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Research article

A characterization for totally real submanifolds using self-adjoint differential operator

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Abstract: In this article, we study totally real submanifolds in Kaehler product manifold with constant scalar curvature using self-adjoint differential operator $\square$. Under this setup, we obtain a characterization result. Moreover, we discuss $\delta-$invariant properties of such submanifolds and get an obstruction result as an application of the inequality derived. The results in the article are supported by non-trivial examples.

Keywords: totally real submanifolds; self-adjoint differential operator; Kaehler product manifold; $\delta-$invariant
Mathematics Subject Classification: 53C05, 53C20, 53C40

1. Introduction

Totally real submanifolds are one of the typical classes of submanifolds of Kaehler manifold. In 1974, B. Y. Chen and K. Ogiue [10] started the study of the totally real submanifolds from the point of view of their curvatures. Due to its geometrical importance, many geometers studied totally real submanifolds from the different point of views and various results were obtained in different ambient spaces [6,9,13,18]. Kaehler product manifold also attracts the attention of geometers toward itself [24]. S. Y. Cheng and S. T. Yau [11] obtained many well-known results introducing a self-adjoint differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - \zeta_{ij}^{n+1}) f_{ij}, \tag{1.1}$$

where $f \in C^2(\mathcal{N}), (f_{ij})$ is Hessian of f, H mean curvature, and ζ_{ij}^{n+1} is the coefficients of second fundamental form ζ.

Using this differential operator H. Li [15] obtained a rigidity result for hypersurfaces in space forms with constant scalar curvature. In 2013, X. Gua and H. Li [17] extended the use of the operator for submanifolds and obtained interesting results for submanifolds with constant scalar curvature in a unit sphere.

Motivated by X. Gua and H. Li, we study the totally real submanifolds of Kaehler product manifold with constant scalar curvature using self-adjoint differential operator $\Box$ and obtain a characterization result.

Further, we study $\delta-$invariant totally real submanifolds in same setting and prove some results.

2. Preliminaries

Let $(\overline{\mathcal{N}}^m, J_m, g_m)$ and $(\overline{\mathcal{N}}^p, J_p, g_p)$ are Kaehler manifolds of complex dimension m and complex dimension p respectively. Let J_m and g_m be almost complex structure and metric tensor on $\overline{\mathcal{N}}^m$ respectively and J_p and g_p almost complex structure and metric tensor on $\overline{\mathcal{N}}^p$ respectively. Further, let us assume $\overline{\mathcal{N}}^m(c_1)$ and $\overline{\mathcal{N}}^p(c_2)$ are complex space forms with constant holomorphic sectional curvatures c_1 and c_2 respectively.

We suppose $\overline{\mathcal{N}}(c_1, c_2) = \overline{\mathcal{N}}^m(c_1) \times \overline{\mathcal{N}}^p(c_2)$ the Kaehlerian product manifold with complex dimension $(m + p)$. Let us denote by $\mathcal{P}$ and $\mathcal{Q}$ the projection operators of the tangent space of $\overline{\mathcal{N}}(c_1, c_2)$ to the tangent spaces of $\overline{\mathcal{N}}^m(c_1)$ and $\overline{\mathcal{N}}^p(c_2)$ respectively. Then,

$$\mathcal{P}^2 = \mathcal{P}, \quad \mathcal{Q}^2 = \mathcal{Q}, \quad \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0.$$

By setting $\mathsf{F} = \mathcal{P} - \mathcal{Q}$, it can be easily shown that $\mathsf{F}^2 = I$. Thus, F is an almost product structure on $\overline{\mathcal{N}}(c_1, c_2)$. Moreover, for a Riemannian metric g on $\overline{\mathcal{N}}(c_1, c_2)$ we have [20]

$$g(E, F) = g_m(\mathcal{P}E, \mathcal{P}F) + g_p(\mathcal{Q}E, \mathcal{Q}F),$$

for all vector fields E, F on $\overline{\mathcal{N}}(c_1, c_2)$. We also have

$$g(\mathsf{F}E, F) = g(\mathsf{F}F, E).$$

If we assume $JE = J_m\mathcal{P}E + J_p\mathcal{Q}E$ for any vector field E of $\overline{\mathcal{N}}(c_1, c_2)$. Then from [20], we see that

$$J_m\mathcal{P} = \mathcal{P}J, \quad J_p\mathcal{Q} = \mathcal{Q}J, \quad \mathsf{F}J = J\mathsf{F},$$
$$J^2 = -I, \quad g(JE, JF) = g(E, F), \quad \overline{\nabla}_E J = 0.$$

Therefore, J is a Kaehlerian structure on $\overline{\mathcal{N}}(c_1, c_2)$. Let $\overline{R}$ be the Riemannian curvature tensor of a Kaehler product manifold $\overline{\mathcal{N}}(c_1, c_2)$. Then [24]

$$\overline{R}(E, F, G, W) = \frac{1}{16}(c_1 + c_2)[g(F, G)g(E, W) - g(E, G)g(F, W)$$

$$\begin{aligned}
&+ g(JF,G)g(JE,W) - g(JE,G)g(JF,W)\\
&+ 2g(JE,F)g(JG,W) + 2g(\digamma F,G)g(\digamma E,W)\\
&- g(\digamma E,G)g(\digamma F,W) + g(\digamma JF,G)g(\digamma JE,W)\\
&- g(\digamma JE,G)g(\digamma JF,W) + 2g(\digamma E,JF)g(\digamma JG,W)]\\
&+ \frac{1}{16}(c_1 - c_2)[g(\digamma F,G)g(E,W) - g(\digamma E,G)g(F,W)\\
&+ g(F,G)g(\digamma E,W) - g(E,G)g(\digamma F,W)\\
&+ g(\digamma JF,G)g(JE,W) - g(\digamma JE,G)g(JF,W)\\
&+ g(JF,G)g(\digamma JE,W) - g(JE,G)g(\digamma JF,W)\\
&+ 2g(\digamma E,JF)g(JG,W) + 2g(E,JF)g(\digamma JG,W)],
\end{aligned} \tag{2.1}$$

for any vector fields E, F and G on $\overline{\mathcal{N}}(c_1, c_2)$.

