Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 25–38.
DOI:10.3934/math.2022002
Received: 19 August 2021
Accepted: 22 September 2021
Published: 29 September 2021

Research article

The general Albertson irregularity index of graphs

Zhen Lin[1,*], **Ting Zhou**[2], **Xiaojing Wang**[2] **and Lianying Miao**[2]

[1] School of Mathematics and Statistics, Qinghai Normal University, Xining, 810008, Qinghai, China
[2] School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, Jiangsu, China

* **Correspondence:** Email: lnlinzhen@163.com.

Abstract: We introduce the general Albertson irregularity index of a connected graph G and define it as $A_p(G) = (\sum_{uv\in E(G)} |d(u) - d(v)|^p)^{\frac{1}{p}}$, where p is a positive real number and $d(v)$ is the degree of the vertex v in G. The new index is not only generalization of the well-known Albertson irregularity index and σ-index, but also it is the Minkowski norm of the degree of vertex. We present lower and upper bounds on the general Albertson irregularity index. In addition, we study the extremal value on the general Albertson irregularity index for trees of given order. Finally, we give the calculation formula of the general Albertson index of generalized Bethe trees and Kragujevac trees.

Keywords: general Albertson irregularity index; tree; generalized Bethe tree; Kragujevac tree
Mathematics Subject Classification: 05C05, 05C07, 05C09, 05C35

1. Introduction

Let G be a simple undirected connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For $v \in V(G)$, $N(v)$ denotes the set of all neighbors of v, and $d(v) = |N(v)|$ denotes the degree of vertex v in G. The minimum and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, or simply δ and Δ, respectively. A pendant vertex of G is a vertex of degree one. A graph G is called (Δ, δ)-semiregular if $\{d(u), d(v)\} = \{\Delta, \delta\}$ holds for all edges $uv \in E(G)$. Denote by P_n and $K_{1,n-1}$ the path and the star with n vertices, respectively.

In 1997, the Albertson irregularity index of a connected graph G, introduced by Albertson [1], is defined as

$$Alb(G) = \sum_{uv\in E(G)} |d(u) - d(v)|.$$

This index has been of interest to mathematicians, chemists and scientists from related fields due to the fact that the Albertson irregularity index plays a major role in irregularity measures of graphs [3, 4, 7, 8, 17], predicting the biological activities and properties of chemical compounds in the QSAR/QSPR modeling [12, 24] and the quantitative characterization of network heterogeneity [9]. By the natural extension of the Albertson irregularity index, Gutman et al., [13] recently proposed the σ-index as follows:

$$\sigma(G) = \sum_{uv\in E(G)} (d(u) - d(v))^2 = F - 2M_2,$$

where F and M_2 are well-known the forgotten topological index and the second Zagreb index of a graph G, respectively. Recently, the σ-index of a connected graph G is studied, such as the characterization of extremal graphs [5] and mathematical relations between the σ-index and other graph irregularity indices [21].

The generalization of topological index is a trend of mathematical chemistry in recent years. Many classical topological indices are generalized, such as the general Randić index [6], the first general Zagreb index [18], the general sum-connectivity index [31], the general eccentric connectivity index [28], etc. Motivated by this fact, we propose the general Albertson irregularity index of a graph G as follows:

$$A_p(G) = \left(\sum_{uv\in E(G)} |d(u) - d(v)|^p \right)^{\frac{1}{p}},$$

where p is a positive real number. Evidently, $A_1(G) = Alb(G)$ and $A_2^2(G) = \sigma(G)$. The other motivation is that the topological index formed from distance function of the degree of vertex has attracted extensive attention of scholars. In 2021, Gutman [10] proposed the Sombor index of a graph G and defined it as $SO(G) = \sum_{uv\in E(G)} \sqrt{d^2(u) + d^2(v)}$, which is the Euclidean norm of $d(u)$ and $d(v)$. According to Gutman [11], it is imaginable to use other distance function to study properties of graphs. Based on this, it is not difficult to find that $A_p(G)$ is the Minkowski norm of $d(u)$ and $d(v)$, which is unification of absolute distance, Euclidean distance and Chebyshev distance. Hence $A_p(G) = \Delta - \delta$ as p becomes infinite. In particular, $A_p(G)$ is the l_p-norm of $d(u)$ and $d(v)$ for $p \geq 1$.

We will first recall some useful notions and lemmas used further in Section 2. In Section 3, upper and lower bounds on the general Albertson irregularity index of graphs are given, and the extremal graphs are characterized. In Section 4, the first two trees with minimum general Albertson irregularity index are determined in all trees of fixed order. In Section 5, the general Albertson index of the well-known generalized Bethe trees and Kragujevac trees is obtained.

2. Preliminaries

Let $u \vee G$ be the graph by adding all edges between the vertex u and $V(G)$, see for example in Figure 1. The first general Zagreb index of a graph G is defined as $Z_p(G) = \sum_{v\in V(G)} d^p(v)$ for any real number p. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of a shortest path between u and v. The eccentricity of v, $\varepsilon(v)$, is the distance between v and any vertex which is furthest from v in G. The line graph $L(G)$ is the graph whose vertex set are the edges in G, where two vertices are adjacent if the corresponding edges in G have a common vertex. Let $\mathcal{T}_n$ be the set of trees with n vertices. A spider is a tree with at most one vertex of degree more than two.

