Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8044–8063.
DOI:10.3934/math.2021467
Received: 27 March 2021
Accepted: 19 May 2021
Published: 24 May 2021

*Research article*

# Synchronization criteria for neutral-type quaternion-valued neural networks with mixed delays

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**Abstract:** In this paper, the problem of synchronization for neutral-type quaternion-valued neural networks (NQVNNs) with mixed delays is investigated. By making full use of the information of the time-delay state, a linear feedback controller and a novel nonlinear feedback controller are constructed to research the global synchronization and finite-time synchronization of the system respectively. In the case where the activation function of the network is not required to be separated into two complex parts or four real parts, the sufficient conditions of synchronization of NQVNNs are acquired based on establishing appropriate Lyapunov-Krasovskii functional, applying the synchronization method of drive-response and some inequality techniques. The obtained delay-dependent synchronization results are less conservative than some existing ones via numerical example comparisons. Two numerical examples with simulations are provided to verify the effectiveness of the obtained results.

**Keywords:** quaternion-valued neural networks; mixed delays; Lyapunov-Krasovskii functional; global synchronization; finite-time synchronization
**Mathematics Subject Classification:** 34D06, 62M45, 93C23

## 1. Introduction

In recent years, with the expansion of the number field, artificial neural networks mainly include real-valued neural networks (RVNNs), complex-valued neural networks (CVNNs) and quaternion-valued neural networks (QVNNs). RVNNs and CVNNs have been widely used in combinatorial optimization, pattern recognition and quantum communication. As is well-known, compared with RVNNs, the advantage of CVNNs is that they can directly deal with two-dimensional data [1–5]. However, we will inevitably encounter high-dimensional data in real life, such as color images, body images, wind direction prediction and so on [6–9]. Although we can transform the

QVNNs into two CVNNs or four RVNNs to address these problems, the decomposition method increases the dimension of the system, which brings difficulties to mathematical analysis [10–15]. Fortunately, quaternion, as an extension of complex number theory in a sense, plays great advantages in these aspects. Some authors have found that the use of quaternions can directly encode and process high-dimensional data, and greatly improve processing efficiency. At present, QVNNs have shown good application prospects in color image compression and satellite attitude control [16–20]. Therefore, it is necessary to study the QVNNs corresponding to quaternion field.

Moreover, in both artificial and biological neural systems, time delays are inevitable because of the finite transmission speed of signal output or input between two neurons. The existence of time delays in neural networks can affect stability and lead to complex dynamical behaviors such as instability, oscillations and chaos [21–25]. The neutral system is an important type of time-delay system. This kind of time-delay system is not only related to the past motion state, but also to the rate of change of the past motion state. Compared with the lag-type time-delay system, the neutral-type time-delay system can describe the dynamic law of things more accurately, so that the study of the system can provide good theoretical support for solving practical problems [26–31]. Most time-delay systems can be regarded as special cases of neutral systems. Many conclusions of neutral time-delay systems can be easily extended to standard time-delay systems. Therefore, the study of neutral time-delay systems has important theoretical value and practical significance.

Since Pecora and Carollt proposed the concept of drive-response synchronization in 1990, the synchronization of neural networks in real and complex fields has been extensively investigated. For example, in [32], the global synchronization of CVNNs without time delay is studied. In [33], the global $\mu$-synchronization of impulsive CVNNs with leakage and mixed time-varying delays is investigated by using Newton Leibniz formula, inequality technique and free weighted matrix method. In [34], the sufficient conditions for global synchronization of QVNNs with mixed delays are obtained by using drive-response synchronization and inequality techniques.In [35], the concept of the Filippov solution for IMQVNNs is introduced by applying differential inclusion theory, and the author designed a nonlinear feedback controller to discuss the global exponential synchronization for delayed inertial memristor-based quaternion-valued neural networks with impulses.

In addition, in some practical situations, synchronization should be achieved in finite time. So, it is necessary to make a study for finite-time synchronization. At present, some results have been achieved in the synchronization of neural networks in finite time. In [36], by dividing the system under investigation into four real-valued parts, the finite-time synchronization problem of quaternion neural networks is studied. In [37], the authors proposed the finite-time synchronization of neural networks with stochastic perturbation term and mixed time delays by using the Ito formula and p-norm. These studies have achieved good results, however, the research on the neutral-type quaternion-valued neural networks with mixed delays has few results.

Motivated by the issues discussed above, a class of neutral-type quaternion-valued neural networks with mixed delays is proposed in this paper. Different form the published papers about global synchronization of QVNNs, the research target of this paper is NQVNNs. Due to the existence of the neutral term, the study of this kind of model is more complicated and difficult than the usual delayed QVNN model, and the results obtained have a wider application range and more research value. The advantage of this paper is the establishment of a linear feedback controller and a novel nonlinear feedback controller, We construct a new Lyapunov–Krasovskii functional and a nonlinear feedback controller that employs the information of the time-delayed state. Based on the synchronization method of drive-response, the sufficient conditions of the global synchronization and finite-time synchronization are derived. In addition, The global synchronization criterion is expressed as a set of linear matrix inequalities, which can be readily tested by using the Matlab LMI toolbox.

The remaining part of the paper is organized as follows. In Section 2, some preliminaries are briefly given. Some sufficient conditions to ensure NQVNNs synchronization are obtained in Section 3. In Section 4, two examples are supplied to substantiate the validity of the obtained results. In last section, conclusion is given.

## 2. Preliminarise

### 2.1. Quaternion Algebra

A quaternion can be written in the form $x = x^r + x^i i + x^j j + x^k k \in Q$, where $x^r, x^i, x^j, x^k \in R$ are real, $i, j, k$ are imaginary unites and obey the following rules:

$$\begin{cases} ij = -ji = k \\ jk = -kj = i \\ ki = -ik = j \\ i^2 = k^2 = j^2 = ijk = -1 \end{cases}$$

From the rules it follows immediately that the quaternion multiplication is not commutative.

For a quaternion $x = x^r + x^i i + x^j j + x^k k \in Q$, then the conjugate of $x$ is defined as $\bar{x} = x^r - x^i i - x^j j - x^k k$, and the modulus of $x$ is defined as:

$$|x| = \sqrt{x\bar{x}} = \sqrt{(x^r)^2 + (x^i)^2 + (x^j)^2 + (x^k)^2}.$$

For another quaternion $y = y^r + y^i i + y^j j + y^k k$, the multiplication of $x$ and $y$ is defined as:

$$\begin{aligned} xy = yx = & (x^r y^r - x^i y^i - x^j y^j - x^k y^k) \\ & +(x^r y^i + x^i y^r + x^j y^k - x^k y^j)i \\ & +(x^r y^j + x^j y^r - x^i y^k + x^k y^i)j \\ & +(x^r y^k + x^k y^r + x^i y^j - x^j y^i)k \end{aligned}$$

Furthermore, for $b = (b_1, b_2, \cdots, b_n)^T \in Q^n$, the vector formed by the modulus of $b_i$ is denoted as $|b|$ and $|b| = (|b_1|, |b_2|, \cdots, |b_n|)^T \in R^n$. The norm of $b$ is defined as $\| b \| = \sqrt{\sum_{i=1}^n |b_i|^2}$. The notation $A^T$ and $A^*$ stand for the transpose and the conjugate transpose of the matrix $A$, respectively. In addition, the notation $A > B (A \geq B)$ means that $A - B$ is positive definite (positive semidefinite). For a positive definite Hermitian matrix $C \in Q^{n\times n}$, $\lambda_{max}(C)$ and $\lambda_{min}(C)$ are defined as the largest and the smallest eigenvalues of $C$, respectively.

### 2.2. Model formulation and basic lemmas

The following NQVNNs model with mixed time delays is considered in this article:

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau_1)) + H\dot{x}(t-\delta) + C\int_{t-\tau_2(t)}^{t} f(x(s))ds + J, \qquad t \geq 0. \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in Q^n$ is the state vector of the NQVNNs with n neurons at time $t$. $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T \in Q^n$ is the quaternion-valued activation function without time delays. $D = diag\{d_1, d_2, \cdots, d_n\} \in R^{n\times n}$ with $d_i > 0, i = 1, 2, \cdots, n$ is the self-feedback connection weight matrix. $A = (a_{ij})_{n\times n} \in Q^{n\times n}, B = (b_{ij})_{n\times n} \in Q^{n\times n}, C = (c_{ij})_{n\times n} \in Q^{n\times n}, H = (h_{ij})_{n\times n} \in Q^{n\times n}$ are the connection weight matrices, $J = [J_1 \quad J_2 \quad \cdots \quad J_n]^T \in Q^n$ is the external input vector; $\tau_1, \delta$, and $\tau_2(t)$ denotes the discrete delay, neutral delay and distributed time-varying delay respectively. Suppose there exist a positive constant $\tau_2$, let $0 \le \tau_2(t) \le \tau_2$.

