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Research article

Einstein solitons with unit geodesic potential vector field

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Abstract: We obtain some results on almost Einstein solitons with unit geodesic potential vector field and provide necessary and sufficient conditions for the soliton to be trivial.

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1. Introduction

An n-dimensional Riemannian manifold (M, g) $(n > 2)$ is an *Einstein soliton* if there exist a vector field ξ and a real constant λ such that

$$\frac{1}{2}£_{\xi}g + \text{Ric} = \left(\lambda + \frac{r}{2}\right)g,$$

where $£_{\xi}$ stands for the Lie derivative operator in the direction of ξ, Ric is the Ricci curvature and r is the scalar curvature of g. Remark that if r is constant, then the notions of Ricci and Einstein soliton coincide. Generalizing these notions by allowing λ to be a function, we talk about an *almost Einstein soliton*. In the particular case when ξ is of gradient type and λ is constant, G. Catino and L. Mazzieri introduced [1] the *gradient Einstein soliton* as a self-similar solution (g, ξ, λ) of the Einstein flow

$$\partial_t g = -2\left(\text{Ric} - \frac{r}{2}g\right).$$

This notion was generalized in the same paper to *gradient ρ-Einstein soliton* as being a data $(g, \xi = \text{grad}(f), \lambda)$ satisfying

$$\text{Hess}(f) + \text{Ric} = (\lambda + \rho r)g,$$

for ρ a nonzero real number. Properties of gradient Einstein and ρ-Einstein solitons can be found in [2–7].

In the present paper, following the ideas developed in [8], we obtain some results on almost Einstein solitons with unit geodesic potential vector field. Moreover, we provide characterization theorems for trivial solitons, which are solitons (g, ξ, λ) with Killing potential vector field. Remark that in the trivial case, Schur's lemma implies that λ must be a constant, hence the scalar curvature will be constant, too. On the other hand, if λ is a constant, then we have an almost Ricci soliton [9] with $\mathrm{div}(\pounds_\xi g) = 0$.

2. Preliminaries

We shall briefly present some properties satisfied by the potential vector field of an almost Einstein soliton immediately deduced from the soliton equation.

Let (M, g) be an n-dimensional Riemannian manifold ($n > 2$).

For any $(1, 1)$-tensor field T_1 and for any symmetric $(0, 2)$-tensor field T_2 on M, we shall denote by $\|\cdot\|$ their norms defined respectively by

$$\|T_1\|^2 := \sum_{1\leq i\leq n} g(T_1E_i, T_1E_i), \quad \|T_2\|^2 := \sum_{1\leq i,j\leq n} (T_2(E_i, E_j))^2,$$

for $\{E_i\}_{1\leq i\leq n}$ a local orthonormal frame field on (M, g).

Consider (g, ξ, λ) an *almost Einstein soliton* defined by the Riemannian metric g, the vector field ξ and the smooth function λ. Then

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} = \left(\lambda + \frac{r}{2}\right) g. \tag{2.1}$$

In particular, if $\pounds_\xi g = 0$, i.e., if ξ is a Killing vector field, then the soliton will be called *trivial*.

Denote by $\eta := i_\xi g$ the dual 1-form of ξ and define the $(1, 1)$-tensor field F by

$$g(FX, Y) := \frac{1}{2}(d\eta)(X, Y),$$

for any $X, Y \in \mathfrak{X}(M)$.

From (2.1) we obtain

$$\nabla\xi = F - Q + \left(\lambda + \frac{r}{2}\right) I, \tag{2.2}$$

where ∇ is the Levi-Civita connection of g, Q is the Ricci operator defined by $g(QX, Y) := \mathrm{Ric}(X, Y)$, for $X, Y \in \mathfrak{X}(M)$ and I is the identity endomorphism on the set of vector fields on M.