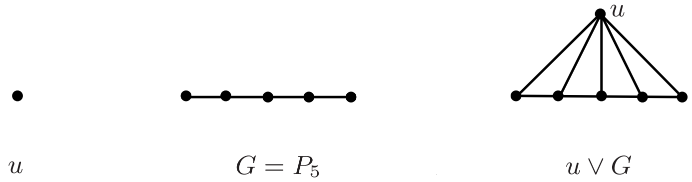

Figure 1. Graph $u \vee G$.

Lemma 2.1. (Power mean inequality) *Let $x_1, x_2, \ldots, x_n$ be positive real numbers and p, q real numbers such that $p > q$. Then,*

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Lemma 2.2. (Hölder inequality) *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers and let p, q be two positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}},$$

with equality if and only if $|a_i|^p = \lambda |b_i|^q$ for some real constant λ, $1 \leq i \leq n$.

Lemma 2.3. ([26]) *If $p \geq 1$ is an integer and $0 \leq x_1, x_2, \ldots, x_n \leq n-1$, then*

$$\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \leq (n-1)^{1-\frac{1}{p}} \sum_{i=1}^{n} x_i^{\frac{1}{p}}.$$

Lemma 2.4. (Minkowski inequality) *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers. If $p \geq 1$. Then*

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}, \tag{2.1}$$

with equality if and only if $a_i = \lambda b_i$ for some real constant λ, $1 \leq i \leq n$. For $0 < p < 1$, the inequality (2.1) *gets reversed.*

Lemma 2.5. ([20]) *Let $x = (x_1, x_2, \ldots, x_k, \ldots)$ be a non-zero vector. Then for $p \geq 2$,*

$$\|x\|_p \leq \|x\|_2,$$

with equality if and only if all but one of the x_i are equal to 0.

Lemma 2.6. *Let G be a connected graph. Suppose there exists a vertex $u \in V(G)$ with $d(u) \geq 3$, $v_1, v_2, \ldots, v_l$ and $w_1, w_2, \ldots, w_t$ are two path components in $G - u$, where $N(u) = \{v_1, w_1, u_1, u_2, \ldots, u_{d(u)-2}\}$. Let $G' = G - uw_1 + v_l w_1$. Then*

(i) *If $p > 0$ and $d(u) > d(u_i)$, then $A_p(G) > A_p(G')$.*

(ii) *If $p \geq 1$ and $d(u) \geq d(u_i)$, then $A_p(G) > A_p(G')$.*

(iii) *If $p > 0$ and $\Delta = d(u) = 3$, then $A_p(G) > A_p(G')$.*

Proof. Let $s = d(u)-2$. Since $d(u) > d(u_i)$, $d(u) \geq 3$, $d(v_1) \leq 2$ and $d(w_1) \leq 2$, we have $|d(u)-d(u_i)|^p - |d(u)-d(u_i)-1|^p > 0$ and $|d(u)-d(v_1)|^p - |d(u)-d(v_1)-1|^p + 1 > 0$. Then

$$\begin{aligned} A_p^p(G) - A_p^p(G') &= \sum_{uv\in E(G)} |d(u)-d(v)|^p - \sum_{u'v'\in E(G')} |d(u')-d(v')|^p \\ &= \sum_{i=1}^{s}(|d(u)-d(u_i)|^p - |d(u)-d(u_i)-1|^p) + |d(u)-d(w_1)|^p \\ &\quad + |d(u)-d(v_1)|^p - |d(u)-d(v_1)-1|^p + 1 \\ &> 0. \end{aligned}$$

Thus we have $A_p(G) > A_p(G')$.

If $p \geq 1$ and $d(u) = d(u_i)$, then we have

$$\begin{aligned} A_p^p(G) - A_p^p(G') &= \sum_{i=1}^{s}(|d(u)-d(u_i)|^p - |d(u)-d(u_i)-1|^p) + |d(u)-d(w_1)|^p \\ &\quad + |d(u)-d(v_1)|^p - |d(u)-d(v_1)-1|^p + 1 \\ &= -(d(u)-2) + |d(u)-d(w_1)|^p + |d(u)-d(v_1)|^p \\ &\quad - |d(u)-d(v_1)-1|^p + 1 \\ &> 0. \end{aligned}$$

Thus we have $A_p(G) > A_p(G')$.

If $p > 0$ and $\Delta = d(u) = 3$, then we have

$$\begin{aligned} A_p^p(G) - A_p^p(G') &= |d(u)-d(u_1)|^p - |d(u)-d(u_1)-1|^p + |d(u)-d(w_1)|^p \\ &\quad + |d(u)-d(v_1)|^p - |d(u)-d(v_1)-1|^p + 1 \\ &\geq -1 + |3-d(w_1)|^p + |3-d(v_1)|^p \\ &\quad - |3-d(v_1)-1|^p + 1 \\ &= |3-d(w_1)|^p + |3-d(v_1)|^p - |2-d(v_1)|^p \\ &> 0. \end{aligned}$$

Thus we have $A_p(G) > A_p(G')$.

