega, \end{cases} \tag{4.1}$$

where $(\mathbf{H}^0, u^0, \phi^0)$ is a solution of (1.1)–(1.8) corresponding to $\mathbf{G}^0 \in U_{ad}$.

The equations for the adjoint states $\mathbf{N}, p$ and ψ run backwards in time.

Theorem 6. *In addition to the assumptions **A(2.1)–A(2.4)**, assume that* $(\mathbf{H}^0, u^0, \phi^0)$ *is the optimal solution of the system* (1.1)–(1.8) *corresponding to the optimal control* $\mathbf{G}^0 \in U_{ad}$. *Then the adjoint system* (4.1) *has a unique solution* $(\mathbf{N}, p, \psi) \in L^2(0,T;X) \times W(0,T) \times W_2^{2,1}(Q_T)$.

Proof. By taking $t = T - \tau$ with $\tau \in [0,T]$, the functions $\mathbf{N}, p, \psi, \mathbf{H}^0, u^0$ and ϕ^0 are transformed into $\tilde{\mathbf{N}}, \tilde{p}, \tilde{\psi}, \tilde{\mathbf{H}}^0, \tilde{u}^0$ and $\tilde{\phi}^0$, respectively. Consequently, the solution of (4.1) is equivalent to the solution to the (forward) parabolic initial-boundary value problem:

$$\begin{cases} \tilde{\mathbf{N}}_\tau + \nabla \times \nabla \times \tilde{\mathbf{N}} = \nabla \times [2\tilde{p}\nabla \times \tilde{\mathbf{H}}^0], (x,\tau) \in Q_T, \\ -\tilde{p}_\tau + \nabla \cdot [k(x)\nabla \tilde{p}] = -2\tilde{\psi}, (x,\tau) \in Q_T, \\ -\tilde{\psi}_\tau + \triangle\tilde{\psi} + \frac{1}{2}\tilde{\psi}[1 - 3(\tilde{\phi}^0)^2] = \frac{1}{2}l\tilde{p}_\tau, (x,\tau) \in Q_T, \\ \mathbf{n} \times \tilde{\mathbf{N}} = \mathbf{0}, (x,\tau) \in S_{\Gamma_1}, \\ \mathbf{n} \times (\nabla \times \tilde{\mathbf{N}}) = \mathbf{n} \times (2\tilde{p}\nabla \times \tilde{\mathbf{H}}^0), (x,\tau) \in S_{\Gamma_2}, \\ \tilde{\mathbf{N}}(x,0) = \mathbf{0}, x \in \Omega, \\ (\tilde{p}_{\mathbf{n}}(x,\tau), \tilde{\psi}_{\mathbf{n}}(x,\tau)) = (0,0), (x,\tau) \in S_\Gamma, \\ (\tilde{p}(x,0), \tilde{\psi}(x,0)) = (\tilde{u}^0(0) - u_T, \tilde{\phi}^0(0) - \phi_T(x) - \frac{1}{2}l\tilde{p}(x,0)), x \in \Omega. \end{cases} \tag{4.2}$$

Note that this is a linear parabolic system, the existence and uniqueness of solution are easily proved. □

4.3. First-order necessary optimality conditions

Theorem 7. *In addition to the assumptions **A(2.1)–A(2.4)**, suppose that* $(\mathbf{H}^0, u^0, \phi^0)$ *is the optimal solution of the system* (1.1)–(1.8) *corresponding to the optimal control* $\mathbf{G}^0 \in U_{ad}$. *Then there exists* $(\mathbf{N}, p, \psi)$*, which satisfies the adjoint system* (4.1). *Moreover, the following inequality is satisfied:*

$$\int_0^T \int_{\Gamma_2} [-\mathbf{n} \times (\mathbf{G} - \mathbf{G}^0) \cdot \Upsilon_T(N) + \lambda \mathbf{G}^0 \cdot (\mathbf{G} - \mathbf{G}^0)] dsdt \geq 0, \forall \mathbf{G} \in U_{ad}.$$

