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Research article

New scenario of decay rate for system of three nonlinear wave equations with visco-elasticities

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Abstract: A system of three semilinear wave equations with strong external forces in $\mathbb{R}^n$ is considered. We use weighted phase spaces, where the problem is well defined, to compensate the lack of Poincare's inequality. Using the Faedo-Galerkin method and some energy estimates, we prove the existence of global solution. By imposing a new appropriate conditions, which are not used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtain an unusual decay rate for the energy function. It is a generalization of similar results in [16, 29, 31]. The work is relevant in the sense that the problem is more complex than what can be found in the literature.

Keywords: damped wave equation; power nonlinearity; global solution; Faedo-Galerkin approximation; decay rate
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1. Introduction and preliminaries

We consider, for $x \in \mathbb{R}^n$, $t > 0$, the following system

$$\begin{cases} u_{tt} - \theta\Delta(u + \int_0^t \varpi_1(t-s)u(s)\,ds) + \alpha u_t = h_1(u,v,w) \\ v_{tt} - \theta\Delta(v + \int_0^t \varpi_2(t-s)v(s)\,ds) + \alpha v_t = h_2(u,v,w) \\ w_{tt} - \theta\Delta(w + \int_0^t \varpi_3(t-s)w(s)\,ds) + \alpha w_t = h_3(u,v,w) \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where $n \geq 3, \alpha > 0$, the functions $h_i(.,.,.) \in (\mathbb{R}^3, \mathbb{R}), i = 1,2,3$ are given by

$$h_1(\xi_1,\xi_2,\xi_3) = (q+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(q-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_1|^{(q-3)/2}\xi_1|\xi_2|^{(q+1)/2}\Big];$$

$$h_2(\xi_1,\xi_2,\xi_3) = (q+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(q-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_2|^{(q-3)/2}\xi_2|\xi_3|^{(q+1)/2}\Big];$$

$$h_3(\xi_1,\xi_2,\xi_3) = (q+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(q-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_3|^{(q-3)/2}\xi_3|\xi_1|^{(q+1)/2}\Big],$$

with $d, e > 0, q > 3$. The function $\frac{1}{\theta(x)} \sim \vartheta(x) > 0$ for all $x \in \mathbb{R}^n$ is a density such that

$$\vartheta \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \le r \le \frac{2n}{n-2}. \tag{1.2}$$

As in [15], it is not hard to see that there exists a function $\mathcal{G} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$uh_1(u,v,w) + vh_2(u,v,w) + wh_3(u,v,w) = (q+1)\mathcal{G}(u,v,w),\ \forall (u,v,w) \in \mathbb{R}^3. \tag{1.3}$$

satisfies

$$(q+1)\mathcal{G}(u,v,w) = |u+v+w|^{q+1} + 2|uv|^{(q+1)/2} + 2|vw|^{(q+1)/2} + 2|wu|^{(q+1)/2}. \tag{1.4}$$

We define the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$, as in [18], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in (L^2(\mathbb{R}^n))^n\},$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\vartheta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\vartheta} = (v,v)^{1/2}_{L^2_\vartheta}$ for

$$(v,w)_{L^2_\vartheta} = \int_{\mathbb{R}^n} \vartheta v w \, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_\vartheta} = \left(\int_{\mathbb{R}^n} \vartheta\, |v|^r \, dx\right)^{\frac{1}{r}}.$$

is the norm of the weighted space $L^r_\vartheta(\mathbb{R}^n)$.

The main aim of this work is to consider an important problem from the point of view of application in sciences and engineering (materials which is something between that of elastic solids and Newtonian fluids), namely, a system of three wave equations having a damping effects in an unbounded domain with strong external forces including damping terms of memory type with past history. Using the Faedo-Galerkin [16] method and some energy estimates, we proved the existence of global solution in $\mathbb{R}^n$ owing to the weighted function. By imposing a new appropriate condition, with the help of some special estimates and generalized Poincaré's inequality, we obtained an unusual decay rate for the energy function. For more detail regarding the single equation, we review the following references [7, 8]. The paper [7] is one of the pioneer in literature for the single equation, which is the source of inspiration of several researches, while the work [8] is a recent generalization of [7] by introducing less dissipative effects.