The initial value of the NQVNNs (2.1) is:

$$x(s) = \varphi(s), s \in [-\tau, 0].$$

where $\varphi(s)$ is bounded and continuous in $[-\tau, 0]$, $\tau = max\{\tau_1, \delta, \tau_2\}$. $\varphi(s) = (\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s))^T \in Q^n$, The norm of $\varphi(s)$ is defined as

$$\| \varphi(s) \| = \sup_{s\in[-\tau,0]} \sqrt{\sum_{i=1}^{n} |\varphi_i(t)|^2}.$$

Consider system (2.1) as the drive system, then choose the response system as below:

$$\dot{y}(t) = -Dy(t) + Af(y(t)) + Bf(y(t-\tau_1)) + H\dot{y}(t-\delta) + C\int_{t-\tau_2(t)}^{t} f(y(s))ds + J + \mu(t), \quad t \ge 0. \tag{2.2}$$

Choose the initial state of system (2.2) as $y(s) = \psi(s), s \in [-\tau, 0]$, $\mu(t)$ is the appropriate controller to be designed. Then let $m(t) = (m_1(t), m_2(t), \cdots, m_n(t))$ be the synchronization error, where $m(t) = x(t) - y(t), g(m(t)) = f(x(t)) - f(y(t))$. Now, the state feedback controller is designed as $\mu(t) = -K_1 m(t)$, where $K_1 \in Q^{n\times n}$ is control gain matrix. Thus, the following error system is achieved:

$$\dot{m}(t) = (K_1 - D)m(t) + Ag(m(t)) + Bg(m(t-\tau_1)) + H\dot{m}(t-\delta) + C\int_{t-\tau_2(t)}^{t} g(m(s))ds, \quad t \ge 0. \tag{2.3}$$

with initial condition $m(s) = \phi(s) = \varphi(s) - \psi(s), s \in [-\tau, 0]$.

Throughout this paper, we assume that the activation functions satisfy the following condition.

**Assumption 1.** For $i = 1, 2, \cdots, n$, the neuron activation function $f_i$ is continuous and satisfies

$$|f_i(\alpha_1) - f_i(\alpha_2)| \le \gamma_i |\alpha_1 - \alpha_2|, \quad \forall \alpha_1, \alpha_2 \in Q.$$

where $\gamma_i$ is a real constant. Moreover, define $\Gamma = diag\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$.

**Definition 1.** The NQVNNs defined by (2.3) are globally stable if there exists a scalar $M > 0$ for any solution of system (2.3) satisfies

$$\| m(t) \| \le M \sup_{s\in[-\tau,0]} \| \varphi(s) - \psi(s) \| .$$

**Definition 2.** The drive system (2.1) and the response system (2.2) are global synchronous if the error system (2.3) is globally stable.

**Definition 3.** The drive system (2.1) and the response system (2.2) are said to be synchronization in finite time if there exists $T > 0$ such that

$$\lim_{t\to T} m(t) = 0, \quad m(t) = 0, \quad \forall t > T.$$

**Lemma 1.** [38] For any positive definite constant Hermitian matrix $W \in Q^{n\times n}, W > 0$ and any scalar function $\omega : [a, b] \to Q^n$ with scalars $a < b$ such that the integrations concerned are well defined,

$$(\int_a^b \omega(s)ds)^* W(\int_a^b \omega(s)ds) \leq (b-a)\int_a^b \omega^*(s)W\omega(s)ds.$$

**Lemma 2.** For any $a, b \in Q^n$, if $P \in Q^{n\times n}$ is a positive definite Hermitian matrix, then

$$a^*b + b^*a \leq a^*Pa + b^*P^{-1}b.$$

**Lemma 3.** Assume that a continuous, positive definite function $V(t)$ satisfies the following differential inequality:

$$\dot{V}(t) \leq -\lambda V^{\alpha}(t), \forall t \geq t_0, V(t_0) \geq 0,$$

where $\lambda > 0, 0 < \alpha < 1$ are all constants. Then, for any given $t_0$, $V(t)$ satisfies the following inequality:

$$V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0) - \lambda(1-\alpha)(t-t_0), t_0 \leq t \leq T,$$

and $V(t) \equiv 0, \forall t \geq T$, with $T$ given by

$$T = t_0 + \frac{V^{1-\alpha}(t_0)}{\lambda(1-\alpha)}.$$

## 3. Main results

**Theorem 1.** Under Assumptions 1, NQVNNs (2.1) and NQVNNs (2.2) are global synchronous, if there exist four positive definite Hermitian matrices $P_1, P_2, P_3, P_4 \in Q^{n\times n}$, and a positive diagonal matrix $Q \in R^{n\times n}$, and three commutative quaternion matrices $S_1, S_2, S_3$, such that the following LMI holds:

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & S_1A & S_1B & S_1C \\ * & \Omega_{22} & \Omega_{23} & S_2A & S_2B & S_2C \\ * & * & \Omega_{33} & S_3A & S_3B & S_3C \\ * & * & * & P_2 + \tau^2 P_4 - Q & 0 & 0 \\ * & * & * & * & -P_2 & 0 \\ * & * & * & * & * & -P_4 \end{bmatrix} < 0 \tag{3.1}$$

where

$$\begin{aligned} \Omega_{11} &= U_1^* + U_1 + \Gamma Q \Gamma - D^*S_1^* - S_1D, \\ \Omega_{12} &= P_1 + U_2^* - D^*S_2^* - S_1, \\ \Omega_{13} &= U_3^* - D^*S_3^* + S_1H, \\ \Omega_{22} &= P_3 - S_2 - S_2^*, \Omega_{23} = -S_3^* + S_2H, \\ \Omega_{33} &= -P_3 + H^*S_3^* + S_3H. \end{aligned}$$

*Proof.* Choose the Lyapunov–Krasovskii functional as following

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{3.2}$$

where

$$\begin{aligned} V_1(t) &= m^*(t)P_1m(t), \\ V_2(t) &= \int_{t-\tau_1}^t g^*(m(s))P_2g(m(s))ds, \\ V_3(t) &= \int_{t-\delta}^t \dot{m}^*(s)P_3\dot{m}(s)ds, \\ V_4(t) &= \tau_2 \int_{-\tau_2}^0 \int_{t+s}^t g^*(m(\theta))P_4g(m(\theta))d\theta ds. \end{aligned}$$

Calculating the derivative of $V_1(t)$ along trajectory of (2.3) and by using Lemma 1, we have

$$\dot{V}_1(t) = \dot{m}^*(t)P_1m(t) + m^*(t)P_1\dot{m}(t) \tag{3.3}$$

$$\dot{V}_2(t) = g^*(m(t))P_2g(m(t)) - g^*(m(t-\tau_1))P_2g(m(t-\tau_1)) \tag{3.4}$$

$$\dot{V}_3(t) = \dot{m}^*(t)P_3\dot{m}(t) - \dot{m}^*(t-\delta)P_3\dot{m}(t-\delta) \tag{3.5}$$

$$\begin{aligned}\dot{V}_4(t) =&\tau_2\int_{-\tau_2}^{0}[g^*(m(t))P_4g(m(t)) - g^*(m(t+s))P_4g(m(t+s))]ds\\ =&\tau_2^2g^*(m(t))P_4g(m(t)) - \tau_2\int_{-\tau_2}^{0}[g^*(m(t+s))P_4g(m(t+s))]ds\\ =&\tau_2^2g^*(m(t))P_4g(m(t)) - \tau_2\int_{t-\tau_2}^{t}g^*(m(s))P_4g(m(s))ds\\ \le&\tau^2g^*(m(t))P_4g(m(t)) - \tau_2(t)\int_{t-\tau_2(t)}^{t}g^*(m(s))P_4g(m(s))ds\\ \le&g^*(m(t))(\tau^2P_4)g(m(t)) + (\int_{t-\tau_2(t)}^{t}g(m(s))ds)^*(-P_4)(\int_{t-\tau_2(t)}^{t}g(m(s))ds)\end{aligned} \tag{3.6}$$