By a direct computation we get the divergence of ξ and $F\xi$, precisely

$$\mathrm{div}(\xi) = \frac{2n\lambda + (n-2)r}{2}, \quad \mathrm{div}(F\xi) = -\|F\|^2 - \sum_{i=1}^{n} g((\nabla_{E_i} F)E_i, \xi), \tag{2.3}$$

for $\{E_i\}_{1\leq i\leq n}$ a local orthonormal frame field on (M, g). Also

$$\mathrm{div}(\lambda\xi) = \xi(\lambda) + \frac{(2n\lambda + (n-2)r)\lambda}{2}$$

and in the compact case, by applying the divergence theorem, we conclude

LEMMA 2.1. *If (g, ξ, λ) is an almost Einstein soliton on the compact n-dimensional smooth manifold M $(n > 2)$, then:*

$$\int_M (2n\lambda + (n-2)r) = 0, \quad \int_M (2\xi(\lambda) + (2n\lambda + (n-2)r)\lambda) = 0,$$

$$\int_M \left(\|F\|^2 + \sum_{i=1}^{n} g((\nabla_{E_i} F)E_i, \xi) \right) = 0.$$

Remark that the Riemann curvature R of ∇ satisfies

$$\begin{aligned} R(X,Y)\xi = {} & (\nabla_X F)Y - (\nabla_Y F)X - (\nabla_X Q)Y + (\nabla_Y Q)X \\ & + X\left(\lambda + \frac{r}{2}\right)Y - Y\left(\lambda + \frac{r}{2}\right)X, \end{aligned} \tag{2.4}$$

for any $X, Y \in \mathfrak{X}(M)$, which by contraction gives

$$\mathrm{Ric}(Y, \xi) = -(n-1)Y(\lambda) - \frac{n-2}{2}Y(r) - \sum_{i=1}^{n} g((\nabla_{E_i} F)E_i, Y)$$

and

$$Q\xi = -(n-1)\,\mathrm{grad}(\lambda) - \frac{n-2}{2}\,\mathrm{grad}(r) - \sum_{i=1}^{n} (\nabla_{E_i} F)E_i, \tag{2.5}$$

for $\{E_i\}_{1 \le i \le n}$ a local orthonormal frame field on (M, g).

3. Almost Einstein solitons with unit geodesic potential vector field

We shall give some properties of almost Einstein solitons with unit geodesic potential vector field and provide, in this case, necessary and sufficient conditions for the soliton to be trivial.

Let (g, ξ, λ) be an almost Einstein soliton on the n-dimensional smooth manifold M $(n > 2)$ and assume that ξ is a unit geodesic vector field, i.e., $\nabla_\xi \xi = 0$. Then Eq (2.2) implies

$$F\xi = Q\xi - \left(\lambda + \frac{r}{2}\right)\xi$$

and

$$\mathrm{Ric}(\xi, \xi) = \left(\lambda + \frac{r}{2}\right)\|\xi\|^2.$$

By a direct computation, taking into account that Q is symmetric and F is skew-symmetric, we get

LEMMA 3.1. *If (g, ξ, λ) is an almost Einstein soliton on the n-dimensional smooth manifold M $(n > 2)$ and ξ is a unit geodesic vector field, then:*

$$\|F\|^2 + \sum_{i=1}^{n} g((\nabla_{E_i} F)E_i, \xi) = \left(\|Q\|^2 - \frac{r^2}{n}\right) + \mathrm{div}(\lambda\xi) + \frac{(n-2)(2n\lambda + (n-2)r)r}{4n}. \tag{3.1}$$

In the compact case, from Lemmas 2.1 and 3.1, we obtain

Proposition 3.2. *If (g, ξ, λ) is an almost Einstein soliton on the compact n-dimensional smooth manifold M ($n > 2$) and ξ is a unit geodesic vector field, then:*

$$\int_M \left(\|Q\|^2 - \frac{r^2}{n}\right) = -\frac{n-2}{4n} \int_M (2n\lambda + (n-2)r)r.$$

Also, from the soliton Eq (2.1), we have

Proposition 3.3. *If (g, ξ, λ) is an almost Einstein soliton on the compact n-dimensional smooth manifold M ($n > 2$), then:*

$$\frac{1}{4}\|\pounds_\xi g\|^2 = \| \mathrm{Ric} \|^2 - \frac{r^2}{n} + \frac{(2n\lambda + (n-2)r)^2}{4n}.$$

We will further deduce necessary and sufficient conditions for an almost Einstein soliton (g, ξ, λ) with unit geodesic potential vector field to be trivial, i.e., $\pounds_\xi g = 0$. In this case, $2n\lambda + (n-2)r = 0$.