Combining the above arguments, we have the proof. □

3. Some bounds for the general Albertson index

Theorem 3.1. *Let G be a connected graph with m edges. If $p > q$, then*

$$A_p(G) \geq m^{\frac{1}{p}-\frac{1}{q}} A_q(G)$$

with equality if and only if G is a regular graph (when G is non-bipartite) or G is a (Δ, δ)-semiregular bipartite graph (when G is bipartite).

Proof. By Lemma 2.1, we have

$$\left(\frac{1}{m}\sum_{uv\in E(G)}|d(u)-d(v)|^p\right)^{\frac{1}{p}} \geq \left(\frac{1}{m}\sum_{uv\in E(G)}|d(u)-d(v)|^q\right)^{\frac{1}{q}},$$

that is,

$$\frac{1}{m^{\frac{1}{p}}}A_p(G) \geq \frac{1}{m^{\frac{1}{q}}}A_q(G),$$

that is,

$$A_p(G) \geq m^{\frac{1}{p}-\frac{1}{q}}A_q$$

with equality if and only if $|d(u)-d(v)|$ is a constant for every edge uv in G, that is, G is a regular graph (when G is non-bipartite) or G is a (Δ,δ)-semiregular bipartite graph (when G is bipartite). □

Corollary 3.2. *Let G be a connected graph with m edges. Then*

$$Alb(G) \leq \sqrt{m(F-2M_2)},$$

with equality if and only if G is a regular graph (when G is non-bipartite) or G is a (Δ,δ)*-semiregular bipartite graph (when G is bipartite).*

Theorem 3.3. *Let G be a connected graph. If* $\frac{1}{p}+\frac{1}{q}=1$*, then*

$$A_p(G)A_q(G) \geq F-2M_2,$$

with equality if and only if $p=2$*, or G is a regular graph (when G is non-bipartite) or G is a* (Δ,δ)*-semiregular bipartite graph (when G is bipartite).*

Proof. For $a_i=b_i=d(u)-d(v)$ and apply Lemma 2.2. Then

$$\sum_{uv\in E(G)}(d(u)-d(v))^2 \leq \left(\sum_{uv\in E(G)}|d(u)-d(v)|^p\right)^{\frac{1}{p}}\left(\sum_{uv\in E(G)}|d(u)-d(v)|^q\right)^{\frac{1}{q}},$$

that is,

$$F-2M_2 \leq A_p(G)A_q(G),$$

with equality if and only if $p=2$, or G is a regular graph (when G is non-bipartite) or G is a (Δ,δ)-semiregular bipartite graph (when G is bipartite). □

Theorem 3.4. *Let G be a connected graph with m edges. If* $p\geq 1$ *is an integer, then*

$$A_p(G) \leq (m-1)^{1-\frac{1}{p}}\left(A_{\frac{1}{p}}(G)\right)^{\frac{1}{p}}.$$

Proof. Let $x_i=|d(u)-d(v)|$ in Lemma 2.3. Then

$$\left(\sum_{uv\in E(G)}|d(u)-d(v)|^p\right)^{\frac{1}{p}} \leq (m-1)^{1-\frac{1}{p}}\sum_{uv\in E(G)}|d(u)-d(v)|^{\frac{1}{p}},$$

that is,

$$A_p(G) \leq (m-1)^{1-\frac{1}{p}}\left(A_{\frac{1}{p}}(G)\right)^{\frac{1}{p}}.$$

□

Theorem 3.5. *Let G be a connected graph with m edges. If $p \geq 1$, then*

$$A_p(G) \geq (m + pAlb(G))^{\frac{1}{p}} - m^{\frac{1}{p}}.$$

If $0 < p < 1$, *then*

$$A_p(G) \leq (m + pAlb(G))^{\frac{1}{p}} - m^{\frac{1}{p}}.$$

Proof. Let $a_i = 1$ and $b_i = |d(u) - d(v)|$ in Lemma 2.4. Then

$$\left(\sum_{uv\in E(G)} (1 + |d(u) - d(v)|)^p\right)^{\frac{1}{p}} \leq A_p(G) + m^{\frac{1}{p}}$$

for $p \geq 1$. By Bernoulli inequality, we have

$$A_p(G) + m^{\frac{1}{p}} \geq \left(\sum_{uv\in E(G)} (1 + p|d(u) - d(v)|)\right)^{\frac{1}{p}} = (m + pAlb(G))^{\frac{1}{p}},$$

that is,

$$A_p(G) \geq (m + pAlb(G))^{\frac{1}{p}} - m^{\frac{1}{p}}.$$

For $0 < p < 1$, by Lemma 2.4 and Bernoulli inequality, we have the proof. □

Theorem 3.6. *Let G be a connected graph. If $p \geq 2$, then*

$$A_p(G) \leq \sqrt{F - 2M_2} \tag{3.1}$$

with equality if and only if $d(u) - d(v) \neq 0$ for unique edge uv and 0 for the other edges in G.