combining with (3.3), (3.4), (3.5), and (3.6), thus, we can infer that

$$\begin{aligned}\dot{V}(t) \le&\dot{m}^*(t)P_1m(t) + m^*(t)P_1\dot{m}(t) + \dot{m}^*(t)P_3\dot{m}(t) + g^*(m(t))(P_2 + \tau^2P_4)g(m(t))\\ &- g^*(m(t-\tau_1))(P_2)g(m(t-\tau_1)) - \dot{m}^*(t-\delta)P_3\dot{m}(t-\delta)\\ &+ (\int_{t-\tau_2(t)}^{t}g(m(s))ds)^*(-P_4)(\int_{t-\tau_2(t)}^{t}g(m(s))ds)\end{aligned} \tag{3.7}$$

under Assumption 1, we get that

$$0 \le m^*(t)\Gamma Q\Gamma m(t) - g^*(m(t))Qg(m(t)) \tag{3.8}$$

From system (2.3), it is obvious that

$$0 = E^*[S_1^*m(t) + S_2^*\dot{m}(t) + S_3^*\dot{m}(t-\delta)] + [S_1^*m(t) + S_2^*\dot{m}(t) + S_3^*\dot{m}(t-\delta)]^*E \tag{3.9}$$

where

$$E = -\dot{m}(t) + (K_1 - D)m(t) + Ag(m(t)) + Bg(m(t-\tau_1)) + H\dot{m}(t-\delta) + C\int_{t-\tau_2(t)}^{t}g(m(s))ds$$

we can obtain

$$\begin{aligned}0 =&\dot{m}^*(t)(-S_1^*)m(t) + \dot{m}^*(t)(-S_2^*)\dot{m}(t) + \dot{m}^*(t)(-S_3^*)\dot{m}(t-\delta) + m^*(t)((K_1-D)^*S_1^*)m(t)\\ &+ m^*(t)((K_1-D)^*S_2^*)\dot{m}(t) + m^*(t)((K_1-D)^*S_3^*)\dot{m}(t-\delta) + g^*(m(t))(A^*S_1^*)m(t)\\ &+ g^*(m(t))(A^*S_2^*)\dot{m}(t) + g^*(m(t))(A^*S_3^*)\dot{m}(t-\delta) + g^*(m(t-\tau_1))(B^*S_1^*)m(t)\\ &+ g^*(m(t-\tau_1))(B^*S_2^*)\dot{m}(t) + g^*(m(t-\tau_1))(B^*S_3^*)\dot{m}(t-\delta) + \dot{m}^*(t-\delta)(H^*S_1^*)m(t)\\ &+ \dot{m}^*(t-\delta)(H^*S_2^*)\dot{m}(t) + \dot{m}^*(t-\delta)(H^*S_3^*)\dot{m}(t-\delta) + \int_{t-\tau_2(t)}^{t}g^*(m(s))ds(C^*S_1^*)m(t)\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-\tau_2(t)}^{t} g^*(m(s))ds(C^*S_2^*)\dot{m}(t)+\int_{t-\tau_2(t)}^{t} g^*(m(s))ds(C^*S_3^*)\dot{m}(t-\delta)\\
&+m^*(t)(-S_1)\dot{m}(t)+m^*(t)(S_1(K_1-D))m(t)+m^*(t)(S_1A)g(m(t))\\
&+m^*(t)(S_1B)g(m(t-\tau_1))+m^*(t)(S_1H)\dot{m}(t-\delta)+m^*(t)(S_1C)\int_{t-\tau_2(t)}^{t} g(m(s))ds\\
&+\dot{m}^*(t)(-S_2)\dot{m}(t)+\dot{m}^*(t)(S_2(K_1-D))m(t)+\dot{m}^*(t)(S_2A)g(m(t))\\
&+\dot{m}^*(t)(S_2B)g(m(t-\tau_1))+\dot{m}^*(t)(S_2H)\dot{m}(t-\delta)+\dot{m}^*(t)(S_2C)\int_{t-\tau_2(t)}^{t} g(m(s))ds\\
&+\dot{m}^*(t-\delta)(-S_3)\dot{m}(t)+\dot{m}^*(t-\delta)(S_3(K_1-D))m(t)+\dot{m}^*(t-\delta)(S_3A)g(m(t))\\
&+\dot{m}^*(t-\delta)(S_3B)g(m(t-\tau_1))+\dot{m}^*(t-\delta)(S_3H)\dot{m}(t-\delta)+\dot{m}^*(t-\delta)(S_3C)\int_{t-\tau_2(t)}^{t} g(m(s))ds\\
=&\dot{m}^*(t)(-S_1^*+S_2(K_1-D))m(t)+\dot{m}^*(t)(-S_2^*-S_2)\dot{m}(t)+\dot{m}^*(t)(-S_3^*+S_2H)\dot{m}(t-\delta)\\
&+\dot{m}^*(t)(S_2A)g(m(t))+\dot{m}^*(t)(S_2B)g(m(t-\tau_1))+\dot{m}^*(t)(S_2C)\int_{t-\tau_2(t)}^{t} g(m(s))ds\\
&+m^*(t)((K_1-D)^*S_1^*+S_1(K_1-D))m(t)+m^*(t)((K_1-D)^*S_2^*-S_1)\dot{m}(t)\\
&+m^*(t)((K_1-D)^*S_3^*+S_1H)\dot{m}(t-\delta)+m^*(t)(S_1A)g(m(t))\\
&+m^*(t)(S_1B)g(m(t-\tau_1))+m^*(t)(S_1C)\int_{t-\tau_2(t)}^{t} g(m(s))ds+g^*(m(t))(A^*S_1^*)m(t)\\
&+g^*(m(t))(A^*S_2^*)\dot{m}(t)+g^*(m(t))(A^*S_3^*)\dot{m}(t-\delta)+g^*(m(t-\tau_1))(B^*S_1^*)m(t)\\
&+g^*(m(t-\tau_1))(B^*S_2^*)\dot{m}(t)+g^*(m(t-\tau_1))(B^*S_3^*)\dot{m}(t-\delta)\\
&+\dot{m}^*(t-\delta)(H^*S_1^*+S_3(K_1-D))m(t)+\dot{m}^*(t-\delta)(H^*S_2^*-S_3)\dot{m}(t)\\
&+\dot{m}^*(t-\delta)(H^*S_3^*+S_3H)\dot{m}(t-\delta)+\int_{t-\tau_2(t)}^{t} g^*(m(s))ds(C^*S_1^*)m(t)\\
&+\int_{t-\tau_2(t)}^{t} g^*(m(s))ds(C^*S_2^*)\dot{m}(t)+\int_{t-\tau_2(t)}^{t} g^*(m(s))ds(C^*S_3^*)\dot{m}(t-\delta)\\
&+\dot{m}^*(t-\delta)(S_3A)g(m(t))+\dot{m}^*(t-\delta)(S_3B)g(m(t-\tau_1))\\
&+\dot{m}^*(t-\delta)(S_3C)\int_{t-\tau_2(t)}^{t} g(m(s))ds
\end{aligned}
\tag{3.10}
$$

It flows from equalities or inequalities (3.7), (3.8) and (3.10) that

$$
\dot{V}(t)\le G^*(t)\Omega G(t), \tag{3.11}
$$

where

$$
G(t)=[m(t)\quad \dot{m}(t)\quad \dot{m}^*(t-\delta)\quad g(m(t))\quad g(m(t-\tau_1))\quad \textstyle\int_{t-\tau_2(t)}^{t} g(m(s))ds]^T
$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & S_1A & S_1B & S_1C \\ * & \Omega_{22} & \Omega_{23} & S_2A & S_2B & S_2C \\ * & * & \Omega_{33} & S_3A & S_3B & S_3C \\ * & * & * & P_2+\tau^2P_4-Q & 0 & 0 \\ * & * & * & * & -P_2 & 0 \\ * & * & * & * & * & -P_4 \end{bmatrix}$$

let $U_1 = S_1K_1, U_2 = S_2K_1, U_3 = S_3K_1$,

$$\begin{aligned} \Omega_{11} &= U_1^* + U_1 + \Gamma Q\Gamma - D^*S_1^* - S_1D, \\ \Omega_{12} &= P_1 + U_2^* - D^*S_2^* - S_1, \\ \Omega_{13} &= U_3^* - D^*S_3^* + S_1H, \\ \Omega_{22} &= P_3 - S_2 - S_2^*, \Omega_{23} = -S_3^* + S_2H, \\ \Omega_{33} &= -P_3 + H^*S_3^* + S_3H. \end{aligned}$$

thus, we get from (3.1) and (3.11) that

$$\dot{V}(t) \leq 0, t \geq 0. \tag{3.12}$$

From (3.12), we know that

$$\begin{aligned} V(t) \leq V(0) =& m^*(0)P_1m(0) + \int_{-\tau_1}^{0} g^*(m(s))P_2g(m(s))ds + \int_{-\delta}^{0} \dot{m}^*(s)P_3\dot{m}(s)ds \\ &+ \tau\int_{-\tau}^{0}\int_{s}^{0} g^*(m(\theta))P_4g(m(\theta))ds \\ \leq&(\| P_1 \| +\tau_1 \| P_2 \| \max_{1\leq i\leq n}\{\gamma_i^2\} + \delta \| P_3 \| +\tau^3 \| P_4 \| \max_{1\leq i\leq n}\{\gamma_i^2\})(\sup_{s\in[-\tau,0]} \| \phi(s) \|)^2 \\ \leq& M_1(\sup_{s\in[-\tau,0]} \| \phi(s) \|)^2, \end{aligned}$$

where

$$M_1 =\| P_1 \| +\tau_1 \| P_2 \| \max_{1\leq i\leq n}\{\gamma_i^2\} + \delta \| P_3 \| +\tau^3 \| P_4 \| \max_{1\leq i\leq n}\{\gamma_i^2\},$$