Theorem 3.4. *Let (g, ξ, λ) be an almost Einstein soliton on the compact n-dimensional smooth manifold M ($n > 2$) with unit geodesic potential vector field ξ. If the scalar curvature is nonzero, then the soliton is trivial if and only if $(2n\lambda + (n-2)r)r \geq 0$.*

Proof. The direct implication is trivial. For the converse implication, notice that from Schwartz's inequality $\|Q\|^2 \geq \frac{r^2}{n}$, by using Proposition 3.2 we deduce $\|Q\|^2 = \frac{r^2}{n}$ and $2n\lambda + (n-2)r = 0$, provided r is nonzero. From Proposition 3.3 we obtain $\pounds_\xi g = 0$, i.e., the soliton is trivial. □

Theorem 3.5. *Let (g, ξ, λ) be an almost Einstein soliton on the compact and connected n-dimensional smooth manifold M ($n > 2$) with unit geodesic potential vector field ξ and nonzero scalar curvature. Then ξ is an eigenvector of the Ricci operator with constant eigenvalue, i.e., $Q\xi = \sigma\xi$, for $\sigma \in \mathbb{R}^*$, satisfying $(n\sigma - r)r \geq 0$, if and only if the soliton is trivial.*

Proof. The converse implication is trivial. Assume now that $Q\xi = \sigma\xi$, $\sigma \in \mathbb{R}^*$. Since ξ is a unit geodesic vector field, then $F\xi = \left(\sigma - \lambda - \frac{r}{2}\right)\xi$ which, for M connected, by taking the inner product with ξ, implies either $\xi = 0$, hence the soliton is trivial, or $\sigma = \lambda + \frac{r}{2}$. In the second case, (g, ξ, λ) is a Ricci soliton and since M is compact, it follows that the soliton is of gradient type [10], hence η is closed and $F = 0$. Then from (2.5), (2.2) and (2.3), we consequently obtain

$$\mathrm{grad}(r) = 2\sigma\xi,$$

$$\mathrm{Hess}(r) = 2\sigma(\sigma g - \mathrm{Ric}),$$

$$\Delta(r) = 2\sigma(n\sigma - r),$$

$$\mathrm{Ric}(\mathrm{grad}(r), \mathrm{grad}(r)) = \sigma\| \mathrm{grad}(r)\|^2.$$

In this case, the Bochner formula [11]

$$\int_M \left(\mathrm{Ric}(\mathrm{grad}(r), \mathrm{grad}(r)) + \| \mathrm{Hess}(r)\|^2 - (\Delta(r))^2\right) = 0$$

becomes

$$\int_M \left(\frac{1}{4\sigma}\| \mathrm{grad}(r)\|^2 + \| \mathrm{Ric} \|^2 + n\sigma^2 - 2\sigma r - (n\sigma - r)^2\right) = 0.$$

But $\Delta(r) = 2\sigma(n\sigma - r)$ and $\operatorname{div}(r \operatorname{grad}(r)) = r\Delta(r) + \| \operatorname{grad}(r)\|^2$ imply

$$\int_M (n\sigma - r) = 0, \quad \int_M (\| \operatorname{grad}(r)\|^2 + 2\sigma r(n\sigma - r)) = 0,$$

which replaced in the previous relation give

$$\int_M \left(\| \operatorname{Ric} \|^2 - \frac{r^2}{n}\right) = -\frac{n-2}{2n} \int_M (n\sigma - r)r.$$

Using Schwartz's inequality we deduce $\|Q\|^2 = \frac{r^2}{n}$ and since r is nonzero, $n\sigma = r$, therefore, $2n\lambda + (n-2)r = 0$. From Proposition 3.3 we obtain $\pounds_\xi g = 0$, i.e., the soliton is trivial. □

Theorem 3.6. *Let* (g, ξ, λ) *be an almost Einstein soliton on the compact and connected n-dimensional smooth manifold M* $(n > 2)$ *with unit geodesic potential vector field* ξ*. Then*

$$\operatorname{Ric}(\xi, \xi) \geq \|F\|^2 + \frac{n-1}{4n}(2n\lambda + (n-2)r)^2$$

if and only if the soliton is trivial.