Definition 2.1. Let $\mathcal{N}$ be a real $n-$dimensional Riemannian manifold isometrically immersed in a $(m+p)-$ dimensional Kaehlerian product manifold $\overline{\mathcal{N}}(c_1, c_2)$. Then $\mathcal{N}$ is said to be totally real submanifold of $\overline{\mathcal{N}}(c_1, c_2)$ if $JT_x(\mathcal{N}) \perp T_x(\mathcal{N})$ for each $x \in \mathcal{N}$ where $T_x(\mathcal{N})$ denotes the tangent space to $\mathcal{N}$ at $x \in \mathcal{N}$.

Let g be the metric tensor field on $\overline{\mathcal{N}}(c_1, c_2)$ as well as that induced on $\mathcal{N}$. Also, we denote by $\overline{\nabla}$(resp. ∇) the Levi-Civita connection on $\overline{\mathcal{N}}(c_1, c_2)$(resp. $\mathcal{N}$). Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_E F = \nabla_E F + \zeta(E, F), \tag{2.2}$$

$$\overline{\nabla}_E N = -\Lambda_N E + \nabla^{\perp}_E N, \tag{2.3}$$

for all E, F tangent to $\mathcal{N}$ and vector field N normal to $\mathcal{N}$, where $\zeta, \nabla^{\perp}_E$ and Λ_N denote the second fundamental form, normal connection and shape operator respectively. The relation between the second fundamental form and the shape operator is given by

$$g(\zeta(E, F), N) = g(\Lambda_N E, F). \tag{2.4}$$

We choose a local field of orthonormal frames $e_1, \ldots, e_n; e_{n+1}, \ldots, e_{m+p}; e_{1^\star} = Je_1, \ldots, e_{n^\star} = Je_n;$ $e_{(n+1)^\star} = Je_{n+1}; e_{(m+p)^\star} = Je_{m+p}$ in $\overline{\mathcal{N}}(c_1, c_2)$ in such a way that restricted to $\mathcal{N}$, the vectors $e_1, \ldots, e_n$ are tangent to $\mathcal{N}$. With respect to this frame field of $\overline{\mathcal{N}}(c_1, c_2)$, let $\omega^1, \ldots, \omega^n; \omega^{n+1}, \ldots, \omega^{m+p}; \omega^{1^\star}, \ldots, \omega^{n^\star}; \omega^{(n+1)^\star}, \ldots, \omega^{(m+p)^\star}$ be the field of dual frames. Unless otherwise stated, we use the following conventions over the range of indices:

$$\begin{aligned}
&A, B, C, D = 1, \ldots, m+p, 1^\star, \ldots, (m+p)^\star;\\
&i, j, k, l, t, s = 1, \ldots, n;\\
&\alpha, \beta, \gamma = n+1, \ldots, m+p; 1^\star, \ldots, (m+p)^\star;\\
&\lambda, \mu, \nu = n+1, \ldots, m+p.
\end{aligned}$$

Then the mean curvature vector H is defined as

$$H = \sum_\alpha H^\alpha e_\alpha, \qquad \text{where} \quad H^\alpha = \frac{1}{n} \sum_i \zeta^\alpha_{ii}. \tag{2.5}$$

Also, the structure equations of $\overline{\mathcal{N}}(c_1, c_2)$ are given by [24]

$$\begin{cases} d\omega^A = -\omega^A_B\omega^B, & \omega^A_B + \omega^B_A = 0, \\ \omega^i_j + \omega^j_i = 0, & \omega^i_j = \omega^{i^\star}_{j^\star}, \qquad \omega^{i^\star}_j = \omega^{j^\star}_i, \\ \omega^\lambda_\mu + \omega^\mu_\lambda = 0, & \omega^\lambda_\mu = \omega^{\lambda^\star}_{\mu^\star}, \qquad \omega^{\lambda^\star}_\mu = \omega^{\mu^\star}_\lambda. \end{cases} \tag{2.6}$$

$$\begin{cases} \omega^i_\lambda + \omega^\lambda_i = 0, & \omega^i_\mu = \omega^{i^\star}_{\mu^\star}, \qquad \omega^{i^\star}_\lambda = \omega^{\lambda^\star}_i, \\ d\omega^A_B = -\omega^A_C\omega^C_B + \phi^A_B, & \phi^A_B = \frac{1}{2}\overline{R}^A_{BCD}\omega^C \wedge \omega^D. \end{cases} \tag{2.7}$$

Restricting these forms to $\mathcal{N}$, we have

$$\omega^\alpha = 0, \tag{2.8}$$

$$d\omega^i = -\omega^i_k \wedge \omega^k, \tag{2.9}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j, \qquad \Omega^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l. \tag{2.10}$$

Since $0 = d\omega^\alpha = -\omega^\alpha_i \wedge \omega^i$, by Cartan's Lemma we get

$$\omega^\alpha_i = \zeta^\alpha_{ij}\omega^j, \qquad \zeta^\alpha_{ij} = \zeta^\alpha_{ji}, \tag{2.11}$$

$$d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta, \qquad \Omega^\alpha_\beta = \frac{1}{2}R^\alpha_{\beta kl}\omega^k \wedge \omega^l. \tag{2.12}$$

From (2.6) and (2.11) we find

$$\zeta^{i^\star}_{jk} = \zeta^{j^\star}_{ik} = \zeta^{k^\star}_{ij}. \tag{2.13}$$

The covariant derivative of ζ^α_{ij} is given by

$$\zeta^\alpha_{ijk} = d\zeta^\alpha_{ij} - \zeta^\alpha_{il}\omega^l_j - \zeta^\alpha_{lj}\omega^l_i + \zeta^\beta_{ij}\omega^\alpha_\beta. \tag{2.14}$$

The Laplacian $\Delta\zeta^\alpha_{ij}$ of ζ^α_{ij} is defined as

$$\Delta\zeta^\alpha_{ij} = \sum_k \zeta^\alpha_{ijkk}, \tag{2.15}$$

where we have put $\zeta^\alpha_{ijkl}\omega^l = d\zeta^\alpha_{ijk} - \zeta^\alpha_{ljk}\omega^l_i - \zeta^\alpha_{ilk}\omega^l_j - \zeta^\beta_{ijk}\omega^\alpha_\beta$.
Now, from [17] we have a trace-free linear map $\Theta^\alpha : T_x\mathcal{N} \to T_x\mathcal{N}$ given by

$$g(\Theta^\alpha E, F) = g(\Lambda^\alpha E, F) - H^\alpha(E, F),$$

where $x \in \mathcal{N}$ and the shape operator Λ^α of e_α is given by

$$\Lambda^\alpha(e_i) = -\sum_j g(\overline{\nabla}_{e_i} e_\alpha, e_j)e_j = \sum_j \zeta_{ij}^\alpha e_j,$$

and Θ is a bilinear map $\Theta : T_x\mathcal{N} \times T_x\mathcal{N} \to T_x^\perp \mathcal{N}$ defined by

$$\Theta(E, F) = \sum_{\alpha=n+1}^{m+p} g(\Theta^\alpha E, F)e_\alpha. \tag{2.16}$$