Obviously,

$$V(t) \geq V_1(t) \geq \lambda_{max}(P_1) \| m(t) \|^2,$$

we get that

$$\| m(t) \| \leq \sqrt{\frac{V(t)}{\lambda_{max}(P_1)}} = \sqrt{\frac{M_1}{\lambda_{max}(P_1)}} \sup_{s\in[-\tau,0]} \| \phi(s) \|, \quad t \geq 0$$

thus

$$\| m(t) \| \leq M \sup_{s\in[-\tau,0]} \| \phi(s) \|, \quad t \geq 0$$

where

$$M = \sqrt{\frac{M_1}{\lambda_{max}(P_1)}}.$$

Therefore, by Definition 1 and Definition 2, the NQVNNs (2.3) is globally stable and the drive system (2.1) and the response system (2.2) are global synchronous.

**Theorem 2.** Suppose Assumption 1 holds, then the drive-response systems (2.1) and (2.2) can reach finite-time synchronization under the controller

$$\begin{aligned}\mu(t) =&K_2 m(t) + \lambda sign(m(t))|m(t)|^{\alpha} + \lambda\Big(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big)\\ &+ \lambda\Big(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big)\\ &+ \lambda\Big(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big).\end{aligned} \tag{3.13}$$

where $\lambda > 0, 0 < \alpha < 1$. Thus, the following error system is achieved:

$$\begin{aligned}\dot{m}(t) =& - Dm(t) + Ag(m(t)) + Bg(m(t-\tau_1)) + H\dot{m}(t-\delta) + C\int_{t-\tau_2(t)}^{t} g(m(s))ds\\ &- K_2 m(t) - \lambda sign(m(t))|m(t)|^{\alpha} - \lambda\Big(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big)\\ &- \lambda\Big(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big)\\ &- \lambda\Big(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds\Big)^{\frac{1+\alpha}{2}}\Big(\frac{m(t)}{\| m(t) \|^2}\Big).\end{aligned} \tag{3.14}$$

If there exist four positive definite Hermitian matrices $R_1, R_2, R_3,$
$R_4 \in Q^{n\times n}$, such that the following conditions holds:

(i) $-2D - K_2^* - K_2 + 2I + AR_1A^* + \Gamma_1(\tau^2 R_1^{-1})\Gamma_1 + BR_2B^* + HR_3H^* + CR_4C^* \leq 0$;

(ii) $\Gamma_2(\tau^2 R_2^{-1})\Gamma_2 - I \leq 0$;

(iii) $R_3^{-1} - I \leq 0$;

(iv) $R_4^{-1} - I \leq 0$.

Then, the synchronization error system $m(t)$ is globally finite-time stability, and the synchronization between system (2.1) and system (2.2) can be obtained in a finite time, and the finite time is estimated by

$$T = t_0 + \frac{V^{1-\frac{1+\alpha}{2}}(t_0)}{\lambda(1 - \frac{1+\alpha}{2})}.$$

*Proof.* Consider the following Lyapunov functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{3.15}$$

where

$$V_1(t) = m^*(t)Im(t),$$
$$V_2(t) = \int_{t-\tau_1}^{t} m^*(s)Im(s)ds,$$
$$V_3(t) = \int_{t-\delta}^{0} \dot{m}^*(s)I\dot{m}(s)ds,$$
$$V_4(t) = \tau_2 \int_{-\tau_2}^{0} \int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds.$$

Calculating the derivative of $V_1(t)$ along trajectory of (3.14) and by using Lemma 1, we have

$$\dot{V}_1(t) = \dot{m}^*(t)Im(t) + m^*(t)I\dot{m}(t) \tag{3.16}$$

$$\dot{V}_2(t) = m^*(t)Im(t) - m^*(t-\tau_1)Im(t-\tau_1) \tag{3.17}$$

$$\dot{V}_3(t) = \dot{m}^*(0)I\dot{m}(0) - \dot{m}^*(t-\delta)I\dot{m}(t-\delta) \tag{3.18}$$

$$\begin{aligned}\dot{V}_4(t) =&\tau_2 \int_{-\tau_2}^{0} [g^*(m(t))Ig(m(t)) - g^*(m(t+s))Ig(m(t+s))]ds \\ =&\tau_2^2 g^*(m(t))Ig(m(t)) - \tau_2 \int_{-\tau_2}^{0} [g^*(m(t+s))Ig(m(t+s))]ds \\ =&\tau_2^2 g^*(m(t))Ig(m(t)) - \tau_2 \int_{t-\tau_2}^{t} g^*(m(s))Ig(m(s))ds \\ \leq&\tau^2 g^*(m(t))Ig(m(t)) - \tau_2(t) \int_{t-\tau_2(t)}^{t} g^*(m(s))Ig(m(s))ds \\ \leq& g^*(m(t))(\tau^2 I)g(m(t)) + (\int_{t-\tau_2(t)}^{t} g(m(s))ds)^*(-I)(\int_{t-\tau_2(t)}^{t} g(m(s))ds)\end{aligned} \tag{3.19}$$

combining with (3.16)—(3.19), thus, we can infer that

$$\begin{aligned}\dot{V}(t) \leq& \dot{m}^*(t)Im(t) + m^*(t)I\dot{m}(t) + m^*(t)Im(t) - m^*(t-\tau_1)Im(t-\tau_1) \\ &+ \dot{m}^*(0)I\dot{m}(0) - \dot{m}^*(t-\delta)I\dot{m}(t-\delta) + g^*(m(t))(\tau^2 I)g(m(t)) \\ &+ (\int_{t-\tau_2(t)}^{t} g(m(s))ds)^*(-I)(\int_{t-\tau_2(t)}^{t} g(m(s))ds)\end{aligned} \tag{3.20}$$

From system (3.14), it is obvious that

$$\begin{aligned}\dot{V}(t) \leq& m^*(t)(-D)m(t) + g^*(m(t))A^*m(t) + g^*(m(t-\tau_1))B^*m(t) + \dot{m}^*(t-\delta)H^*m(t) \\ &+ (\int_{t-\tau_2(t)}^{t} g^*(m(s))ds)C^*m(t) + m^*(t)(-K_2^*)m(t) - \lambda[sign(m(t))|m(t)|^{\alpha}]^*m(t) \\ &- \lambda(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2}) - \lambda(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2}) \\ &- \lambda(\tau_2 \int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2}) + m^*(t)(-D)m(t) \\ &+ m^*(t)Ag(m(t)) + m^*(t)Bg(m(t-\tau_1)) + m^*(t)H\dot{m}(t-\delta) \\ &+ m^*(t)C(\int_{t-\tau_2(t)}^{t} g(m(s))ds) + m^*(t)(-K_2)m(t) - \lambda m^*(t)[sign(m(t))|m(t)|^{\alpha}]\end{aligned}$$

$$-\lambda(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2}) - \lambda(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2})$$
$$-\lambda(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds)^{\frac{1+\alpha}{2}}(\frac{m^*(t)m(t)}{\| m(t) \|^2})$$
$$+\dot{m}^*(0)I\dot{m}(0) - \dot{m}^*(t-\delta)I\dot{m}(t-\delta) + g^*(m(t))(\tau^2 I)g(m(t))$$
$$+(\int_{t-\tau_2(t)}^{t} g(m(s))ds)^*(-I)(\int_{t-\tau_2(t)}^{t} g(m(s))ds) \tag{3.21}$$