Proof. From (2.2) we get

$$\|\nabla\xi\|^2 = \|F\|^2 + \|Q\|^2 - \frac{r^2}{n} + \frac{(2n\lambda + (n-2)r)^2}{4n}.$$

Using Proposition 3.3 and Bochner formula [11]

$$\int_M \left(\operatorname{Ric}(\xi, \xi) + \frac{1}{2}\|\pounds_\xi g\|^2 - \|\nabla\xi\|^2 - (\operatorname{div}(\xi))^2\right) = 0$$

we obtain

$$\int_M \left(\|Q\|^2 - \frac{r^2}{n}\right) = \int_M \left(\|F\|^2 + \frac{n-1}{4n}(2n\lambda + (n-2)r)^2 - \operatorname{Ric}(\xi, \xi)\right).$$

Using Schwartz's inequality we deduce $\|Q\|^2 = \frac{r^2}{n}$, hence $Q = \frac{r}{n}I$. Therefore,

$$\frac{r}{n}\xi = Q\xi = F\xi + \frac{2\lambda + r}{2}\xi \Rightarrow F\xi = -\frac{2n\lambda + (n-2)r}{2n}\xi$$

which, by taking the inner product with ξ, implies either $\xi = 0$ or $2n\lambda + (n-2)r = 0$. In both of the cases we deduce $\pounds_\xi g = 0$, i.e., the soliton is trivial. □

Theorem 3.7. *Let* (g, ξ, λ) *be an almost Einstein soliton on the connected n-dimensional smooth manifold M* $(n > 2)$ *with unit geodesic potential vector field* ξ*. Then the soliton is trivial if and only if* $\operatorname{Ric}(\xi, \xi) \geq \|F\|^2$ *and the function* $2n\lambda + (n-2)r$ *is constant on the integral curves of* ξ*.*

Proof. From (2.4) we get

$$\operatorname{Ric}(\xi, \xi) = \|F\|^2 - \left\|Q - \left(\lambda + \frac{r}{2}\right)I\right\|^2 - \frac{1}{2}\xi(2n\lambda + (n-2)r). \tag{3.2}$$

If the soliton is trivial, from (2.1) we obtain $Q = (\lambda + \frac{r}{2})I$ and $2n\lambda + (n-2)r = 0$, so we get the conclusion. Conversely, if $2n\lambda + (n-2)r$ is constant on the integral curves of ξ, then $\operatorname{Ric}(\xi, \xi) = \|F\|^2 - \left\|Q - \left(\lambda + \frac{r}{2}\right)I\right\|^2 \geq \|F\|^2$ implies $Q = (\lambda + \frac{r}{2})I$, which from the soliton Eq (2.1) gives $\pounds_\xi g = 0$, i.e., the soliton is trivial. □

Corollary 3.8. *Let (g, ξ, λ) be an almost Einstein soliton on the compact and connected n-dimensional smooth manifold M ($n > 2$) with unit geodesic potential vector field ξ. Then the soliton is trivial if and only if*

$$\mathrm{Ric}(\xi,\xi) \geq \|F\|^2 + \frac{(2n\lambda + (n-2)r)^2}{4}.$$

Proof. If the soliton is trivial, from (2.1) we get $Q = (\lambda + \frac{r}{2})I$ and $2n\lambda + (n-2)r = 0$, and from Theorem 3.7 we obtain the conclusion. Conversely, taking into account that $\xi(2n\lambda + (n-2)r) = \mathrm{div}((2n\lambda + (n-2)r)\xi) - \frac{(2n\lambda+(n-2)r)^2}{2}$, from (3.2), by integration we get

$$\int_M \left\| Q - \left(\lambda + \frac{r}{2}\right) I \right\|^2 = \int_M \left(\|F\|^2 + \frac{(2n\lambda + (n-2)r)^2}{4} - \mathrm{Ric}(\xi,\xi) \right) \leq 0,$$

hence $Q = (\lambda + \frac{r}{2})I$, which from the soliton Eq (2.1) gives $£_\xi g = 0$, i.e., the soliton is trivial. □