Proof. By Lemma 2.5, we have

$$A_p(G) \leq A_2(G) = \left(\sum_{uv\in E(G)} |d(u) - d(v)|^2\right)^{\frac{1}{2}} = \sqrt{\sigma(G)} = \sqrt{F - 2M_2}$$

with equality if and only if $d(u) - d(v) \neq 0$ for only one edge uv and 0 for the other edges in G. □

Remark 3.7. *There exist many graphs such that the equality in* (3.1) *holds, the following graphs are examples (See Figure 2).*

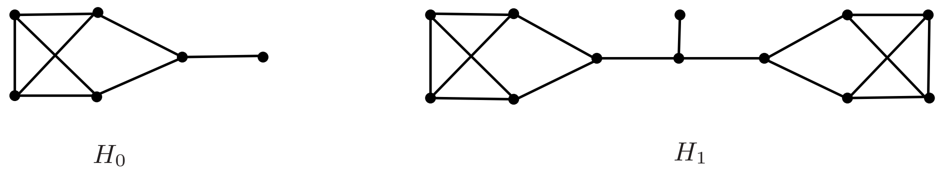

Figure 2. Graphs H_0 and H_1.

Theorem 3.8. *Let G be a connected graph with n vertices. Then*

$$A_p(G) \leq [Z_p(L(G))]^{\frac{1}{p}}$$

with equality if and only if $G \cong K_{1,\, n-1}$.

Proof. By definition of $A_p(G)$, we have

$$\begin{aligned}
A_p(G) &= \left(\sum_{uv\in E(G)} |d(u) - d(v)|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{uv\in E(G)} |(d(u) - 1) - (d(v) - 1)|^p\right)^{\frac{1}{p}} \\
&\leq \left[\sum_{uv\in E(G)} (d(u) + d(v) - 2)^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{v\in V(L(G))} (d(v))^p\right]^{\frac{1}{p}} \\
&= [Z_p(L(G))]^{\frac{1}{p}}
\end{aligned}$$

with equality if and only if $(d(u) - 1)(d(v) - 1) \leq 0$, that is, $d(v) = 1$ for every edge uv in G, that is, G is a star $K_{1,\, n-1}$. □

Corollary 3.9. *Let G be a connected graph with n vertices and m edges. Then*

$$Alb(G) \leq Z_1(L(G)) = Z_2(G) - 2m \quad \textit{and} \quad \sigma(G) \leq Z_2(L(G))$$

with equality if and only if $G \cong K_{1,\, n-1}$.

Theorem 3.10. *Let u be a vertex and G be a connected graph. Then*

$$A_p(u \vee G) = [Z_p(\overline{G}) + A_p^p(G)]^{\frac{1}{p}},$$

where $\overline{G}$ *is the complement of G.*

Proof. By definition of $u \vee G$, we have

$$\begin{aligned}
A_p(u \vee G) &= \left(\sum_{wv\in E(u\vee G)} |d(w) - d(v)|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{v\in V(G)} |n - d(v) - 1|^p + \sum_{wv\in E(G)} |d(w) - d(v)|^p\right)^{\frac{1}{p}} \\
&= \left[\sum_{v\in V(\overline{G})} d^p(v) + A_p^p(G)\right]^{\frac{1}{p}} \\
&= [Z_p(\overline{G}) + A_p^p(G)]^{\frac{1}{p}}.
\end{aligned}$$

□

Corollary 3.11. *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} Alb(u \vee G) - Alb(G) &= n(n-1) - 2m, \\ \sigma(u \vee G) - \sigma(G) + Z_2(G) &= n(n-1)^2 - 4m(n-1). \end{aligned}$$

Theorem 3.12. *Let u be a pendant vertex of a connected graph G with $n \geq 3$ vertices. If $G + P_t$ $(t \geq 1)$ is the graph by adding a new (pendant) path to u, then*

(i) $A_p(G + P_t) > A_p(G)$ *for* $0 < p < 1$.
(ii) $A_p(G + P_t) = A_p(G)$ *for* $p = 1$.
(iii) $A_p(G + P_t) < A_p(G)$ *for* $p > 1$.

Proof. Let v be the unique neighbour of u in G. Since $a^p + b^p > (a+b)^p$ for $a > 0, b > 0$ and $0 < p < 1$, we have

$$\begin{aligned} A_p^p(G + P_t) &= \sum_{rs \in E(G+e)} |d(r) - d(s)|^p \\ &= (d(v) - 2)^p + (2 - 1)^p + \sum_{rs \in E(G), r,s \neq u} |d(r) - d(s)|^p \\ &> (d(v) - 1)^p + \sum_{rs \in E(G), r,s \neq u} |d(r) - d(s)|^p \\ &= A_p^p(G), \end{aligned}$$

for $0 < p < 1$. Thus $A_p(G + P_t) > A_p(G)$. By a similar reasoning as above, we have the proof of (ii) and (iii). □

Corollary 3.13. *Let u be a pendant vertex of a connected graph G with $n \geq 3$ vertices. If $G + P_t$ $(t \geq 1)$ is the graph by adding a new (pendant) path to u, then*

$$\sigma(G + P_t) < \sigma(G).$$

4. The general Albertson index of trees

Theorem 4.1. *Let $T_n \in \mathcal{T}_n$. Then*

$$2^{\frac{1}{p}} \leq A_p(T_n) \leq (n-2)(n-1)^{\frac{1}{p}}.$$

The lower bound is attained if and only if $T_n \cong P_n$. The upper bound is attained if and only if $T_n \cong K_{1,n-1}$.