Then we have $|\Theta|^2 = |\Lambda|^2 - nH^2$, where $H^2 = \sum_\alpha (H^\alpha)^2$.
The Gauss equation is given by

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (\zeta_{ik}^\alpha \zeta_{jl}^\alpha - \zeta_{il}^\alpha \zeta_{jk}^\alpha). \tag{2.17}$$

From (2.1) and Gauss equation, we obtain

$$2\tau - \frac{1}{16}(c_1 + c_2)n(n+1) = n^2H^2 - |\Lambda|^2, \tag{2.18}$$

where τ is scalar curvature.

The Codazzi and the Ricci equation are respectively

$$\zeta_{ij,k}^\alpha = \zeta_{ik,l}^\alpha, \tag{2.19}$$

$$R_{\alpha\beta ij}^\perp = \sum_k (\zeta_{ik}^\alpha \zeta_{kj}^\beta - \zeta_{jk}^\alpha \zeta_{ki}^\beta). \tag{2.20}$$

Then, by using Codazzi equation one can easily see that the operator $\Box$ is self-adjoint. That is

$$\int_{\mathcal{N}} \Box f dv = 0, f \in C^2(\mathcal{N}). \tag{2.21}$$

Since we have constant scalar curvature, Eq (2.18) implies that

$$|\nabla\Lambda|^2 = n^2|\nabla H^2|. \tag{2.22}$$

We can choose a unit normal vector field e_{n+1} which is parallel to H. Hence we have [16]

$$H^{n+1} = H, \qquad H^\alpha = 0 \qquad (n+2 \le \alpha \le m+p), \tag{2.23}$$

$$\Theta_{ij}^{n+1} = \zeta_{ij}^{n+1} - H\delta_{ij}, \quad \Theta_{ij}^\alpha = \zeta_{ij}^\alpha, \quad (n+2 \le \alpha \le m+p). \tag{2.24}$$

Now, we quote the following lemmas for later use.

Lemma 2.2. *[19] Let $B : R^n \to R^n$ be a symmetric linear map such that $trB = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}|B|^3 \le trB^3 \le \frac{n-2}{\sqrt{n(n-1)}}|B|^3,$$

where $|B|^2 = trB^2$, and the equality holds if and only if at least $n-1$ eigenvalues of B are equal.

Lemma 2.3. *[21] Let $C, B : R^n \to R^n$ be a symmetric linear map such that $[C, B] = 0$ and $trC = trB = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}|C|^2|B| \leq tr(C^2 B) \leq \frac{n-2}{\sqrt{n(n-1)}}|C|^2|B|.$$

Lemma 2.4. *[14] Let $B^1, B^2, \ldots, B^m$, be symmetric $(n \times n)-$matrices.*
Set $S_{\alpha\beta} = tr(B^\alpha B^\beta), S_\alpha = S_{\alpha\alpha}, S = \sum_\alpha S_\alpha$, then

$$\sum_{\alpha,\beta} |B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S^2_{\alpha\beta} \leq \frac{3}{2}(\sum_\alpha S_\alpha)^2.$$

3. Main Theorem

This section is devoted to the proof of main result.

Theorem 3.1. *Let $\mathcal{N}^n$ be a totally real submanifold in Kaehlerian product manifold $\overline{\mathcal{N}}(c_1, c_2) = \overline{\mathcal{N}}^m_1(c_1) \times \overline{\mathcal{N}}^p_2(c_2)$, $c_1, c_2 > 0$, with constant scalar curvature. If trF vanishes, then $\mathcal{N}$ is totally geodesic.*

For proving that result, we need to prove the following preliminary Lemmas. Since F is symmetric and J is skew-symmetric, following result is obvious.

Lemma 3.2. *Let $\mathcal{N}$ be a totally real submanifold in Kaehler product manifold $\overline{\mathcal{N}}(c_1, c_2) = \overline{\mathcal{N}}^m_1(c_1) \times \overline{\mathcal{N}}^p_2(c_2)$, then $trFJ = trJF = 0$.*

Lemma 3.3. *Let $\mathcal{N}$ be a totally real submanifold in Kaehler product manifold $\overline{\mathcal{N}}(c_1, c_2) = \overline{\mathcal{N}}^m_1(c_1) \times \overline{\mathcal{N}}^p_2(c_2)$, then*

$$\begin{aligned}
\frac{1}{2}\Delta|\Lambda|^2 &= |\nabla\Lambda|^2 + \sum_{\alpha,i,j,k} \zeta^\alpha_{ij}\zeta^\alpha_{kkij} \\
&+ \frac{1}{16}(c_1 + c_2)\sum_\alpha [(n + 9 + 6(trF)^2)tr\Lambda^2_\alpha - (3 + (trF)^2)] \\
&+ \frac{1}{16}(c_1 - c_2)\sum_\alpha [(n + 1)(trF)tr\Lambda^2_\alpha - 2(trF)(tr\Lambda_\alpha)^2] \\
&+ \frac{1}{16}(c_1 + c_2)\sum_t [(4(trF)^2 - 2)tr\Lambda^2_t - (1 + (trF)^2)(tr\Lambda_t)^2] \\
&+ \frac{1}{16}(c_1 - c_2)\sum_t [2(trF)tr\Lambda^2_t - 2(trF)(tr\Lambda_t)^2] \\
&- \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2 - \sum_{\alpha,\beta}(\zeta^\alpha_{ij}\zeta^\beta_{kl})^2 + \sum_{\alpha,\beta} nH^\beta\zeta^\beta_{kl}\zeta^\alpha_{jl}\zeta^\alpha_{jk}.
\end{aligned} \tag{3.1}$$