To enrich our topic, it is necessary to review previous works regarding the nonlinear coupled system of wave equations, from a qualitative and quantitative study. Let us begin with the single wave equation treated in [13], where the aim goal was mainely on the system

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega\Delta u_t = u \ln |u|, \ (x, t) \in \Omega \times (0, \infty) \\ u(x, t) = 0, x \in \partial\Omega, t \geq 0 \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The author firstly constructed a local existence of weak solution by using contraction mapping principle and of course showed the global existence, decay rate and infinite time blow up of the solution with condition on initial energy.

Next, a nonexistence of global solutions for system of three semi-linear hyperbolic equations was introduced in [3]. A coupled system for semi-linear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{q-1}|v|^{q+1}u, f_2 = |v|^{q-1}|u|^{q+1}v$. (Please, see [2, 5, 9, 14, 24, 29]).

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [16], where the considered equation as follows

$$u_{tt} + au_t - \phi(x)\Delta\left(u + \omega u_t - \int_0^t g(t - s)u(s)\, ds\right) = u|u|^{q-1}, \tag{1.6}$$

with initial data

$$\begin{cases} u(x, 0) = u_0(x), \\ u_t(x, 0) = u_1(x). \end{cases} \tag{1.7}$$

The authors showed the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result by considering the relaxation function is strictly convex, for more results related to decay rate of solution of this type of problems, please see [6, 17, 25, 26, 30, 31].

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling some of the work recently published. Baowei Feng et *al.* considered in [10], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain ($(x, t) \in \Omega \times (0, \infty)$) with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - s)\Delta u(s)\, ds + u_t = f_1(u, v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t - s)\Delta v(s)\, ds + v_t = f_2(u, v). \end{cases} \tag{1.8}$$

Here, the authors concerned with a system in $\mathbb{R}^n (n = 1, 2, 3)$. Under appropriate hypotheses, they established a general decay result by multiplication techniques to extends some existing results for a single equation to the case of a coupled system.

It is worth noting here that there are several studies in this field and we particularly refer to the generalization that Shun et *al.* made in studying a complicate non-linear case with degenerate damping term in [22]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\kappa + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v) \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{1.9}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.

The lack of existence (Blow up) is considered one of the most important qualitative studies that must be spoken of, given its importance in terms of application in various applied sciences. Concerning the nonexistence of solution for a more degenerate case for coupled system of wave equations with different damping, we mention the papers [19–21, 23, 27].

In m-equations, paper in [1] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1, i\neq j}^{m} |u_j|^{p_j}|u_i|^{p_i}u_i,\ i = 1, 2, \ldots, m, \tag{1.10}$$

where the absence of global solutions with positive initial energy was investigated.

We introduce a very useful Sobolev embedding and generalized Poincaré inequalities.

Lemma 1.1. *[16] Let ϑ satisfy (1.2). For positive constants $C_\tau > 0$ and $C_P > 0$ depending only on ϑ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\vartheta} \leq C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 1.2. *[12] Let ϑ satisfy (1.2), then the estimates*

$$\|v\|_{L^r_\vartheta} \leq C_r \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \|\vartheta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