Proof. Substituting $P(\mathbf{G^0}) = (\mathbf{H^0}, u^0, \phi^0)$ into (1.9) , one can obtain the reduced cost functional f,

$$J(\mathbf{G^0}; \mathbf{H^0}, u^0, \phi^0) = J(\mathbf{G^0}; P(\mathbf{G^0})) =: f(\mathbf{G^0}).$$

By Theorem 5, the Fréchet derivative of $f(\mathbf{G})$ is given by

$$\begin{aligned}&f'(\mathbf{G}^0)(\mathbf{G}-\mathbf{G}^0)\\&=\int_\Omega [u^0(T)-u_T]\bar{u}(T)dx+\int_\Omega [\phi^0(T)-\phi_T]\bar{\psi}(T)dx+\lambda\int_0^T\int_{\Gamma_2}\mathbf{G}^0\cdot(\mathbf{G}-\mathbf{G}^0)dsdt,\end{aligned}\tag{4.3}$$

where $(\bar{\mathbf{H}},\bar{u},\bar{\psi})$ is the solution of the problem (3.15) with $\bar{\mathbf{G}}=\mathbf{G}-\mathbf{G}^0$

Multiplying the second equation of problem (3.15) by the adjoint state p, integrating over $\Omega\times[0,T]$ and integrating by parts, one can find that

$$\begin{aligned}&\int_\Omega [u^0(T)-u_T]\bar{u}(T)dx-\int_0^T\int_\Omega [p_t+\nabla\cdot[k(x)\nabla p]]\bar{u}dxdt\\&=\int_0^T\int_\Omega 2\nabla\times\bar{\mathbf{H}}\cdot\nabla\times\mathbf{H}^0pdxdt-\frac{l}{2}\int_\Omega\bar{\phi}(T)p(T)dxdt+\frac{l}{2}\int_0^T\int_\Omega\bar{\phi}p_tdxdt.\end{aligned}\tag{4.4}$$

Substituting the second equation in (4.1) into (4.4), it can be seen that

$$\begin{aligned}&\int_\Omega [u^0(x,T)-u_T]\bar{u}(T)dx\\&=\int_0^T\int_\Omega (2\nabla\times\bar{\mathbf{H}}\cdot\nabla\times\mathbf{H}^0p+\frac{l}{2}\bar{\phi}p_t-2\psi\bar{u})dxdt-\frac{l}{2}\int_\Omega\bar{\phi}(T)[u^0(T))-u_T]dxdt.\end{aligned}\tag{4.5}$$

Multiplying the first equation in (4.1) and the first equation of (3.15) by $\bar{\mathbf{H}}$ and $\mathbf{N}$, respectively, integrating over $\Omega\times[0,T]$ and integrating by parts, we obtain

$$\int_0^T\int_\Omega \mathbf{N}\cdot\bar{\mathbf{H}}_tdxdt+\int_0^T\int_\Omega\nabla\times\mathbf{N}\cdot\nabla\times\bar{\mathbf{H}}dxdt=\int_0^T\int_\Omega 2p\nabla\times\mathbf{H}^0\cdot\nabla\times\bar{\mathbf{H}}dxdt\tag{4.6}$$

and

$$\int_0^T\int_\Omega \mathbf{N}\cdot\bar{\mathbf{H}}_tdxdt+\int_0^T\int_\Omega\nabla\times\mathbf{N}\cdot\nabla\times\bar{\mathbf{H}}dxdt=-\int_0^T\int_{\Gamma_2}\mathbf{n}\times(\mathbf{G}-\mathbf{G}^0)\cdot\Upsilon_T(\mathbf{N})dsdt.\tag{4.7}$$

From (4.5)–(4.7), one has

$$\begin{aligned}&\int_\Omega [u^0(x,T)-u_T]\bar{u}(T)dx\\&=\int_0^T\int_\Omega [\frac{l}{2}\bar{\phi}p_t-2\psi\bar{u}]dxdt-\frac{l}{2}\int_\Omega\bar{\phi}(T)[u^0(T))-u_T]dxdt-\int_0^T\int_{\Gamma_2}\mathbf{n}\times(\mathbf{G}-\mathbf{G}^0)\cdot\Upsilon_T(\mathbf{N})dsdt.\end{aligned}$$