From Assumption 1 and lemma 2, we know that

$$\begin{aligned} g^*(m(t))A^*m(t) + m^*(t)Ag(m(t)) &\le m^*(t)AR_1A^*m(t) + g^*(m(t))R_1^{-1}g(m(t)) \\ &\le m^*(t)AR_1A^*m(t) + m^*(t)\Gamma_1 R_1^{-1}\Gamma_1 m(t) \end{aligned} \tag{3.22}$$

$$\begin{aligned} g^*(m(t-\tau_1))B^*m(t) + m^*(t)Bg(m(t-\tau_1)) &\le m^*(t)BR_2B^*m(t) + g^*(m(t-\tau_1))R_2^{-1}g(m(t-\tau_1)) \\ &\le m^*(t)BR_2B^*m(t) + m^*(t-\tau_1)\Gamma_2 R_2^{-1}\Gamma_2 m(t-\tau_1) \end{aligned} \tag{3.23}$$

$$\dot{m}^*(t-\delta)H^*m(t) + m^*(t)H\dot{m}(t-\delta) \le m^*(t)HR_3H^*m(t) + \dot{m}^*(t-\delta)R_3^{-1}\dot{m}(t-\delta) \tag{3.24}$$

Also, we are able to get the following inequality:

$$\begin{aligned} -\lambda[sign(m(t))|m(t)|^\alpha]^*m(t) - \lambda m^*(t)[sign(m(t))|m(t)|^\alpha] &\le -2\lambda|m(t)|^*|m(t)|^\alpha \\ &\le -2\lambda|m^*(t)m(t)|^{\frac{1+\alpha}{2}} \end{aligned} \tag{3.25}$$

apply (3.22)—(3.25) to (3.21), thus, we can infer that

$$\begin{aligned} \dot{V}(t) \le\, & m^*(t)[-2D - K_2^* - K_2 + 2I + AR_1A^* + \Gamma_1(\tau^2R_1^{-1})\Gamma_1 + BR_2B^* \\ & + HR_3H^* + CR_4C^*]m(t) + m^*(t-\tau_1)[\Gamma_2(\tau^2R_2^{-1})\Gamma_2 - I]m(t-\tau_1) \\ & + \dot{m}^*(t-\delta)[R_3^{-1} - I]\dot{m}(t-\delta) + \int_{t-\tau_2(t)}^{t} g^*(m(s))ds[R_4^{-1} - I]\int_{t-\tau_2(t)}^{t} g(m(s))ds \\ & - 2\lambda|m^*(t)m(t)|^{\frac{1+\alpha}{2}} - 2\lambda(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds)^{\frac{1+\alpha}{2}} - 2\lambda(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds)^{\frac{1+\alpha}{2}} \\ & - 2\lambda(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds)^{\frac{1+\alpha}{2}} \end{aligned} \tag{3.26}$$

according to the conditions (i)—(iv) of Theorem 2, we can get

$$\begin{aligned} \dot{V}(t) \le & -2\lambda|m^*(t)Im(t)|^{\frac{1+\alpha}{2}} - 2\lambda(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds)^{\frac{1+\alpha}{2}} - 2\lambda(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds)^{\frac{1+\alpha}{2}} \\ & - 2\lambda(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds)^{\frac{1+\alpha}{2}} \\ = & -2\lambda V^{\frac{1+\alpha}{2}}(t) \end{aligned} \tag{3.27}$$

Therefore, we have

$$V^{1-\frac{1+\alpha}{2}}(t) \le V^{1-\frac{1+\alpha}{2}}(t_0) - \lambda(\frac{1+\alpha}{2})(t-t_0), \tag{3.28}$$

and $V(t) = 0, \forall t \ge T$ with $t_1$ given by

$$T = t_0 + \frac{V^{1-\frac{1+\alpha}{2}}(t_0)}{\lambda(1-\frac{1+\alpha}{2})}.$$

Therefore, the synchronization error system $m(t)$ is globally finite-time stability, and the synchronization between system (2.1) and system (2.2) can be obtained in a finite time, and the finite time is estimated by

$$T = t_0 + \frac{V^{1-\frac{1+\alpha}{2}}(t_0)}{\lambda(1-\frac{1+\alpha}{2})}.$$

**Remark 1.** In Refs. [31, 34, 36], the global synchronization or finite-time synchronization of several quaternion-valued neural networks under the controller is studied respectively. Refs. [34, 36] do not consider the effect of neutral delay on synchronization. Although Ref. [31] considered the above factors, it did not consider the distributed time-varying delay. Therefore, in this sense, the models studied in Refs. [31, 34, 36] are all special cases of this paper. The models studied in this paper are general and more in line with the actual situation.
**Remark 2.** The synchronization method of drive-response is widely used in the study of neural network synchronization. The sufficient conditions for system synchronization are obtained by constructing a suitable Lyapunov functional. The difference between this paper and the existing results is that most of the other literatures use linear controller to study the synchronization. In this paper, a novel nonlinear feedback controller is constructed to research the finite-time synchronization of the system, and good synchronization results are obtained, which provides more possibilities for the study of such issues and is more in line with the actual situation.

## 4. Simulation example

In this section, the validity of the obtained outcomes is confirmed by two example.
**Example 1.** Consider the following two-neuron NQVNNs with mixed delays:

$$\dot{m}(t) =(K_1 - D)m(t) + Ag(m(t)) + Bg(m(t-\tau_1)) + H\dot{m}(t-\delta) + C\int_{t-\tau_2(t)}^{t} g(m(s))ds, \quad t \geq 0 \tag{4.1}$$

The parameters are shown in Ref. [34], where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \ C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \ D = \begin{bmatrix} 3.6910 & 0 \\ 0 & 1.2339 \end{bmatrix}, \ J = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$a_{11} = -1.2171 + 1.8709i - 0.4868j + 0.7484k, a_{12} = 0.4057 + 0.8205i + 0.1623j + 0.3282k,$
$a_{21} = -0.4057 + 1.8338i - 0.1623j + 0.7335k, a_{22} = 1.6228 + 1.7873i + 0.6491j + 0.7149k;$
$b_{11} = -0.0347 + 0.7057i - 0.0278j + 0.5646k, b_{12} = 0.0695 + 0.0197i + 0.0556j + 0.0158k,$
$b_{21} = 0.0232 + 1.6263i + 0.0185j + 1.3011k, b_{22} = -0.0463 + 0.2778i - 0.0371j + 0.2222k;$
$c_{11} = -0.2468 + 0.3794i - 0.0987j + 0.1517k, c_{12} = 0.0823 + 0.1664i + 0.0329j + 0.0666k,$
$c_{21} = -0.0823 + 0.3718i - 0.0329j + 0.1487k, c_{22} = 0.3291 + 0.3624i + 0.1316j + 0.1450k;$
$h_{11} = 0.89 + 0.13i - 0.84j - 0.19k, h_{12} = -0.95 + 0.25i + 0.24j - 0.21k,$
$h_{21} = -0.54 + 0.84i - 0.92j - 0.61k, h_{22} = 0.13 - 0.25i - 0.35j + 0.47k.$

Define $g_1(m(t)) = g_2(m(t)) = 0.2tanh(m(t)), \forall m(t) \in Q^n$, $\delta = 0.2, \tau_1 = 0.2, \tau_2(t) = 0.1|sin(7t)|$. The following feasible solutions to the LMIs (3.1) in Theorem 1 are found in MATLAB:

$$P_1 = 10^{-10}\begin{bmatrix} P_1^{11} & P_1^{12} \\ P_1^{21} & P_1^{22} \end{bmatrix},\ P_2 = 10^{-11}\begin{bmatrix} P_2^{11} & P_2^{12} \\ P_2^{21} & P_2^{22} \end{bmatrix},\ P_3 = 10^{-11}\begin{bmatrix} P_3^{11} & P_3^{12} \\ P_3^{21} & P_3^{22} \end{bmatrix},$$

$$P_4 = 10^{-11}\begin{bmatrix} P_4^{11} & P_4^{12} \\ P_4^{21} & P_4^{22} \end{bmatrix},\ S_1 = 10^{-10}\begin{bmatrix} S_1^{11} & S_1^{12} \\ S_1^{21} & S_1^{22} \end{bmatrix},\ S_2 = 10^{-11}\begin{bmatrix} S_2^{11} & S_2^{12} \\ S_2^{21} & S_2^{22} \end{bmatrix},$$

$$S_3 = 10^{-11}\begin{bmatrix} S_3^{11} & S_3^{12} \\ S_3^{21} & S_3^{22} \end{bmatrix},\ U_1 = 10^{-10}\begin{bmatrix} U_1^{11} & U_1^{12} \\ U_1^{21} & U_1^{22} \end{bmatrix},\ U_2 = 10^{-10}\begin{bmatrix} U_2^{11} & U_2^{12} \\ U_2^{21} & U_2^{22} \end{bmatrix},$$

$$U_3 = 10^{-10}\begin{bmatrix} U_3^{11} & U_3^{12} \\ U_3^{21} & U_3^{22} \end{bmatrix},\ Q_1 = 10^{-8}\begin{bmatrix} 0.2306 & 0 \\ 0 & 0.0838 \end{bmatrix}.$$