Proof. If $\Delta \geq 3$, then T_n has at least three pendant vertices. Thus $A_p(T_n) > 3^{\frac{1}{p}} > 2^{\frac{1}{p}} = A_p(P_n)$. In addition, $A_p(T_n) \leq (\Delta - 1)(n-1)^{\frac{1}{p}} \leq (n-2)(n-1)^{\frac{1}{p}} = A_p(K_{1,n-1})$. □

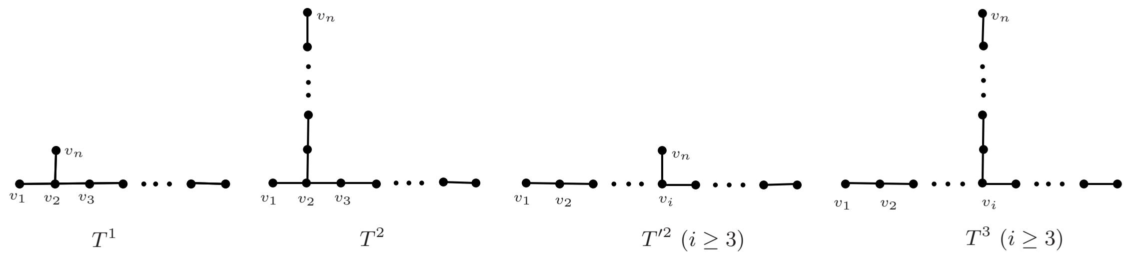

Figure 3. Graphs T^1, T^2, T'^2, T^3.

Theorem 4.2. *Let $n \geq 10$, $T_n \in \mathcal{T}_n - \{P_n\}$, T^1 and T^3 shown as in Figure 3.*
(i) *If $p > 1$, then $A_p(T_n) \geq 6^{\frac{1}{p}}$ with equality if and only if $T_n \cong T^1$.*
(ii) *If $p = 1$, then $A_p(T_n) \geq 6$ with equality if and only if T_n is a spider with $\Delta = 3$.*
(iii) *If $0 < p < 1$, then $A_p(T_n) \geq (2^{p+1} + 2)^{\frac{1}{p}}$ with equality if and only if $T_n \cong T^3$.*

Proof. Let $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$ be the degree sequence of T_n, and let k be the number of non-pendant edges uv with $d(u) \neq d(v)$. Then

$$A_p^p(T_n) = \sum_{uv \in E(G)} |d(u) - d(v)|^p \geq d_1 + (d_2 - 2) + (d_3 - 2) + k.$$

If $d_1 \geq 7$, then $A_p^p(T_n) > 7$.
If $d_1 \geq 6$ and $d_2 \geq 3$, then $A_p^p(T_n) > 6 + (3 - 2) = 7$.
If $d_1 \geq 6$ and $d_2 = 2$, then $A_p^p(T_n) > 6 + 1 = 7$.
If $d_1 = d_2 = 5$, then $A_p^p(T_n) > 5 + (5 - 2) = 8$.
If $d_1 = 5$ and $d_2 = 4$, then $A_p^p(T_n) > 5 + (4 - 2) = 7$.
If $d_1 = 5$ and $d_2 = 3$, then $A_p^p(T_n) > 5 + (3 - 2) + 1 = 7$.

If $d_1 = 5$ and $d_2 = 2$, then $A_p^p(T_n) = \begin{cases} 4^{p+1} + 3^p + 1, \\ 3 \cdot 4^p + 2 \cdot 3^p + 2, \\ 2 \cdot 4^p + 3^{p+1} + 3, \\ 4^p + 4 \cdot 3^p + 4, \\ 5 \cdot 3^p + 5. \end{cases}$ Thus $A_p^p(T_n) > 6$.

If $d_1 = d_2 = 4$, then $A_p^p(T_n) \geq 4 + (4 - 2) + 1 > 7$.
If $d_1 = 4$ and $d_2 = d_3 = d_4 = 3$, then $A_p^p(T_n) > 4 + (3 - 2) + (3 - 2) + (3 - 2) = 7$.
If $d_1 = 4$ and $d_2 = d_3 = 3$, then $A_p^p(T_n) > 4 + (3 - 2) + (3 - 2) + 1 = 7$.
If $d_1 = 4$ and $d_2 = 3$, then $A_p^p(T_n) > 4 + (3 - 2) + 1 = 6$.

If $d_1 = 4$ and $d_2 = 2$, then $A_p^p(T_n) = \begin{cases} 3^{p+1} + 2^p + 1, \\ 2 \cdot 3^p + 2^{p+1} + 2, \\ 3^p + 3 \cdot 2^p + 3, \\ 4 \cdot 2^p + 4. \end{cases}$

If $d_1 = 3$, we can applying Lemma 2.6 repeatedly to the vertices with degree three. Thus the minimum value of T_n has four cases, shown as in Figure 3. By direct computing, we have

$$A_p^p(T^1) = 2^{p+1} + 2, \quad A_p^p(T^2) = A_p^p(T'^2) = 2^p + 4, \quad A_p^p(T^3) = 6.$$

By comparing the above cases, we have that T_1, T^2, T'^2 and T^3 are the candidates with minimum general Albertson index among $\mathcal{T}_n - \{P_n\}$. Further, we have $A_p^p(T^1) > A_p^p(T^2) = A_p^p(T'^2) > A_p^p(T^3)$ for $p > 1$, $A_p^p(T^1) = A_p^p(T^2) = A_p^p(T'^2) = A_p^p(T^3)$ for $p = 1$, and $A_p^p(T^1) < A_p^p(T^2) = A_p^p(T'^2) < A_p^p(T^3)$ for $0 < p < 1$. □

Theorem 4.3. *Let T_n be a tree with n vertices. If $p \geq 1$, then*

$$A_p(T_n) \geq (\Delta \varepsilon^{1-p}(v_\Delta))^{\frac{1}{p}}(\Delta - 1),$$

where v_Δ is a vertex of the maximum degree.