We assume that the kernel functions $\varpi_1, \varpi_2, \varpi_3 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$\begin{cases} 1 - \overline{\varpi_1} = l > 0 \quad \text{for} \quad \overline{\varpi_1} = \int_0^{+\infty} \varpi_1(s)\,ds,\ \varpi_1'(t) \leq 0, \\ 1 - \overline{\varpi_2} = m > 0 \quad \text{for} \quad \overline{\varpi_2} = \int_0^{+\infty} \varpi_2(s)\,ds,\ \varpi_2'(t) \leq 0, \\ 1 - \overline{\varpi_3} = \nu > 0 \quad \text{for} \quad \overline{\varpi_3} = \int_0^{+\infty} \varpi_3(s)\,ds,\ \varpi_3'(t) \leq 0, \end{cases} \tag{1.11}$$

we mean by $\mathbb{R}^+$ the set $\{\tau \mid \tau \geq 0\}$. Noting by

$$\varpi(t) = \max_{t\geq 0} \Big\{\varpi_1(t), \varpi_2(t), \varpi_3(t)\Big\}, \tag{1.12}$$

and

$$\varpi_0(t) = \min_{t\geq 0} \Big\{ \int_0^t \varpi_1(s)ds, \int_0^t \varpi_2(s)ds, \int_0^t \varpi_3(s)ds\Big\}. \tag{1.13}$$

We assume that there is a function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\varpi_i'(t) + \chi(\varpi_i(t)) \leq 0, \quad \chi(0) = 0, \quad \chi'(0) > 0,\ i = 1, 2, 3, \tag{1.14}$$

for any $\xi \geq 0$.
Hölder and Young's inequalities give

$$\begin{aligned} \|uv\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} &\leq \Big(\|u\|^2_{L_\vartheta^{(q+1)}} + \|v\|^2_{L_\vartheta^{(q+1)}}\Big)^{(q+1)/2} \\ &\leq \Big(l\|u\|^2_{\mathcal{H}} + m\|v\|^2_{\mathcal{H}}\Big)^{(q+1)/2}, \end{aligned} \tag{1.15}$$

and

$$\|vw\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} \leq \Big(m\|v\|^2_{\mathcal{H}} + \nu\|w\|^2_{\mathcal{H}}\Big)^{(q+1)/2}, \tag{1.16}$$

and

$$\|wu\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} \leq \Big(\nu\|w\|^2_{\mathcal{H}} + l\|u\|^2_{\mathcal{H}}\Big)^{(q+1)/2}. \tag{1.17}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned} \|u+v+w\|_{L_\vartheta^{(q+1)}}^{(q+1)} &\leq c\Big(\|u\|^2_{L_\vartheta^{(q+1)}} + \|v\|^2_{L_\vartheta^{(q+1)}} + \|w\|^2_{L_\vartheta^{(q+1)}}\Big)^{(q+1)/2} \\ &\leq c\Big(\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}\Big)^{(q+1)/2}. \end{aligned}$$

Then there exist $\eta > 0$ such that

$$\begin{aligned} &\|u+v+w\|_{L_\vartheta^{(q+1)}}^{(q+1)} + 2\,\|uv\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} + 2\,\|vw\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} + 2\,\|wu\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} \\ &\leq \eta\Big(l\|u\|^2_{\mathcal{H}} + m\|v\|^2_{\mathcal{H}} + \nu\|w\|^2_{\mathcal{H}}\Big)^{(q+1)/2}. \end{aligned} \tag{1.18}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(q-1)} \quad \text{and} \quad \mathcal{E}_0 = \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\eta^{-2/(q-1)}. \tag{1.19}$$

The mainely aim of the present paper is to obtain a novel decay rate of solution from the convexity property of the function χ given in Theorem 3.1.

We denote as in [18, 28] an eigenpair $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ of

$$-\theta(x)\Delta e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$, $(\theta(x))^{-1} \equiv \vartheta(x)$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 1.3. *The triplet functions (u, v, w) is said a weak solution to (1.1) on $[0, T]$ if satisfies for $x \in \mathbb{R}^n$,*