Let us further assume that (g, ξ, λ) is an almost Einstein soliton on a compact and connected n-dimensional smooth manifold M ($n > 2$). Then

$$\nabla\xi = \left(\lambda + \frac{r}{2}\right) I - Q + F. \tag{3.3}$$

As ξ is a unit geodesic vector field, we have

$$Q\xi = \left(\lambda + \frac{r}{2}\right)\xi + v, \tag{3.4}$$

where $v = F\xi$. Then v is a closed vector field, as for the smooth function $h = \frac{1}{2}\|\xi\|^2$, we get $v = -\frac{1}{2}\mathrm{grad}(h)$. Moreover, we have

$$\nabla_X v = (\nabla_X F)\,\xi + \left(\lambda + \frac{r}{2}\right) FX - FQX + F^2 X,$$

for any $X \in \mathfrak{X}(M)$. Since v is closed, we get

$$g\left((\nabla_X F)\,\xi, Y\right) - g\left((\nabla_Y F)\,\xi, X\right) + 2\left(\lambda + \frac{r}{2}\right) g\left(FX, Y\right)$$
$$- g\left(FQX, Y\right) + g\left(FQY, X\right) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$. Taking $Y = \xi$ and using

$$g\left((\nabla_X F)\,\xi, \xi\right) = 0,$$

we conclude

$$-g\left(\left(\nabla_\xi F\right)\xi, X\right) - 2\left(\lambda + \frac{r}{2}\right) g\left(F\xi, X\right) + g\left(QF\xi, X\right) + g\left(FQ\xi, X\right) = 0,$$

for any $X \in \mathfrak{X}(M)$, therefore

$$-\left(\nabla_\xi F\right)\xi = 2\left(\lambda + \frac{r}{2}\right) v - Qv - FQ\xi.$$

Since, ξ is a unit geodesic vector field, using (3.4), we obtain

$$\nabla_\xi v = -\left(\lambda + \frac{r}{2}\right)v + Qv + Fv. \tag{3.5}$$

Note that v being a closed vector field, we can define a symmetric operator A by

$$g(AX, Y) := \frac{1}{2}(\pounds_v g)(X, Y),$$

for any $X, Y \in \mathfrak{X}(M)$, which is precisely

$$A = \nabla v, \tag{3.6}$$

and satisfies trace$(A) = \operatorname{div}(v)$. Also, for any $X, Y \in \mathfrak{X}(M)$, we get

$$R(X, Y)v = (\nabla_X A)\, Y - (\nabla_Y A)\, X$$

and

$$\operatorname{Ric}(Y, v) = g\left(Y, \sum_{i=1}^{n} (\nabla_{E_i} A)\, E_i\right) - Y(\operatorname{div}(v)). \tag{3.7}$$

Note that

$$\operatorname{div}(Av) = \|A\|^2 + g\left(v, \sum_{i=1}^{n} (\nabla_{E_i} A)\, E_i\right)$$

and inserting it into (3.7), we have

$$\operatorname{Ric}(v, v) = \operatorname{div}(Av) - \|A\|^2 - v(\operatorname{div}(v)).$$

Integrating the above equation and using $\operatorname{div}((\operatorname{div}(v)v) = v(\operatorname{div}(v)) + (\operatorname{div}(v))^2$, we conclude

$$\int_M \left(\|A\|^2 - \frac{1}{n}(\operatorname{div}(v))^2\right) = \int_M \left(\frac{n-1}{n}(\operatorname{div}(v))^2 - \operatorname{Ric}(v, v)\right).$$

Under the assumption $\operatorname{Ric}(v, v) \geq \frac{n-1}{n}(\operatorname{div}(v))^2$, the above integral implies

$$A = \frac{1}{n}\,(\operatorname{div}(v))\, I,$$

and combining it with (3.6), we conclude

$$\nabla_\xi v = \frac{1}{n}\,(\operatorname{div}(v))\, \xi.$$

Taking now its inner product with ξ and noticing that v is orthogonal to ξ and that ξ is a unit geodesic vector field, we get

$$(\operatorname{div}(v))\, \|\xi\|^2 = 0.$$

If $\xi = 0$, then the soliton is trivial, so we have $\operatorname{div}(v) = 0$, that is, $A = 0$. Thus, Eq (3.5) implies

$$Qv = \left(\lambda + \frac{r}{2}\right)v - Fv.$$

Also, equation (3.7) gives $\text{Ric}(Y, v) = 0$, that is $Qv = 0$. Thus, we have $Fv = (\lambda + \frac{r}{2})v$, and taking the inner product with ξ, we conclude

$$\|v\|^2 = 0.$$

Now, the Eq (3.4) takes the form

$$Q\xi = \left(\lambda + \frac{r}{2}\right)\xi.$$

Taking the covariant derivative in the above equation and using (3.3), we have

$$(\nabla_X Q)\,\xi = X\left(\lambda + \frac{r}{2}\right)\xi + \left(Q - \left(\lambda + \frac{r}{2}\right)I\right)^2(X) + \left(\lambda + \frac{r}{2}\right)FX - QFX$$

which yields

$$\left\|Q - \left(\lambda + \frac{r}{2}\right)I\right\|^2 + \xi(\lambda) = 0.$$

Integrating this relation and using Lemma 2.1, we conclude

$$\int_M \left\|Q - \left(\lambda + \frac{r}{2}\right)I\right\|^2 = \frac{1}{2}\int_M ((2n\lambda + (n-2)r)\lambda). \tag{3.8}$$