Proof. Let $v_\Delta v_1 v_2 \dots v_{l_1}$ be the path from the vertex v_Δ to pendant vertex v_{l_1}. Then $d(v_\Delta, v_{l_1}) = l_1$. Let $f(x) = x^p$. Since $f(x)$ is a increasing and convex function for $x > 0$ and $p \geq 1$, we have

$$\begin{aligned}
&\frac{1}{l_1}(f(|d(v_\Delta) - d(v_1)|) + f(|d(v_1) - d(v_2)|) + \cdots + f(|v_{l_1-1} - d(v_{l_1})|)) \\
&\geq f\left(\frac{|d(v_\Delta) - d(v_1)| + |d(v_1) - d(v_2)| + \cdots + |v_{l_1-1} - d(v_{l_1})|}{l_1}\right) \\
&\geq f\left(\frac{(d(v_\Delta) - d(v_1) + d(v_1) - d(v_2) + \cdots + v_{l_1-1} - d(v_{l_1}))}{l_1}\right) \\
&= f\left(\frac{\Delta - 1}{l_1}\right),
\end{aligned}$$

that is,

$$|d(v_\Delta) - d(v_1)|^p + \cdots + |v_{l_1-1} - d(v_{l_1})|^p \geq l_1^{1-p}(\Delta - 1)^p.$$

Since T_n has at least Δ pendant vertices, we have

$$A_p(T_n) \geq (l_1^{1-p} + l_2^{1-p} + \cdots + l_\Delta^{1-p})^{\frac{1}{p}}(\Delta - 1),$$

where $l_1, l_2, \dots, l_\Delta$ is the distance from maximum degree vertex v_Δ to pendant vertex v_{l_i}, $1 \leq i \leq \Delta$. Note that $\varepsilon(v_\Delta) = \max_{v \in V(G)} d(v_\Delta, v) \geq l_i$ for $1 \leq i \leq \Delta$. Thus we have $A_p(T_n) \geq (\Delta \varepsilon^{1-p}(v_\Delta))^{\frac{1}{p}}(\Delta - 1)$. □

Corollary 4.4. *Let T_n be a tree with n vertices. Then*

$$Alb(T_n) \geq \Delta(\Delta - 1)$$

with equality if and only if G is a spider.

5. The general Albertson index of generalized Bethe trees and Kragujevac trees

In this section, we give the calculation formula of the general Albertson index of generalized Bethe trees and Kragujevac trees which are a wide range of applications in the field of mathematics [22, 25], cheminformatics [15, 27, 29], statistical mechanics [19], etc.

A generalized Bethe tree [23] is a rooted tree in which vertices of the same level (height) have the same degree. We usually use B_k to denote the generalized Bethe tree with k levels with the root at the level 1. More specifically, $B_{k,d}$ denotes a Bethe tree [16] of k levels with the root degree d, and the vertices between the level 2 and $k - 1$ all have degree $d + 1$. A regular dendrimer tree [14] $T_{k,d}$ is a special case of B_k, where the degrees of all internal vertices are d.

Theorem 5.1. *Let B_k be the generalized Bethe tree where the degree of each level is $d_1 \geq d_2 \geq \cdots \geq d_{k-1}, d_k = 1$. Then*

$$A_p(B_k) = d_1^{\frac{1}{p}} \left(|d_1 - d_2|^p + \sum_{i=2}^{k} |d_i - d_{i-1}|^p \prod_{j=2}^{i} (d_j - 1) \right)^{\frac{1}{p}}.$$

Proof. By definition of the generalized Bethe tree, we have

$$\begin{aligned} A_p^p(B_k) &= \sum_{uv \in E(G)} |d(u) - d(v)|^p \\ &= d_1[|d_1 - d_2|^p + |d_2 - d_3|^p(d_2 - 1) + |d_3 - d_4|^p(d_3 - 1)(d_2 - 1) + \cdots \\ &\quad + |d_{k-1} - d_k|^p(d_{k-1} - 1) \cdots (d_2 - 1)] \\ &= d_1 \left(|d_1 - d_2|^p + \sum_{i=2}^{k} |d_i - d_{i-1}|^p \prod_{j=2}^{i} (d_j - 1) \right). \end{aligned}$$

Thus we have the proof. □

Corollary 5.2. *Let $B_{k,d}$ and $T_{k,d}$ be the Bethe tree and a regular dendrimer tree, respectively. Then*

$$A_p(B_{k,d}) = (d + d^{p+k-1})^{\frac{1}{p}} \quad \text{and} \quad A_p(T_{k,d}) = [d(d-1)^{p+k-2}]^{\frac{1}{p}}.$$

Let P_3 be the 3-vertex tree, rooted at one of its terminal vertices, see Figure 4. For $k = 2, 3, \ldots$, construct the rooted tree R_k by identifying the roots of k copies of P_3. The vertex obtained by identifying the roots of P_3-trees is the root of R_k. Let $d \geq 2$ be an integer and $\gamma_1, \gamma_2, \ldots, \gamma_d$ be rooted trees, i.e., $\gamma_1, \gamma_2, \ldots, \gamma_d \in \{R_2, R_3, \ldots\}$. A Kragujevac tree KT [15] is a tree possessing a vertex of degree d, adjacent to the roots of $\gamma_1, \gamma_2, \ldots, \gamma_d$. This vertex is said to be the central vertex of KT, whereas d is the degree of KT. The subgraphs $\gamma_1, \gamma_2, \ldots, \gamma_d$ are the branches of KT. Recall that some (or all) branches of KT may be mutually isomorphic.