$$\begin{cases} \int_{\mathbb{R}^n} \vartheta(x)(u_{tt} + \alpha u_t)\varphi dx + \int_{\mathbb{R}^n} \nabla u \nabla\varphi dx - \int_0^t \varpi_1(t-s)\nabla u(s)\, ds \nabla\varphi dx \\ = \int_{\mathbb{R}^n} \vartheta(x) h_1(u, v, w)\varphi dx, \\ \\ \int_{\mathbb{R}^n} \vartheta(x)(v_{tt} + \alpha v_t)\psi dx + \int_{\mathbb{R}^n} \nabla v \nabla\psi dx - \int_0^t \varpi_2(t-s)\nabla v(s)\, ds \nabla\psi dx \\ = \int_{\mathbb{R}^n} \vartheta(x) h_2(u, v, w)\psi dx, \\ \\ \int_{\mathbb{R}^n} \vartheta(x)(w_{tt} + \alpha w_t)\Psi dx + \int_{\mathbb{R}^n} \nabla v \nabla\Psi dx - \int_0^t \varpi_3(t-s)\nabla w(s)\, ds \nabla\Psi dx \\ = \int_{\mathbb{R}^n} \vartheta(x) h_3(u, v, w)\Psi dx, \end{cases} \tag{1.20}$$

for all test functions $\varphi, \psi, \Psi \in \mathcal{H}$ for almost all $t \in [0, T]$.

2. Local and global existence

The next Theorem is concerned on the local solution (in time $[0, T]$).

Theorem 2.1. *(Local existence) Assume that*

$$1 < q \leq \frac{n+2}{n-2} \quad \text{and that} \quad n \geq 3. \tag{2.1}$$

Let $(u_0, v_0, w_0) \in \mathcal{H}^3$ and $(u_1, v_1, w_3) \in L^2_\vartheta(\mathbb{R}^n) \times L^2_\vartheta(\mathbb{R}^n) \times L^2_\vartheta(\mathbb{R}^n)$. Under the assumptions (1.2)–(1.17) and (1.11)–(1.14). Then (1.1) admits a unique local solution (u, v, w) such that

$$(u, v, w) \in \mathcal{X}_T^3, \ \mathcal{X}_T \equiv C([0, T]; \mathcal{H}) \cap C^1([0, T]; L^2_\vartheta(\mathbb{R}^n)),$$

for sufficiently small $T > 0$.

We prove the existence of global solution in time. Let us introduce the potential energy $J : \mathcal{H}^3 \to \mathbb{R}$ defined by

$$\begin{aligned} J(u, v, w) \ &= \ \left(1 - \int_0^t \varpi_1(s)\, ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\ &+ \ \left(1 - \int_0^t \varpi_2(s)\, ds\right) \|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v) \\ &+ \ \left(1 - \int_0^t \varpi_3(s)\, ds\right) \|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w)\,, \end{aligned} \tag{2.2}$$

where

$$\left(\varpi_j \circ w\right)(t) = \int_0^t \varpi_j(t-s)\,\|w(t) - w(s)\|_{\mathcal{H}}^2\, ds,$$

for any $w \in L^2(\mathbb{R}^n)$, $j = 1, 2, 3$. The modified energy is defined by

$$\mathcal{E}(t) = \frac{1}{2}\Big(\,\|u_t\|_{L^2_\vartheta}^2 + \|v_t\|_{L^2_\vartheta}^2 + \|w_t\|_{L^2_\vartheta}^2\,\Big) + \frac{1}{2}J(u, v, w) - \int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u, v, w)dx, \tag{2.3}$$

Theorem 2.2. *(Global existence) Let (1.2)–(1.17) and (1.11)–(1.14) hold. Under (2.1) and for sufficiently small* $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L^2_\vartheta(\mathbb{R}^n)$*, problem (1.1) admits a unique global solution* (u, v, w) *such that*

$$(u, v, w) \in \mathcal{X}^3,\ \mathcal{X} \equiv C([0, +\infty); \mathcal{H}) \cap C^1([0, +\infty); L^2_\vartheta(\mathbb{R}^n)). \tag{2.4}$$

The next, Lemma will play an important role in the sequel.