Proof. From [12], we have

$$\begin{aligned}
\sum_{\alpha,i,j}\zeta^\alpha_{ij}\Delta\zeta^\alpha_{ij} = \sum_{\alpha,i,j,k}(&\zeta^\alpha_{ij}\zeta^\alpha_{kkij} - \overline{R}^\alpha_{ij\beta}\zeta^\alpha_{ij}\zeta^\beta_{kk} + 4\overline{R}^\alpha_{\beta ki}\zeta^\beta_{jk}\zeta^\alpha_{jk} \\
&- \overline{R}^\alpha_{k\beta k}\zeta^\alpha_{ij}\zeta^\beta_{ij} + 2\overline{R}^l_{kik}\zeta^\alpha_{lj}\zeta^\alpha_{ij} + 2\overline{R}^l_{ijk}\zeta^\alpha_{lk}\zeta^\alpha_{ij})
\end{aligned}$$

$$-\sum_{\alpha,\beta,i,j,k,l}(\zeta^{\alpha}_{ik}\zeta^{\beta}_{jk}-\zeta^{\alpha}_{jk}\zeta^{\beta}_{ik})(\zeta^{\alpha}_{il}\zeta^{\beta}_{jl}-\zeta^{\alpha}_{jl}\zeta^{\beta}_{il})$$
$$-\sum_{\alpha,\beta,i,j,k,l}\zeta^{\alpha}_{ij}\zeta^{\alpha}_{kl}\zeta^{\beta}_{ij}\zeta^{\beta}_{kl}+\sum_{\alpha,\beta,i,j,k,l}\zeta^{\alpha}_{ji}\zeta^{\alpha}_{ki}\zeta^{\beta}_{kj}\zeta^{\beta}_{ll}. \tag{3.2}$$

On the other hand, one has

$$\sum_{\alpha,i,j}\zeta^{\alpha}_{ij}\Delta\zeta^{\alpha}_{ij}=\frac{1}{2}\Delta|\Lambda|^2-|\nabla\Lambda|^2. \tag{3.3}$$

By using Eqs (2.5), (2.20) and (3.3) in (3.2), we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\Lambda|^2 &= |\nabla\Lambda|^2+\sum_{\alpha,i,j,k}\zeta^{\alpha}_{ij}\zeta^{\alpha}_{kkij}\\ &+\sum_{\alpha,i,j,k}(-\overline{R}^{\alpha}_{ij\beta}\zeta^{\alpha}_{ij}\zeta^{\beta}_{kk}+4\overline{R}^{\alpha}_{\beta ki}\zeta^{\beta}_{jk}\zeta^{\alpha}_{jk}-\overline{R}^{\alpha}_{k\beta k}\zeta^{\alpha}_{ij}\zeta^{\beta}_{ij}\\ &+2\overline{R}^{l}_{kik}\zeta^{\alpha}_{lj}\zeta^{\alpha}_{ij}+2\overline{R}^{l}_{ijk}\zeta^{\alpha}_{lk}\zeta^{\alpha}_{ij})-\sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2\\ &-\sum_{\alpha,\beta}(\zeta^{\alpha}_{ij}\zeta^{\beta}_{kl})^2+\sum_{\alpha,\beta}nH^{\beta}\zeta^{\beta}_{kj}\zeta^{\alpha}_{ji}\zeta^{\alpha}_{ki}.\end{aligned} \tag{3.4}$$

Using (2.1) and Lemma 3.2, we now compute the values of curvature terms involving $\overline{R}$ of the Eq (3.4) as follows:

$$\begin{aligned}\overline{R}^{\alpha}_{ij\beta}\zeta^{\alpha}_{ij}\zeta^{\beta}_{kk} &= g(\overline{R}(e_j,e_\beta)e_i,e_\alpha)\zeta^{\alpha}_{ij}\zeta^{\beta}_{kk}\\ &=\frac{1}{16}(c_1+c_2)[\sum_{\alpha}tr\Lambda^2_{\alpha}-3\sum_{t}(tr\Lambda_t)^2\\ &-3\sum_{t}(trF)^2(tr\Lambda_t)^2-\sum_{\alpha}(trF)^2(tr\Lambda_\alpha)^2]\\ &+\frac{1}{16}(c_1-c_2)[-2(trF)\sum_{\alpha}(tr\Lambda_\alpha)^2\\ &-\sum_{t}6(trF)(tr\Lambda_t)^2].\end{aligned} \tag{3.5}$$

Similarly we obtain,

$$\begin{aligned}\overline{R}^{\alpha}_{\beta ki}\zeta^{\beta}_{jk}\zeta^{\alpha}_{ij} &= g(\overline{R}(e_k,e_i)e_\beta,e_\alpha)\zeta^{\beta}_{jk}\zeta^{\alpha}_{ij}\\ &=\frac{1}{16}(c_1+c_2)[\sum_{t}(tr\Lambda^2_t)-\sum_{t}(tr\Lambda_t)^2+\sum_{\alpha}(tr\Lambda^2_\alpha)\\ &+\sum_{t}(trF)^2(tr\Lambda^2_t)-\sum_{t}(trF)^2(tr\Lambda_t)^2]\\ &+\frac{1}{16}(c_1-c_2)[2\sum_{t}(trF)(tr\Lambda^2_t)-2\sum_{t}(trF)(tr\Lambda_t)^2],\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
\overline{R}^{\alpha}_{k\beta k}\zeta^{\alpha}_{ij}\zeta^{\beta}_{ij} &= g(\overline{R}(e_\beta, e_k)e_k, e_\alpha)\zeta^{\alpha}_{ij}\zeta^{\beta}_{ij} \\
&= \frac{1}{16}(c_1 + c_2)[(n-1)\sum_{\alpha}(tr\Lambda^2_\alpha) + \sum_{t} 6(tr\Lambda^2_t) \\
&+ 2\sum_{\alpha}(trF)^2(tr\Lambda^2_\alpha)] \\
&+ \frac{1}{16}(c_1 - c_2)[(n+1)\sum_{\alpha}(trF)(tr\Lambda^2_\alpha) \\
&+ \sum_{t} 6(trF)(tr\Lambda^2_t)],
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\overline{R}^{l}_{kik}\zeta^{\alpha}_{lj}\zeta^{\alpha}_{ij} &= g(\overline{R}(e_i, e_k)e_k, e_l)\zeta^{\alpha}_{lj}\zeta^{\alpha}_{ij} \\
&= \frac{1}{16}(c_1 + c_2)[\sum_{\alpha} n(tr\Lambda^2_\alpha) + \sum_{\alpha} 2(trF)(tr\Lambda^2_\alpha) \\
&+ \sum_{\alpha}(tr\Lambda^2_\alpha)] + \frac{1}{16}(c_1 - c_2)[n\sum_{\alpha}(trF)(tr\Lambda^2_\alpha) \\
&- \sum_{\alpha}(trF)(tr\Lambda^2_\alpha)],
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\overline{R}^{l}_{ijk}\zeta^{\alpha}_{lk}\zeta^{\alpha}_{ij} &= g(\overline{R}(e_j, e_k)e_i, e_l)\zeta^{\alpha}_{lk}\zeta^{\alpha}_{ij} \\
&= \frac{1}{16}(c_1 + c_2)[\sum_{\alpha}(tr\Lambda^2_\alpha) - \sum_{\alpha}(tr\Lambda_\alpha)^2 \\
&+ \sum_{\alpha}(trF)^2(tr\Lambda^2_\alpha) - \sum_{\alpha}(trF)^2(tr\Lambda_\alpha)^2] \\
&+ \frac{1}{16}(c_1 - c_2)[2\sum_{\alpha}(trF)(tr\Lambda^2_\alpha) \\
&- 2\sum_{\alpha}(trF)(tr\Lambda_\alpha)^2].
\end{aligned}
\tag{3.9}
$$