$P_1^{11} = 0.1362 + 0.0000i + 0.3300j + 0.0000k, P_1^{12} = -0.0017 + 0.0113i - 0.0912j + 0.0058k,$
$P_1^{21} = -0.0017 - 0.0113i - 0.0912j - 0.0058k,, P_1^{22} = 0.0615 + 0.0000i + 0.0805j + 0.0000k;$
$P_2^{11} = -0.6439 + 0.0000i - 0.8784j + 0.0000k, P_2^{12} = -0.0993 + 0.0441i + 0.1171j - 0.1908k,$
$P_2^{21} = -0.0993 - 0.0441i + 0.1171j + 0.1908k, P_2^{22} = -0.1377 + 0.0000i - 0.5447j + 0.0000k;$
$P_3^{11} = -0.7756 + 0.0000i + 0.7665j + 0.0000k, P_3^{12} = -0.1074 + 0.0408i - 0.1548j - 0.1418k,$
$P_3^{21} = -0.1074 - 0.0408i - 0.1548j + 0.1418k, P_3^{22} = -0.7217 + 0.0000i + 0.9485j + 0.0000k;$
$P_4^{11} = -0.2449 + 0.0000i + 25.960j + 0.0000k, P_4^{12} = -0.0363 + 0.0521i + 0.5400j - 5.4600k,$
$P_4^{21} = -0.0363 - 0.0521i + 0.5400j + 5.4600k, P_4^{22} = -0.0581 + 0.0000i + 37.940j + 0.0000k;$
$S_1^{11} = -0.1632 + 0.0000i + 0.3570j + 0.0000k, S_1^{12} = 0.0170 - 0.0030i - 0.1397j - 0.0297k,$
$S_1^{21} = 0.0170 + 0.0030i - 0.1397j + 0.0297k, S_1^{22} = -0.0582 + 0.0000i + 0.0998j + 0.0000k;$
$S_2^{11} = 0.1047 + 0.0000i + 1.2660j + 0.0000k, S_2^{12} = -0.1236 - 0.1157i - 0.7080j - 0.1350k,$
$S_2^{21} = -0.1236 + 0.1157i - 0.7080j + 0.1350k, S_2^{22} = 0.1804 + 0.0000i + 0.4660j + 0.0000k;$
$S_3^{11} = -0.3870 + 0.0000i + 2.4170j + 0.0000k, S_3^{12} = -0.3367 + 0.0289i - 1.2250j - 0.2970k,$
$S_3^{21} = -0.3367 - 0.0289i - 1.2250j + 0.2970k, S_3^{22} = 0.0026 + 0.0000i + 0.8710j + 0.0000k;$
$U_1^{11} = -0.0357 + 0.0000i + 0.2349j + 0.0000k, U_1^{12} = -0.0221 + 0.0042i - 0.0678j - 0.0032k,$
$U_1^{21} = -0.0221 - 0.0042i - 0.0678j + 0.0032k, U_1^{22} = -0.1349 + 0.0000i - 0.0707j + 0.0000k;$
$U_2^{11} = 0.1362 + 0.0000i + 0.3300j + 0.0000k, U_2^{12} = -0.0017 + 0.0113i - 0.0912j + 0.0058k,$
$U_2^{21} = -0.0017 - 0.0113i - 0.0912j - 0.0058k, U_2^{22} = 0.0615 + 0.0000i + 0.0805j + 0.0000k;$
$U_3^{11} = 0.2932 + 0.0000i + 1.1120j + 0.0000k, U_3^{12} = 0.0025 + 0.0417i - 0.2810j - 0.0320k,$
$U_3^{21} = 0.0025 - 0.0417i - 0.2810j + 0.0320k, U_3^{22} = 0.1123 + 0.0000i + 0.2560j + 0.0000k.$

Therefore, the controller coefficient $K_1$ can be obtained as follow:

$$K_1 = \begin{bmatrix} K_1^{11} & K_1^{12} \\ K_1^{21} & K_1^{22} \end{bmatrix}$$

$K_1^{11} = 0.6299 + 0.0360i - 0.2343j + 0.0550k, K_1^{12} = 0.3615 + 0.0977i + 0.6895j + 0.0726k,$
$K_1^{21} = 0.1970 + 0.0130i - 0.0988j + 0.2064k, K_1^{22} = 0.8278 - 0.0360j + 1.9143j - 0.0550k.$

So under Theorem 1, the NQVNNs (2.1) and the NQVNNs (2.2) are global synchronous.

The initial values of (4.1) are chosen as:

$m_1(t) = 1.6655 + 7.6979i + 6.4402j - 0.5671k,$

$m_2(t) = -2.9953 + 3.1760i + 1.8176j + 0.4517k.$

Figure 1 shows the synchronization errors $m_1(t), m_2(t)$ between drive system (2.1) and response system (2.2) without control input. For $\delta = 0.2$, and $\tau_1 = 0.2$, Figure 2 shows the state trajectories of $m_1^R, m_1^I, m_1^J, m_1^K, m_2^R, m_2^I, m_2^J, m_2^K$ with control. When we choose $\delta = 0.7$, and $\tau_1 = 0.5$, the state trajectories of the errors $m_1(t)$, $m_2(t)$ is still synchronized with control in Figure 3.

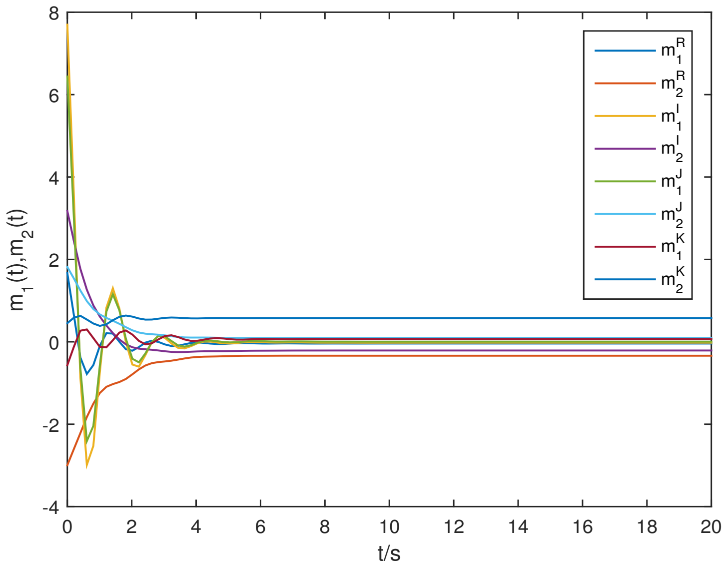


**Figure 1.** For $\delta = 0.2$ and $\tau_1 = 0.2$, the dynamical behavior of system (4.1) without control in Example 1.