Lemma 2.3. *For* $(u, v, w) \in \mathcal{X}^3_T$*, the functional* $\mathcal{E}(t)$ *associated with problem (1.1) is a decreasing energy.*

Proof. For $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1)\\
&= \int_{t_1}^{t_2} \frac{d}{dt}\mathcal{E}(t)\, dt\\
&= -\frac{1}{2}\int_{t_1}^{t_2} \Big(\varpi_1(t)\,\|u\|_{\mathcal{H}}^2 - (\varpi_1' \circ u)\Big)\, dt\\
&- \frac{1}{2}\int_{t_1}^{t_2} \Big(\varpi_2(t)\,\|v\|_{\mathcal{H}}^2 - (\varpi_2' \circ v)\Big)\, dt\\
&- \frac{1}{2}\int_{t_1}^{t_2} \Big(\varpi_3(t)\,\|w\|_{\mathcal{H}}^2 - (\varpi_3' \circ w)\Big)\, dt\\
&- \alpha\Big(\,\|u_t\|_{L^2_\vartheta}^2 + \|v_t\|_{L^2_\vartheta}^2 + \|w_t\|_{L^2_\vartheta}^2\,\Big)\\
&\le 0,
\end{aligned}$$

owing to (1.11)–(1.14). □

We sketch here the outline of the proof for local solution by a standard procedure(See [4, 11, 31]).

Proof. (Of Theorem 2.1.) Let $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L^2_\vartheta(\mathbb{R}^n)$. For any $(u, v, w) \in \mathcal{X}^3_T$, we can obtain weak solution of the related system

$$\begin{cases}
\vartheta(x)(z_{tt} + \alpha z_t) - \Delta z = -\int_0^t \varpi_1(t-s)\Delta u(s)\, ds + \vartheta(x)h_1(u, v, w)\\
\vartheta(x)(y_{tt} + \alpha y_t) - \Delta y = -\int_0^t \varpi_2(t-s)\Delta v(s)\, ds + \vartheta(x)h_2(u, v, w)\\
\vartheta(x)(\zeta_{tt} + \alpha \zeta_t) - \Delta \zeta = -\int_0^t \varpi_3(t-s)\Delta w(s)\, ds + \vartheta(x)h_3(u, v, w)\\
z(x, 0) = u_0(x), y(x, 0) = v_0(x), \zeta(x, 0) = w_0(x)\\
z_t(x, 0) = u_1(x), y_t(x, 0) = v_1(x), \zeta_t(x, 0) = w_1(x).
\end{cases} \tag{2.5}$$

We reduces problem (2.5) to Cauchy problem for system of ODE by using the Faedo-Galerkin approximation. We then find a solution map $\top : (u, v, w) \mapsto (z, y, \zeta)$ from $\mathcal{X}_T^3$ to $\mathcal{X}_T^3$. We are now ready show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}_T^3$ for a small $T > 0$. Hence $\top$ has a fixed point $\top(u, v, w) = (u, v, w)$, which gives a unique solution in $\mathcal{X}_T^3$. □

We will show the global solution. By using conditions on functions $\varpi_1, \varpi_2, \varpi_3$, we have

$$\begin{aligned}
\mathcal{E}(t) &\geq \frac{1}{2}J(u,v,w) - \int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u,v,w)dx \\
&\geq \frac{1}{2}J(u,v,w) - \frac{1}{q+1}\,\|u+v+w\|_{L_\vartheta^{(q+1)}}^{(q+1)} \\
&- \frac{2}{q+1}\Big(\|uv\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} + \|vw\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} + \|wu\|_{L_\vartheta^{(q+1)/2}}^{(q+1)/2} \Big) \\
&\geq \frac{1}{2}J(u,v,w) - \frac{\eta}{q+1}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \Big]^{(q+1)/2} \\
&\geq \frac{1}{2}J(u,v,w) - \frac{\eta}{q+1}\Big(J(u,v,w)\Big)^{(q+1)/2} \\
&= G(\varsigma), \qquad (2.6)
\end{aligned}$$

here $\varsigma^2 = J(u,v,w)$, for $t \in [0,T)$, where

$$G(\xi) = \frac{1}{2}\xi^2 - \frac{\eta}{q+1}\xi^{(q+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (1.19). Then

$$\begin{cases} G(\xi) \geq 0 & in \;\; \xi \in [0, \lambda_0] \\ G(\xi) < 0 & in \;\; \xi > \lambda_0. \end{cases} \qquad (2.7)$$