Thus, making use of Eqs (3.5)–(3.9) in (3.4), we get (3.1). □

Proof of Theorem 3.1. From (1.1) and (2.22), we obtain

$$
\Box(nH) = \frac{1}{2}\Delta|\Lambda|^2 - n^2|\nabla H|^2 - \sum n\zeta_{ij}H_{,ij}. \tag{3.10}
$$

Now, using (3.1) in the above equation, we get

$$
\begin{aligned}
\Box(nH) &= |\nabla\Lambda|^2 + \sum_{\alpha,i,j,k}\zeta^{\alpha}_{ij}\zeta^{\alpha}_{kkij} \\
&+ \frac{1}{16}(c_1 + c_2)\sum_{\alpha}[(n + 9 + 6(trF)^2)tr\Lambda^2_\alpha - (3 + (trF)^2)]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{16}(c_1-c_2)\sum_{\alpha}\left[(n+1)(trF)tr\Lambda_\alpha^2-2(trF)(tr\Lambda_\alpha)^2\right]\\
&+\frac{1}{16}(c_1+c_2)\sum_{t}\left[(4(trF)^2-2)tr\Lambda_t^2-(1+(trF)^2)(tr\Lambda_t)^2\right]\\
&+\frac{1}{16}(c_1-c_2)\sum_{t}\left[2(trF)tr\Lambda_t^2-2(trF)(tr\Lambda_t)^2\right]\\
&-\sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2-\sum_{\alpha,\beta}(\zeta^{\alpha}_{ij}\zeta^{\beta}_{kl})^2+\sum_{\alpha,\beta}nH^{\beta}\zeta^{\beta}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk}\\
&-n^2|\nabla H|^2-\sum n\zeta_{ij}H_{,ij},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\Box(nH)=&\frac{1}{16}(c_1+c_2)\sum_{\alpha}\left[(n+9+6(trF)^2)tr\Lambda_\alpha^2-(3+(trF)^2)\right]\\
&+\frac{1}{16}(c_1-c_2)\sum_{\alpha}\left[(n+1)(trF)tr\Lambda_\alpha^2-2(trF)(tr\Lambda_\alpha)^2\right]\\
&+\frac{1}{16}(c_1+c_2)\sum_{t}\left[(4(trF)^2-2)tr\Lambda_t^2-(1+(trF)^2)(tr\Lambda_t)^2\right]\\
&+\frac{1}{16}(c_1-c_2)\sum_{t}\left[2(trF)tr\Lambda_t^2-2(trF)(tr\Lambda_t)^2\right]\\
&-\sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2-\sum_{\alpha,\beta}(\zeta^{\alpha}_{ij}\zeta^{\beta}_{kl})^2+\sum_{\alpha,\beta}nH^{\beta}\zeta^{\beta}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk}. \qquad (3.11)
\end{aligned}
$$

A direct computation gives

$$
\sum_{\alpha}(tr\Lambda_\alpha)^2=\sum_{\alpha}\zeta^{\alpha}_{ii}\zeta^{\alpha}_{jj}=n^2H^2. \qquad (3.12)
$$

Moreover, it is easy to see that

$$
\sum_{\alpha}(tr\Lambda_\alpha^2)=\sum_{\alpha}\zeta^{\alpha}_{ij}\zeta^{\alpha}_{ij}=|\Theta|^2+nH^2 \qquad (3.13)
$$

and

$$
\sum_{\alpha}(tr\Lambda_t^2)=\|\zeta^{\star}\|^2, \qquad (3.14)
$$

where $\zeta^{\star}_{ij}=g(\zeta(e_i,e_j),e_{t^{\star}})$ and $\zeta_{ij}=\overline{\zeta}_{ij}\oplus\zeta^{\star}_{ij}$.
Also we have

$$
\begin{aligned}
\sum_{\alpha,\beta}(\zeta^{\alpha}_{ij}\zeta^{\beta}_{kl})^2+\sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2=&\sum_{\alpha,\beta}\left[tr(\Lambda^{\alpha}\Lambda^{\beta})\right]^2\\
&+\sum_{\alpha\neq n+1,\beta\neq n+1,i,j}(R^{\perp}_{\alpha\beta ij})^2. \qquad (3.15)
\end{aligned}
$$