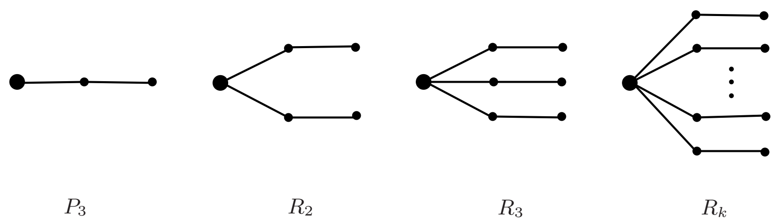

Figure 4. Graphs P_3, R_2, R_3, R_k.

Theorem 5.3. *Let KT be a Kragujevac tree with n vertices and $\gamma_i \cong R_{k_i}$, $i = 1, 2, \ldots, d$. Then*

$$A_p(KT) = \left[\frac{n-d-1}{2} + \sum_{i=1}^{d} (k_i(k_i - 1)^p + |k_i - d + 1|^p) \right]^{\frac{1}{p}}.$$

Proof. Since $1 + \sum_{i=1}^{d} (2k_i + 1) = n$, by definition of the Kragujevac tree, we have

$$A_p^p(KT) = \sum_{uv \in E(G)} |d(u) - d(v)|^p$$

$$= \sum_{i=1}^{d}[k_i + k_i(k_i + 1 - 2)^p + |d - (k_i + 1)^p|]$$
$$= \frac{n-d-1}{2} + \sum_{i=1}^{d}(k_i(k_i - 1)^p + |k_i - d + 1|^p).$$

Thus we have the proof. □

Corollary 5.4. *Let KT be a Kragujevac tree with n vertices and $\gamma_i \cong R_k$, $i = 1, 2, \ldots, d$. Then*

$$A_p(KT) = \left[\frac{n-d-1}{2} + dk(k-1)^p + d|k - d + 1|^p\right]^{\frac{1}{p}}.$$

6. Conclusions

In this paper, we propose the general Albertson irregularity index which extends classical Albertson irregularity index and σ-index. The tight bounds of the general Albertson irregularity index are established. Additionally, the general Albertson irregularity index of trees are studied. In 2014, the total irregularity of a graph G, introduced by Abdo, Brandt and Dimitrov [2], is defined as $\mathrm{irr}_t(G) = \sum_{\{u,v\}\subseteq V(G)} |d(u) - d(v)|$. For measuring the non-self-centrality of a graph, the non-self-centrality number of G was introduced in [30] as $N(G) = \sum_{\{u,v\}\subseteq V(G)} |\varepsilon(u) - \varepsilon(v)|$. Based on these, we can propose the general total irregularity and the general non-self-centrality number of a graph G as follows:

$$\mathrm{irr}_p(G) = \left(\sum_{\{u,v\}\subseteq V(G)} |d(u) - d(v)|^p\right)^{\frac{1}{p}} \quad \text{and} \quad N_p(G) = \left(\sum_{\{u,v\}\subseteq V(G)} |\varepsilon(u) - \varepsilon(v)|^p\right)^{\frac{1}{p}},$$

where the summation goes over all the unordered pairs of vertices in G. The research interaction among $A_p(G)$, $\mathrm{irr}_p(G)$ and $N_p(G)$ will be carried out in the near future.

Acknowledgments

The authors are grateful to the anonymous referee for careful reading and valuable comments which result in an improvement of the original manuscript. This work was supported by the Qinghai science and technology plan project (No. 2021-ZJ-703) and the National Natural Science Foundation of China (No. 11771443).

Conflict of interest

The authors declare no conflict of interest.