Now we turn to the integral $\int_\Omega(\phi^0(T))-\phi_T)\bar{\psi}(T)dx$. Analogously, multiplying the third of (3.15) by ψ and then integrating by parts, it yields that

$$\begin{aligned}&\int_\Omega [\phi^0(T))-\phi_T]\bar{\psi}(T)dx\\&=\int_0^T\int_\Omega [\bar{\phi}\psi_t+\bar{\phi}\Delta\psi+\frac{1}{2}\bar{\phi}(1-3(\phi^0)^2)\psi+2\bar{u}\psi]dxdt+\frac{l}{2}\int_\Omega P(T)\bar{\phi}(T)dx.\end{aligned}\tag{4.8}$$

Multiplying the third equation in (4.1) by $\bar{\phi}$ and integrating over $\Omega \times [0, T]$, we have

$$\int_0^T \int_\Omega [\bar{\phi}\psi_t + \bar{\phi}\Delta\psi + \frac{1}{2}\bar{\phi}(1 - 3(\phi^0)^2)\psi]dxdt = -\frac{l}{2}\int_0^T \int_\omega p_t\bar{\phi}dxdt. \tag{4.9}$$

Substituting (4.9) into (4.8), we see that

$$\int_\Omega [\phi^0(T)) - \phi_T]\bar{\psi}(T)dx = \int_0^T \int_\Omega (2\bar{u}\psi - \frac{l}{2}p_t\bar{\phi})dxdt + \frac{l}{2}\int_\Omega P(T)\bar{\phi}(T)dx.$$

Recall that $p(T) = u^0(T) - u_T$ in (4.1). Therefore,

$$\int_\Omega [u^0(T)) - u_T]\bar{u}(T)dx + \int_\Omega [\phi^0(T)) - \phi_T]\bar{\psi}(T)dx = -\int_0^T \int_{\Gamma_2} \mathbf{n} \times (\mathbf{G} - \mathbf{G}^0) \cdot \Upsilon_T(N)dsdt. \tag{4.10}$$

Note that $(\mathbf{H}^0, u^0, \phi^0)$ is the optimal solution of the system (1.1)–(1.8) corresponding to the optimal control $\mathbf{G}^0 \in U_{ad}$. Therefore, it can be concluded from (4.3) and (4.10) that

$$\begin{aligned} 0 \leq & f'(\mathbf{G}^0)(\mathbf{G} - \mathbf{G}^0) \\ = & \int_0^T \int_{\Gamma_2} [-\mathbf{n} \times (\mathbf{G} - \mathbf{G}^0) \cdot \Upsilon_T(N) + \lambda\mathbf{G}^0 \cdot (\mathbf{G} - \mathbf{G}^0)]dsdt, \forall \mathbf{G} \in U_{ad}. \end{aligned}$$

The proof is completed. □

5. Conclusions

In this paper, we study an optimal control problem arising from a metal melting process by using a induction heating method. The controlled system is a nonlinear coupled system given by Maxwell's equations, heat equation and phase field equation. The goal of optimal control is to find the electric field action on a part of the boundary such that the temperature profile at the final stage has a relative uniform distribution and minimum energy consumption. The new existence and uniqueness theorem on the solution of controlled system is established in the case that the resistivity $\rho(x, t) = 1$ or ρ does not depend on x. By defined a control-to-state operator P and studied its properties, we showed that there is at least one optimal control and derived the first order optimality condition.

When the resistivity $\rho(x, t)$ depends on position x, the heat source generated by the electromagnetic field is $\rho|\nabla\times\mathbf{H}|^2 \in L^1(Q_T)$, which cannot be guaranteed the requirement $L^2(Q_T)$ in proof of the existence of solutions for heat equation. This is a challenging problem. We intend to further work on this, as well as make a numerical simulation for the problem, in the near future.

Acknowledgments

The authors would like to thank anonymous referees for helpful comments and suggestions which improved the original version of the paper. This work is supported by the National Natural Science Foundation of China (Nos. 11761021, 12001131), the Natural Science Foundation of Guizhou Province (No. [2018]1118) and the Foundation of Research Project for Graduate of Guizhou Province (No. YJSCXJH [2020]083).

Conflict of interest

The authors declare that they have no competing interests.

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