Using Lemma 2.4 in (3.15), we get

$$\begin{aligned}
\sum_{\alpha,\beta}(\zeta^{\alpha}_{ij}\zeta^{\beta}_{kl})^2 + \sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2 \leq{}& [tr(\Lambda^{n+1}\Lambda^{n+1})]^2 \\
&+ 2\sum_{\beta\neq n+1}(tr\Lambda^{n+1}\Lambda^{\beta})^2 + \frac{3}{2}[\sum_{\beta\neq n+1}|\Theta^{\beta}|^2]^2 \\
&= \frac{5}{2}|\Theta^{n+1}|^4 + 2nH^2|\Theta^{n+1}|^2 + n^2H^4 \\
&+ 2\sum_{\beta\neq n+1}(tr\Theta^{n+1}\Theta^{\beta})^2 - 2(tr\Theta^{n+1}\Theta^{n+1})^2 \\
&+ \frac{3}{2}|\Theta|^4 - 3|\Theta|^2|\Theta^{n+1}|^2 \\
&\leq \frac{5}{2}|\Theta^{n+1}|^4 + 2nH^2|\Theta^{n+1}|^2 + n^2H^4 \\
&+ 2|\Theta^{n+1}|^2(|\Theta|^2 - |\Theta^{n+1}|^2) + \frac{3}{2}|\Theta|^4 \\
&- 3|\Theta|^2|\Theta^{n+1}|^2 \\
&= \frac{1}{2}|\Theta^{n+1}|^4 + 2nH^2|\Theta^{n+1}|^2 + n^2H^4 \\
&- |\Theta|^2|\Theta^{n+1}|^2 + \frac{3}{2}|\Theta|^4. \qquad (3.16)
\end{aligned}$$

Taking into account the Eq (2.24), we derive

$$\begin{aligned}
\sum_{\alpha,\beta,i,j,k} H^{\beta}\zeta^{\beta}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk} ={}& \sum_{\alpha,i,j,k} H\zeta^{n+1}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk} \\
&= Htr(\Theta^{n+1})^3 + 3H^2(\Theta^{n+1})^2 + nH^4 \\
&+ 3tr\Theta^{n+1}H^2 + \sum_{\alpha=n+2}^{m+p} H^2|\Theta^{\alpha}|^2 \\
&+ \sum_{\alpha=n+2}^{m+p}\sum_{i,j,k} H\Theta^{n+1}_{ij}\Theta^{\alpha}_{jk}\Theta^{\alpha}_{ki}. \qquad (3.17)
\end{aligned}$$

Taking Lemma 2.2 and Eq (3.17) into account, we have

$$\begin{aligned}
\sum_{\alpha,\beta,i,j,k} H^{\beta}\zeta^{\beta}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk} \geq{}& -\frac{n-2}{\sqrt{n(n-1)}}|\Theta^{n+1}|^3|H| + 3H^2|\Theta^{n+1}|^2 \\
&+ nH^4 + (\sum_{\alpha=n+2} H^2|\Theta^{\alpha}|^2 - H^2|\Theta^{n+1}|^2) \\
&+ \sum_{\alpha=n+2} Htr(\Theta^{n+1})(\Theta^{\alpha})^2. \qquad (3.18)
\end{aligned}$$

Which by virtue of Lemma 2.3 and (3.18), yields

$$\sum_{\alpha,\beta,i,j,k} H^{\beta}\zeta^{\beta}_{kl}\zeta^{\alpha}_{jl}\zeta^{\alpha}_{jk} \geq -\frac{n-2}{\sqrt{n(n-1)}}|\Theta^{n+1}|^3|H| + 3H^2|\Theta^{n+1}|^2 + nH^4$$

$$
\begin{aligned}
&+ H^2|\Theta|^2 - \frac{n-2}{\sqrt{n(n-1)}} \sum_{\alpha=n+2} |\Theta^{n+1}||\Theta^{\alpha}|^2|H| \\
&= 2H^2|\Theta^{n+1}|^2 + H^2|\Theta|^2 + nH^4 \\
&- \frac{n-2}{\sqrt{n(n-1)}}|\Theta^{n+1}||\Theta|^2|H|.
\end{aligned} \tag{3.19}
$$

Now, substituting (3.12)–(3.14), (3.16) and (3.19) in (3.11), we find

$$
\begin{aligned}
\square(nH) \geq\ & \frac{1}{16}(c_1 + c_2)[(n + 9 + 6(trF)^2)(|\Theta|^2 + nH^2) - 3 - (trF)^2] \\
&+ \frac{1}{16}(c_1 - c_2)[(trF)(n|\Theta|^2 - n^2H^2 + |\Theta|^2 + nH^2)] \\
&+ \frac{1}{16}(c_1 + c_2)[(4(trF)^2 - 2)\|\zeta^{\star}\|^2] \\
&+ \frac{1}{16}(c_1 - c_2)[2(trF)\|\zeta^{\star}\|^2] - \frac{n-2}{\sqrt{n(n-1)}}|\Theta^{n+1}|\Theta|^2||H| \\
&+ nH^2|\Theta|^2 - \frac{1}{2}|\Theta^{n+1}|^4 + |\Theta|^2|\Theta^{n+1}|^2 - \frac{3}{2}|\Theta|^4 \\
=\ & \frac{1}{16}(c_1 + c_2)[(n + 9 + 6(trF)^2)(|\Theta|^2 + nH^2) - 3 - (trF)^2] \\
&+ \frac{1}{16}(c_1 - c_2)[(trF)(n|\Theta|^2 - n^2H^2 + |\Theta|^2 + nH^2)] \\
&+ \frac{1}{16}(c_1 + c_2)[(4(trF)^2 - 2)\|\zeta^{\star}\|^2] \\
&+ \frac{1}{16}(c_1 - c_2)[2(trF)\|\zeta^{\star}\|^2] \\
&+ |\Theta|^2\Big[- \frac{n-2}{\sqrt{n(n-1)}}|\Theta||H| + nH^2 - |\Theta|^2\Big] \\
&+ (|\Theta| - |\Theta^{n+1}|)\Big[\frac{n-2}{\sqrt{n(n-1)}}|\Theta|^2|H| \\
&- \frac{1}{2}(|\Theta| - |\Theta^{n+1}|)(|\Theta| + |\Theta^{n+1}|)^2\Big].
\end{aligned} \tag{3.20}
$$

It is known that [17],

$$
(|\Theta| - |\Theta^{n+1}|)\Big[\frac{n-2}{\sqrt{n(n-1)}}|\Theta|^2|H| - \frac{1}{2}(|\Theta| - |\Theta^{n+1}|)(|\Theta| + |\Theta^{n+1}|)^2\Big] \geq 0.
$$