References

1. M. O. Albertson, The irregularity of a graph, *Ars Combin.*, **46** (1997), 219–225.

2. H. Abdo, S. Brandt, D. Dimitrov, The total irregularity of a graph, *DMTCS*, **16** (2014), 201–206.
3. H. Abdoa, N. Cohenb, D. Dimitrov, Graphs with maximal irregularity, *Filomat*, **28** (2014), 1315–1322. doi: 10.2298/FIL1407315A.
4. H. Abdo, D. Dimitrov, The irregularity of graphs under graph operations, *Discuss. Math. Graph T.*, **34** (2014), 263–278. doi: 10.7151/dmgt.1733.
5. H. Abdo, D. Dimitrov, I. Gutman, Graphs with maximal σ irregularity, *Discrete Appl. Math.*, **250** (2018), 57–64. doi: 10.1016/j.dam.2018.05.013.
6. B. Bollobás, P. Erdős, Graphs of extremal weights, *Ars Combin.*, **50** (1998), 225–233.
7. X. D. Chen, Y. P. Hou, F. G. Lin, Some new spectral bounds for graph irregularity, *Appl. Math. Comput.*, **320** (2018), 331–340. doi: 10.1016/j.amc.2017.09.038.
8. D. Dimitrov, T. Réti, Graphs with equal irregularity indices, *Acta Polytech. Hung.*, **11** (2014), 41–57.
9. E. Estrada, Quantifying network heterogeneity, *Phys. Rev. E Stat. Nonlin. Soft Matter Phys.*, **82** (2010), 066102. doi: 10.1103/PhysRevE.82.066102.
10. I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.
11. I. Gutman, Some basic properties of Sombor indices, *Open J. Discret. Appl. Math.*, **4** (2021), 1–3. doi: 10.30538/psrp-odam2021.0047.
12. I. Gutman, P. Hansen, H. Mélot, Variable neighborhood search for extremal graphs. 10. comparison of irregularity indices for chemical trees, *J. Chem. Inf. Model.*, **45** (2005), 222–230. doi: 10.1021/ci0342775.
13. I. Gutman, M. Togan, A. Yurttas, A. S. Cevik, I. N. Cangul, Inverse problem for sigma index, *MATCH Commun. Math. Comput. Chem.*, **79** (2018), 491–508.
14. I. Gutman, Y.N. Yeh, S.L. Lee, J.C. Chen, Wiener numbers of dendrimers, *MATCH Commun. Math. Comput. Chem.*, **30** (1994), 103–115.
15. S. A. Hosseini, M. B. Ahmadi, I. Gutman, Kragujevac trees with minimal atom-bond connectivity index, *MATCH Commun. Math. Comput. Chem.*, **71** (2014), 5–20.
16. O. J. Heilmann, E. H. Lieb, Theory of monomer-dimer systems, *Commun. Math. Phys.*, **25** (1972), 190–232. doi: 10.1007/BF01877590.
17. M. A. Henninga, D. Rautenbach, On the irregularity of bipartite graphs, *Discrete Math.*, **307** (2007), 1467–1472. doi: 10.1016/j.disc.2006.09.038.
18. X. L. Li, J. Zheng, A unifled approach to the extremal trees for difierent indices, *MATCH Commun. Math. Comput. Chem.,* **54** (2005), 195–208.
19. M. Ostilli, Cayley trees and Bethe lattices: A concise analysis for mathematicians and physicists, *Physica A*, **391** (2012), 3417–3423. doi: 10.1016/j.physa.2012.01.038.
20. I. Rivin, Counting cycles and finite dimensional L^p norms, *Adv. Appl. Math.*, **29** (2002), 647–662. doi: 10.1016/S0196-8858(02)00037-4.
21. T. Réti, On some properties of graph irregularity indices with a particular regard to the σ-index, *Appl. Math. Comput.*, **344–345** (2019), 107–115. doi: 10.1016/j.amc.2018.10.010.

22. O. Rojo, R. D. J. Alarcón, Line graph of combinations of generalized Bethe trees: Eigenvalues and energy, *Linear Algebra Appl.*, **435** (2011), 2402–2419. doi: 10.1016/j.laa.2010.10.008.

23. O. Rojo, M. Robbiano, An explicit formula for eigenvalues of Bethe trees and upper bounds on the largest eigenvalue of any tree, *Linear Algebra Appl.*, **427** (2007), 138–150. doi: /10.1016/j.laa.2007.06.024.

24. T. Réti, R. Sharafdini, H. Haghbin, Á. Drégelyi-Kiss, Graph irregularity indices used as molecular descriptors in QSPR studies, *MATCH Commun. Math. Comput. Chem.*, **79** (2018), 509–524.

25. M. Robbianoa, V. Trevisan, Applications of recurrence relations for the characteristic polynomials of Bethe trees, *Comput. Math. Appl.*, **59** (2010), 3039–3044. doi: 10.1016/j.camwa.2010.02.023.

26. L. A. Székely, L. H. Clark, R. C. Entringer, An inequality for degree sequences, *Discrete Math.*, **103** (1992), 293–300. doi: 10.1016/0012-365X(92)90321-6.

27. M. K. Siddiqui, M. Imran, M. A. Iqbal, Molecular descriptors of discrete dynamical system in fractal and Cayley tree type dendrimers, *J. Appl. Math. Comput.*, **61** (2019), 57–72. doi: 10.1007/s12190-019-01238-1.

28. T. Vetrík, M. Masre, General eccentric connectivity index of trees and unicyclic graphs, *Discrete Appl. Math.,* **284** (2020), 301–315. doi: 10.1016/j.dam.2020.03.051.

29. Y. Wu, F. Y. Wei, B. L. Liu, Z. Jia, The generalized (terminal) Wiener polarity index of generalized Bethe trees and coalescence of rooted trees, *MATCH Commun. Math. Comput. Chem.*, **70** (2013), 603–620.

30. K. X. Xu, K. C. Das, A. D. Maden, On a novel eccentricity-based invariant of a graph, *Acta Math. Sin.*, **32** (2016), 1477–1493. doi: 10.1007/s10114-016-5518-z.

31. B. Zhou, N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.*, **47** (2010), 210–218. doi: 10.1007/s10910-009-9542-4.

AIMS Press