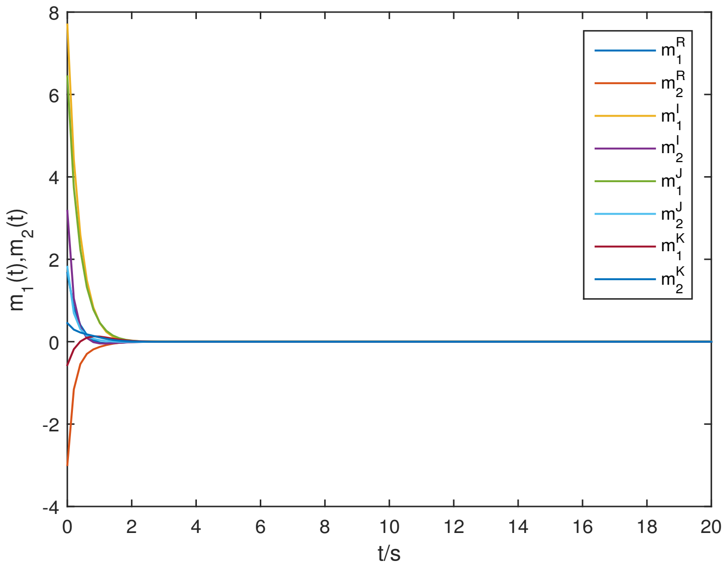


**Figure 2.** For $\delta = 0.2$ and $\tau_1 = 0.2$, the dynamical behavior of system (4.1) with control in Example 1.

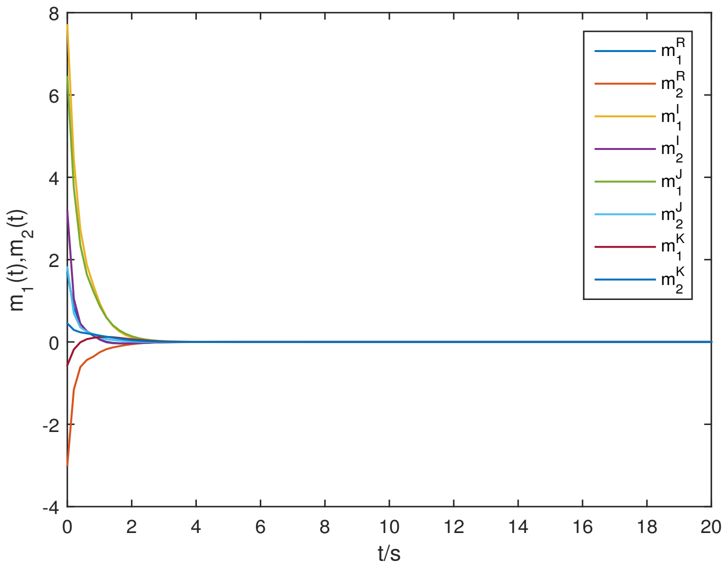


**Figure 3.** For $\delta = 0.7$ and $\tau_1 = 0.5$, the dynamical behavior of system (4.1) with control in Example 1.

According to the simulation results, the drive system (2.1) and the response system (2.2) can achieve global synchronization under the controller $\mu(t) = -K_1 m(t)$, and the system can still maintain synchronization when the time lag is increased. Compared with Ref. [34], the conservativeness of the results is reduced, thus verifying the validity of Theorem 1.

**Example 2.** Consider the following two-neuron NQVNNs with mixed delays:

$$\begin{aligned}\dot{m}(t) =& -Dm(t) + Ag(m(t)) + Bg(m(t-\tau_1)) + H\dot{m}(t-\delta) + C\int_{t-\tau_2(t)}^{t} g(m(s))ds \\ & - K_2 m(t) - \lambda sign(m(t))|m(t)|^{\alpha} - \lambda(\int_{t-\tau_1}^{t} m^*(s)Im(s)ds)^{\frac{1+\alpha}{2}}(\frac{m(t)}{\| m(t) \|^2}) \\ & - \lambda(\int_{t-\delta}^{t} \dot{m}^*(s)I\dot{m}(s)ds)^{\frac{1+\alpha}{2}}(\frac{m(t)}{\| m(t) \|^2}) \\ & - \lambda(\tau_2\int_{-\tau_2}^{0}\int_{t+s}^{t} g^*(m(\theta))Ig(m(\theta))d\theta ds)^{\frac{1+\alpha}{2}}(\frac{m(t)}{\| m(t) \|^2}).\end{aligned} \tag{4.2}$$

The required parameters are the same as in example 1. For numerical simulation, we select $\lambda = 6, \alpha = 0.9, \delta = 0.2$. After calculation,we obtain the $T = 75$, and plot the synchronization errors curve, as shown in Figure 4. As the time t goes to infinity, the synchronization error system (4.2) is stable in finite time. Hence, the drive system (2.1) and the response system (2.2) are synchronized in a finite time interval $[0, 75]$. When $\delta = 1$, the system (4.2) can still maintain synchronization within a finite time, as shown in Figure 5. When a smaller value is taken for constant $\lambda = 4$, the finite time of synchronization may become larger, as shown in Figure 6.

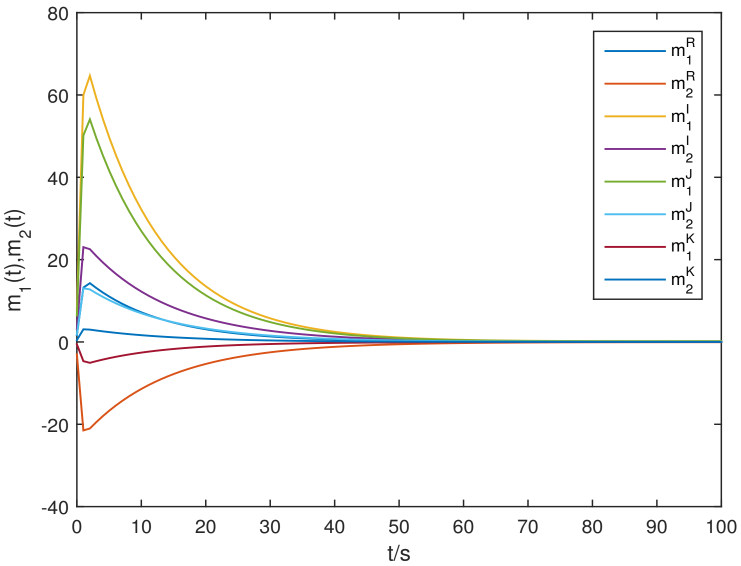


**Figure 4.** For $\lambda = 6,\alpha = 0.9$ and $\delta = 0.2$, the dynamical behavior of system (4.2) in Example 2.

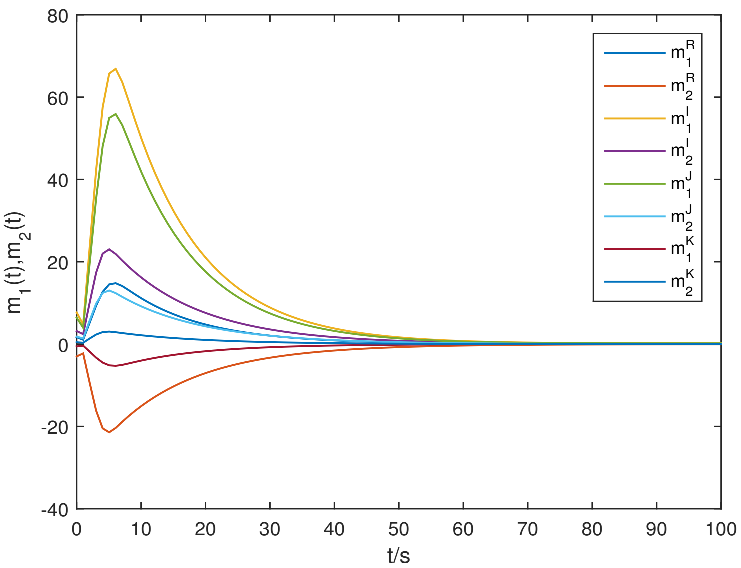


**Figure 5.** For $\lambda = 6,\alpha = 0.9$ and $\delta = 1$, the dynamical behavior of system (4.2) in Example 2.

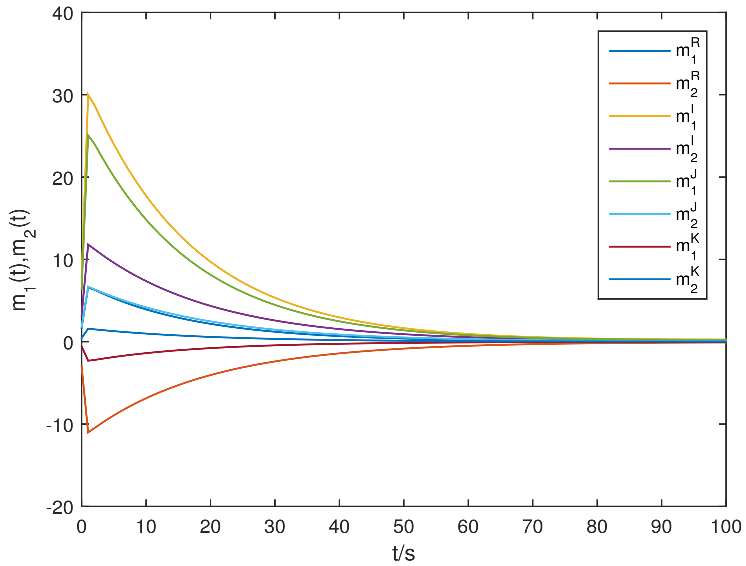


**Figure 6.** For $\lambda = 4$ , $\alpha = 0.9$ and $\delta = 0.2$ , the dynamical behavior of system (4.2) in Example 2.