Thus, we are ready to prove

Theorem 3.9. *Let (g, ξ, λ) be an almost Einstein soliton on the compact and connected n-dimensional smooth manifold M $(n > 2)$ with unit geodesic potential vector field ξ. Then the Ricci curvature satisfies* $\text{Ric}(F\xi, F\xi) \geq \frac{n-1}{n}(\text{div}(F\xi))^2$ *and* $(2n\lambda + (n-2)r)\lambda \leq 0$*, if and only if the soliton is trivial.*

Proof. Equation (3.8) gives $Q = (\lambda + \frac{r}{2})I$, that implies $\lambda + \frac{r}{2}$ is a constant (as $n > 2$). Also, we find that $r = n(\lambda + \frac{r}{2})$ is a constant, hence $\lambda = -\frac{n-2}{2n}r$ is a constant. Moreover, by Proposition 3.3, we conclude $\pounds_\xi g = 0$, i.e., the soliton is trivial. The converse implication is trivial. □

We end these considerations by providing two examples.

Example 3.10. Let S^{2n+1} be the odd dimensional unit sphere with the usual Sasakian structure (φ, ξ, η, g). Then it follows that the Reeb vector field ξ, being a unit Killing vector field, is a geodesic vector field and consequently

$$\frac{1}{2}\pounds_\xi g + \text{Ric} = \left(\lambda + \frac{r}{2}\right)g,$$

holds, where the scalar curvature is $r = 2n(2n+1)$ and $\lambda = -n(2n-1)$. Thus, (g, ξ, λ) is a trivial Einstein soliton on the sphere S^{2n+1} with ξ a unit geodesic vector field.

Example 3.11. Let (M, g) be an n-dimensional complete quasi-Einstein manifold $n \geq 3$ (cf. [12]) whose Ricci curvature is given by

$$\text{Ric} = ag + b\alpha \otimes \alpha,$$

where a, b are smooth functions and α is a 1-form on M. Let f be the distance function on M (cf. [13]) and we choose $b = \frac{1}{f}$ and $\alpha = df$. Note that f being the distance function, we have $\left\|\text{grad}(f)\right\| = 1$ and that integral curves of the vector field grad(f) are geodesics (cf. [13]). Thus, choosing $\xi = \text{grad}(f)$, we get

$$\nabla_\xi \xi = 0,$$

that is, ξ is a unit geodesic vector field. Also, by using geodesic coordinates on a normal neighborhood, we find that

$$\xi = \frac{1}{f}\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

and that

$$\nabla_X \xi = -\frac{1}{f^2}X(f)\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + \frac{1}{f}X = -\frac{1}{f}X(f)\xi + \frac{1}{f}X.$$

Thus, we have

$$\frac{1}{2}\left(\pounds_\xi g\right)(X,Y) = -\frac{1}{f}X(f)Y(f) + \frac{1}{f}g\,(X,Y) = -\frac{1}{f}\alpha(X)\alpha(Y) + \frac{1}{f}g\,(X,Y)$$

and we get

$$\frac{1}{2}\pounds_\xi g + \text{Ric} = \left(a + \frac{1}{f}\right)g.$$

Moreover, the scalar curvature $r = na + \frac{1}{f}$ and in view of this, we have

$$\frac{1}{2}\pounds_\xi g + \text{Ric} = \left(\lambda + \frac{r}{2}\right)g,$$

where

$$\lambda = \frac{1}{2}\left(\frac{1}{f} - (n-2)a\right).$$

Hence, (g, ξ, λ) is an almost Einstein soliton on the considered quasi-Einstein manifold (M, g) with ξ a unit geodesic vector field.

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Conflict of interest

The authors declare that they have no conflict of interest.

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