Therefore, from (3.20) we have

$$
\begin{aligned}
\square(nH) \geq\ & \frac{1}{16}(c_1 + c_2)[(n + 9 + 6(trF)^2)(|\Theta|^2 + nH^2) - 3 - (trF)^2] \\
&+ \frac{1}{16}(c_1 - c_2)[(trF)(n|\Theta|^2 - n^2H^2 + |\Theta|^2 + nH^2)] \\
&+ \frac{1}{16}(c_1 + c_2)[(4(trF)^2 - 2)\|\zeta^{\star}\|^2]
\end{aligned}
$$

$$+\frac{1}{16}(c_1-c_2)[2(trF)\|\zeta^\star\|^2]$$
$$+|\Theta|^2\Big[-\frac{n-2}{\sqrt{n(n-1)}}|\Theta||H|+nH^2-|\Theta|^2\Big]. \tag{3.21}$$

Since, $c_1, c_2 > 0$ and $trF = 0$. Then the above inequality implies the following inequality

$$\begin{aligned}\square(nH) \geq\ & \frac{1}{16}(c_1+c_2)[(n+9)(|\Theta|^2+nH^2)-3-2\|\zeta^\star\|^2]\\ &+|\Theta|^2\Big[-\frac{n-2}{\sqrt{n(n-1)}}|\Theta||H|+nH^2-|\Theta|^2\Big]\\ \geq\ & \frac{1}{16}(c_1+c_2)[(n+9)(|\Theta|^2+nH^2)]+nH^2|\Theta|^2. \end{aligned}\tag{3.22}$$

From (2.21) we have $\int_{\mathcal{N}} \square(nH)dv = 0$. Thus we have following two cases:

Case 1:

$$|\Theta|^2+nH^2=0 \quad \textit{and} \quad nH^2|\Theta|^2=0$$

which yields $\Lambda = 0$ and $H = 0$. Thus, the submanifold is totally geodesic.

Case 2:

$$\frac{1}{16}(c_1+c_2)(n+9)(|\Theta|^2+nH^2)=-nH^2|\Theta|^2$$

which implies that $\Lambda = 0$ and $H = 0$. It is again totally geodesic.

Hence, we have our assertion. □

Now, we give an example in the support of the Theorem 3.1.

Example. It is known that the real projective space $\mathbb{R}P^n(1)$ is totally geodesic submanifold of the complex projective space $\mathbb{C}P^n(4)$ [3]. Also from [23] we know that, if N_1 is any submanifolds of Kaehler manifold M_1 and N_2 is any submanifold of Kaehler manifold M_2, then the natural product $N = N_1 \times N_2$ is a submanifold of the Kaehler product manifold $M = M_1 \times M_2$. Hence, $\mathbb{R}P^n(1) \times \mathbb{R}P^n(1)$ is a submanifolds of the Kaehler product manifold $\mathbb{C}P^n(4) \times \mathbb{C}P^n(4)$, which satisfies all the hypothesis of the Theorem 3.1 and indeed totally geodesic.

Remark 3.4. In the above example, it can be noticed that *trF* vanishes, due to the fact that the projection operators $\mathcal{P}$ and $\mathcal{Q}$ coincide.

4. $\delta-$invariant totally real submanifold in Kaehler product manifold

Let $\mathcal{N}$ be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of $\mathcal{N}$ of the plane section $\pi \subset T_x\mathcal{N}$ at a point $x \in \mathcal{N}$. Let $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2(m+p)}\}$ be the orthonormal basis of $T_x\mathcal{N}$ and $T_x^\perp\mathcal{N}$ at any $x \in \mathcal{N}$, then the scalar curvature τ at that point is given by

$$\tau(x)=\sum_{1\leq i<j\leq n}K(e_i\wedge e_j).$$

If we consider that L is an r-dimensional subspace of $T\mathcal{N}$, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L. Then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1\leq\gamma<\beta\leq r} K(e_\gamma \wedge e_\beta), \qquad 1 \leq \gamma, \beta \leq r, \tag{4.1}$$

for $n \geq 3$ and $k \geq 1$. Let us assume $\mathfrak{S}(n,k)$ the finite set consisting of $k-$tuples $(n_1, \ldots, n_k)$ of integers satisfying

$$2 \leq n_1, \ldots, n_k < n \text{ and } n_1 + \cdots + n_k \leq n.$$

Also denote by $\mathfrak{S}(n)$ the union $\bigcup_{k\geq 1} \mathfrak{S}(n,k)$.
For each $(n_1, \ldots, n_k) \in \mathfrak{S}(n)$ and each point $x \in \mathcal{N}$, B. Y. Chen [8] introduced a Riemannian invariant $\delta(n_1, \ldots, n_k)(x)$ defined by

$$\delta(n_1, \ldots, n_k)(x) = \tau(x) - inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{4.2}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of $T_x\mathcal{N}$ such that $dimL_j = \sum n_j$, $j = 1, \ldots, k$.

We recall the following Lemma [7]:

Lemma 4.1. *Let $a_1, \ldots, a_n, a_{n+1}$ be $n+1$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)(a_{n+1} + \sum_{i=1}^{n} a_i^2). \tag{4.3}$$

Then $2a_1a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

In this section we state and prove the following.

Theorem 4.2. *Let $\mathcal{N}$ be a totally real submanifold in Kaehler product manifold $\overline{\mathcal{N}}(c_1, c_2) = \overline{\mathcal{N}}_1^m(c_1) \times \overline{\mathcal{N}}_2^p(c_2)$ and if $tr\mathcal{P}$ coincides with $tr\mathcal{Q}$, then*

$$\begin{aligned}\delta(n_1, \ldots, n_k) \leq\ & \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 \\ & + \frac{1}{32}\Big[n(n+1) - \sum_{j=1}^{k} n_j(n_j+1)\Big](c_1 + c_2),\end{aligned} \tag{4.4}$$

and the equality holds in (4.4) if and only if at a point $x \in \mathcal{N}$ there exists an orthonormal basis $e_1, \ldots, e_{2(m+p)}$ at x such that the shape operator of $\mathcal{N}$ in $\overline{\mathcal{N}}(c_1, c_2)$ at x takes the forms:

$$\Lambda_r = \begin{pmatrix} \Lambda_1^r & \ldots & 0 & \\ \vdots & \ddots & \vdots & O \\ 0 & \ldots & \Lambda_k^r & \\ & O & & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, 2(m+p), \tag{4.5}$$

where O is a null matrix, I is an identity matrix and each Λ_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$tr(\Lambda_1^r) = \cdots = tr(\Lambda_k^r) = \mu_r. \tag{4.6}$$

Proof. We put

$$\varepsilon = 2\tau - \frac{1}{16}(c_1 + c_2)n(n+1) - \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}H^2. \tag{4.7}$$