## 5. Conclusions

The synchronization criteria for neutral-type quaternion-val-ued neural networks with mixed delays is investigated in this paper. By making full use of the information of the time-delay state, a linear feedback controller and a novel nonlinear feedback controller are constructed. Based on the synchronization method of drive-response and establishing appropriate Lyapunov-Krasovskii functional, the sufficient conditions for the global synchronization and finite-time synchronization of NQVNNs are given in the form of algebra and LMIs. Finally, the validity of the obtained results are confirmed by two examples. Due to the existence of the neutral term, the study of this kind of model is more complicated and diffcult than the usual delayed QVNN model, and the results obtained have a wider application range and more research value. Considering the advantages of NQVNNs in image processing and associative memory, we will further study the application of NQVNNs in these aspects.


## Acknowledgements

The work described in this paper was supported by the Sich-uan Province science and technology department application foundation project (2016JY0238), Research Fund of Leshan Normal University (LZD003).


## Conflict of interest

This work does not have any conflicts of interest.

## References


1. X. M. Wang, Introduction to neural networks, *China Sci. J.*, 2017.
2. X. F. Chen, L. J. Li, Z. S. Li, Robust stability analysis of quaternion-valued neural networks via LMI approach, *Adv. Diff. Equ.*, **2018** (2018), 131.
3. R. Rakkiyappan, K. Udhayakumar, G. Velmurugan, J. D. Cao, A. Ahmed, Stability and Hopf bifurcation analysis of fractional-order complex-valued neural networks with time delays, *Adv. Diff. Equ.*, **2017** (2017), 225.
4. X. X. Zhang, C. D. Li, T. W. Huang, Impacts of state-dependent impulses on the stability of switching Cohen-Grossberg neural networks, *Adv. Diff. Equ.*, **2017** (2017), 316.
5. X. M. Yu, X. M. Wang, S. M. Zhong, K. B. Shi, Further results on delay-dependent stability for continuous system with two additive time-varying delay components, *Isa T.* **65** (2016), 9–18.
6. G. Rajchakit, P. Chanthorn, M. Niezabitowski, R. Raja, D. Baleanu, A. Pratap, Impulsive effects on stability and passivity analysis of memristor-based fractional-order competitive neural networks, *Neurocomputing*, **417** (2020), 290–301.
7. U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, Global stability analysis of fractional-order quaternion-valued bidirectional associative memory neural networks, *Mathematics*, **8** (2020), 801.
8. G. Rajchakit, P. Chanthorn, P. Kaewmesri, R. Sriraman, C. P. Lim, Global mittag-leffler stability and stabilization analysis of fractional-order quaternion-valued memristive neural networks, *Mathematics*, **8** (2020), 422.
9. U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, Stochastic memristive quaternion-valued neural networks with time delays: An analysis on mean square exponential input-to-state stability, *Mathematics*, **8** (2020), 815.
10. T. Isokawa, T. Kusakabe, N. Matsui, F. Peper, Quaternion neural network and its application, In: International Conference on Knowledge-Based and Intelligent Information and Engineering Systems, Berlin, Heidelberg: Springe, 2003.
11. T. Minemoto, T. Isokawa, H. Nishimura, N. Matsui, Quaternionic multistate Hopfield neural network with extended projection rule, *Artif. Life Robotics*, **21** (2016), 106–111.
12. X. F. Chen, Q. K. Song, Z. S. Li, Design and analysis of quaternion-valued neural networks for associative memories, *IEEE T. Syst. Cybern. Syst.*, **48** (2018), 2305–2314.
13. Y. Liu, D. D. Zhang, J. G. Lou, J. Q. Lu, J. D. Cao, Stability analysis of quaternion-valued neural networks: Decomposition and direct approaches, *IEEE T. Neur. Net. Lear.*, **29** (2017), 4201–4211.
14. G. Rajchakit, R. Sriraman, Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays, *Neural Process Lett.*, **53** (2021), 581–606.
15. R. Sriraman, G. Rajchakit, C. P. Lim, P. Chanthorn, R. Samidurai, Discrete-time stochastic quaternion-valued neural networks with time delays: An asymptotic stability analysis, *Symmetry*, **12** (2020), 936.
16. S. Gupta, Linear quaternion equations with application to spacecraft attitude propagation, *IEEE Aerosp. Conf. Proc.*, **1** (1998), 69–76.

17. L. C. Luo, H. Feng, L. J. Ding, Color image compression based on quaternion neural network principal component analysis, *2010 International Conference on Multimedia Technology*, 2010, 1-4.

18. H. Kusamichi, T. Isokawa, N. Matsui, Y. Ogawa, K. Maeda, A new scheme for colour night vision by quaternion neural network, *2nd International Conference on Autonomous Robots and Agents*, 2004, 101–106.

19. C. Maharajan, R. Raja, J. D. Cao, G. Rajchakit, Z. W. Tu, A. Alsaedi, LMI-based results on exponential stability of BAM-type neural networks with leakage and both time-varying delays: A non-fragile state estimation approach, *Appl. Math. Comput.*, **326** (2018), 33–55.

20. J. Liu, J. G. Jian, B. X. Wang, Stability analysis for BAM quaternion-valued inertial neural networks with time delay via nonlinear measure approach, *Math. Comput. Simul.*, **174** (2020), 134–152.

21. X. F. Chen, Z. S. Li, Q. K. Song, J. Hu, Y. S. Tan, Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties, *Neural Networks*, **91** (2017), 55–65.

22. X. W. Liu, T. P. Chen, Global exponential stability for complex-valued recurrent neural networks with asynchronous time delays, *IEEE T. Neur. Net. Lear.*, **27** (2016), 593–606.

23. W. L. Lu, T.P. Chen, Synchronization of coupled connected neural networks with delays, *IEEE Transactions on Circuits & Systems I Regular Papers*, **51** (2004), 2491–2503.

24. J. Liu, J. G. Jian, Global dissipativity of a class of quaternion-valued BAM neural networks with time delay, *Neurocomputing*, **349** (2019), 123–132.

25. L. Li, W. S. Chen, Exponential stability analysis of quaternion-valued neural networks with proportional delays and linear threshold neurons: Continuous-time and discrete-time cases, *Neurocomputing*, **381** (2020), 152–166.

26. Q. K. Song, Y. X. Chen, Z. J. Zhao, Y. R. Liu, F. E. Alsaadi, Robust stability of fractional-order quaternion-valued neural networks with neutral delays and parameter uncertainties, *Neurocomputing*, **420** (2021), 70–81.

27. J. L. Shu, L. L. Xiong, T. Wu, Z. X. Liu, Stability analysis of quaternion-valued neutral-type neural networks with time-varying delay, *Mathematics*, **7** (2019), 101.

28. D. Y. Liu, Stability analysis of switched neutral systems, *University of Electronic Science and Technology of China*, 2010.

29. Z. Tu, X. Yang, L. Wang, B. Ding, Stability and stabilization of quaternion-valued neural networks with uncertain time-delayed impulses: Direct quaternion method, *Physica A.*, **535** (2019), 122358.

30. Z. W. Tu, D. D. Wang, X. S. Yang, J. D. Cao, Lagrange stability of memristive quaternion-valued neural networks with neutral items, *Neurocomputing*, **399** (2020), 380–389.

31. Q. Y. Zhu, Adaptive synchronization control of mode-dependent stochastic neutral-type neural network, *Donghua University*, 2014.

32. H. Zhang, X. Y. Wang, X. H. Lin, Synchronization of complex-valued neural network with sliding mode control, *J. Franklin I.*, **353** (2016), 345–358.

33. B. X. Hu, Q. K. Song, K. L. Li, Z. J. Zhao, Y. R. Liu, F. E. Alsaadie, Global $\mu$-synchronization of impulsive complex-valued neural networks with leakage delay and mixed time-varying delays, *Neurocomputing*, **307** (2018), 106–116.

34. L. B. Liu, X. X. You, X. P. GAO, Global synchronization control of quaternion neural networks with mixed delays, *Control Theory Appl.*, **36** (2019), 1360–1368.

35. D. Y. Lin, X. F. Chen, G. P. Yu, Z. S. Li, X. N. Xia, Global exponential synchronization via nonlinear feedback control for delayed inertial memristor-based quaternion-valued neural networks with impulses, *Appl. Math. Comput.*, **401** (2021), 126093.

36. H. Deng, H. B. Bao, Fixed-time synchronization of quaternion-valued neural networks, *Physica A.*, **527** (2019), 121351.

37. H. Pu, L. Q. Wang, Control synchronization of random perturbation neural network with reaction diffusion term in finite time, *Anhui Normal University*, **42** (2019), 442–450.

38. X. F. Chen, Q. K. Song, Global stability of complex-valued neural networks with both leakage time delay and discrete time delay on time scales, *Neurocomputing.*, **121** (2013), 254–264.