By combining (2.18) and (4.7), we obtain

$$n^2H^2 = (n+k-\sum n_j)(\varepsilon + |\Lambda|^2).$$

With respect to the orthonormal basis, the last equation can be written as

$$\begin{aligned}(\sum_{i=1}^{n}\zeta_{ii}^{n+1})^2 = (n+k-\sum n_j)(\varepsilon + \sum_{i=1}^{n}(\zeta_{ii}^{n+1})^2 \\ + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2(m+p)}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2),\end{aligned} \tag{4.8}$$

which implies that

$$\begin{aligned}(\sum_{i=1}^{n}a_i)^2 = (n+k-\sum n_j)(\varepsilon + \sum_{i=1}^{n}(a_i)^2 \\ + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2(m+p)}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2).\end{aligned} \tag{4.9}$$

Now, let us set

$$\Delta_1 = \{1,\ldots,n_1\},\ldots,\Delta_k = \{n_1+\cdots+n_{k-1}+1,\ldots,n_1+\cdots+n_k\},$$

and

$$\begin{aligned}&\overline{a}_1 = a_1, \overline{a}_2 = a_2+\cdots+a_{n_1},\\ &\overline{a}_3 = a_{n_1+1}+\cdots+a_{n_1+n_2},\ldots,\overline{a}_{k+1} = a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_{n_1+\cdots+n_k},\\ &\overline{a}_{k+2} = a_{n_1+\cdots+n_k+1},\ldots,\overline{a}_{n+k+1-\sum n_j} = a_n.\end{aligned}$$

Then Eq (4.9) is equivalent to

$$\begin{aligned}(\sum_{i=1}^{n+k+1-\sum n_j}\overline{a}_i)^2 = (n+k-\sum n_j)(\varepsilon + \sum_{i=1}^{n+k+1-\sum n_j}(\overline{a}_i)^2 \\ + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2(m+p)}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2) \\ - \sum_{2\leq\alpha_1\neq\beta_1\leq n_1}a_{\alpha_1}a_{\beta_1} - \sum_{\alpha_1\neq\beta_1}a_{\alpha_2}a_{\beta_2}\end{aligned}$$

$$-\cdots-\sum_{\alpha_k\neq\beta_k} a_{\alpha_k}a_{\beta_k}, \tag{4.10}$$

where $\alpha_2,\beta_2 \in \Delta_2,\ldots,\alpha_k,\beta_k \in \Delta_k$.
Using Lemma 4.1 in (4.10) yields

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} - \cdots - \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}$$
$$\geq \frac{\varepsilon}{2} + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2(m+p)}\sum_{i,j=1}^{n}(\zeta_{ij}^r)^2, \tag{4.11}$$

where $\alpha_j,\beta_j \in \Delta_j$, $j=1,\ldots,k$.
Furthermore, combining (4.1) with the Gauss equation, we obtain

$$\tau(L_j) = \frac{1}{32}n_j(n_j+1)(c_1+c_2) + \sum_{r=n+1}^{2(m+p)}\sum_{\alpha_j<\beta_j}(\zeta^r_{\alpha_j\alpha_j}\zeta^r_{\beta_j\beta_j} - (\zeta^r_{\alpha_j\beta_j})^2). \tag{4.12}$$

Combining (4.11) and (4.12) gives

$$\tau(L_1)+\cdots+\tau(L_k) \geq \frac{\varepsilon}{2} + \frac{1}{32}\sum_{j=1}^{k} n_j(n_j+1)(c_1+c_2)$$
$$+\frac{1}{2}\sum_{r=n+1}^{2(m+p)}\sum_{(\alpha,\beta)\notin\Delta^2}(\zeta^r_{\alpha\beta})^2 + \frac{1}{2}\sum_{r=n+2}^{2(m+p)}\sum_{j=1}^{k}\Big(\sum_{\alpha_j\in\Delta_j}\zeta^r_{\alpha_j\alpha_j}\Big)^2$$
$$\geq \frac{\varepsilon}{2} + \frac{1}{32}\sum_{j=1}^{k} n_j(n_j+1)(c_1+c_2), \tag{4.13}$$

where $\Delta = \Delta_1 \cup \cdots \cup \Delta_k$, $\Delta^2 = (\Delta_1\times\Delta_1)\cup\cdots\cup(\Delta_k\times\Delta_k)$.

Thus, Eqs (4.2), (4.7) and (4.13) imply (4.4).

Moreover, equality in (4.11) and (4.13) holds at a point x, if it holds for (4.4) at a point x. In this case from Lemma 4.1 and Eqs (4.10)–(4.13), we have (4.5) and (4.6). A straightforward computation yields the converse part. □

Example. Due to the fact that the real hyperbolic space $H^n(1)$ can be isometrically embedded in the complex hyperbolic space $\mathbb{C}H^n(-4)$ as a totally real totally geodesic submanifold of minimal codimension [22]. It follows that $N = RP^n(1)\times RP^n(1)$ is a totally real submanifold of $M = HP^n(4)\times HP^n(4)$. This submanifold satisfies all hypotheses of Theorem 4.2. In this case the inequality is satisfied with equality at all points.

Theorem 4.2 yields the following obstruction result.

Corollary 4.3. *Let $\mathcal{N}$ be a totally real submanifold in Kaehler product manifold $\overline{\mathcal{N}}(c_1,c_2) = \overline{\mathcal{N}}_1^n(c_1)\times\overline{\mathcal{N}}_2^p(c_2)$ and if trF vanishes, then for $c_1+c_2=0$, $\mathcal{N}$ can not be minimally immersed in $\overline{\mathcal{N}}(c_1,c_2)$.*

5. Conclusions

We characterized totally real submanifold using self-adjoint differential operator. The self-adjoint differential operators are mainly used in functional analysis and quantum mechanics. In quantum mechanics their importance lies in the Drac-Von Neumann formulation of quantum mechanics in which momentum, angular momentum and spin are represented by self-adjoint operators on Hilbert space. A self-adjoint differential operator is an important class of unbounded operators. Therefore, we can use such operator for infinite dimensional cases and we resemble the finite dimensional case. Thus, use of the operator for such characterization may open a new path to link results in differential geometry with quantum mechanics as well as well with functional analysis.

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Conflict of interest

The authors declare that there is no conflict of interests.

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