Moreover, $\lim_{\xi \to +\infty} G(\xi) \to -\infty$. Then, we have the following lemma

Lemma 2.4. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.
(i) *If* $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 < \lambda_0^2$, *then local solution of (1.1) satisfies*

$$J(u,v,w) < \lambda_0^2, \;\; \forall t \in [0,T).$$

(ii) *If* $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 > \lambda_0^2$, *then local solution of (1.1) satisfies*

$$\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2 > \lambda_1^2, \;\; \forall t \in [0,T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.
The case (i)**.** By (2.6), we have

$$G(J(u_0, v_0, w_0)) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_0, v_0, w_0) \leq \xi_1^2$. Then we claim that $J(u, v, w) \leq \xi_1^2,\ \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u(t_0), v(t_0), w(t_0)) < \xi_2^2.$$

Then

$$G(J(u(t_0), v(t_0), w(t_0))) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 2.3, which contradicts (2.6). Hence we have

$$J(u, v, w) \leq \xi_1^2 < \lambda_0^2,\ \forall t \in [0, T).$$

The case (*ii*)**.** We can now show that

$$\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 \geq \xi_2^2,$$

and

$$\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2,$$

in the same way as (*i*). □

Proof. (*Of Theorem 2.2.*) Let $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L^2_\vartheta(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and

$$\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 < \lambda_0^2.$$

By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
&\frac{1}{2}\Big(\|u_t\|_{L^2_\vartheta}^2 + \|v_t\|_{L^2_\vartheta}^2 + \|w_t\|_{L^2_\vartheta}^2 \Big) + l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \\
&\leq \frac{1}{2}\Big(\|u_t\|_{L^2_\vartheta}^2 + \|v_t\|_{L^2_\vartheta}^2 + \|w_t\|_{L^2_\vartheta}^2 \Big) + \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) \\
&+ \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + (\varpi_2 \circ v) + \left(1 - \int_0^t \varpi_3(s)\,ds\right)\|w\|_{\mathcal{H}}^2 + (\varpi_3 \circ w) \\
&\leq 2\mathcal{E}(t) + \frac{2\eta}{q+1}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|u\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2\Big]^{(q+1)/2} \\
&\leq 2\mathcal{E}(0) + \frac{2\eta}{q+1}\Big(J(u, v, w)\Big)^{(q+1)/2} \\
&\leq 2\mathcal{E}_0 + \frac{2\eta}{q+1}\lambda_0^{q+1} \\
&= \eta^{-2/(q-1)}.
\end{aligned} \tag{2.8}$$

This completes the proof. □

Let

$$\begin{aligned}
\Lambda(u, v, w) &= \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + \frac{1}{2}\,(\varpi_1 \circ u) \qquad (2.9) \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}\,(\varpi_2 \circ v)
\end{aligned}$$

$$+\quad \frac{1}{2}\left(1-\int_0^t \varpi_3(s)\,ds\right)\|w\|_{\mathcal{H}}^2+\frac{1}{2}\left(\varpi_3\circ w\right)-\int_{\mathbb{R}^n}\vartheta(x)\mathcal{G}(u,v,w)dx,$$

$$\begin{aligned}\Pi(u,v,w) &= \left(1-\int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2+(\varpi_1\circ u)\\ &+ \left(1-\int_0^t \varpi_2(s)\,ds\right)\|v\|_{\mathcal{H}}^2+(\varpi_2\circ v)\\ &+ \left(1-\int_0^t \varpi_3(s)\,ds\right)\|w\|_{\mathcal{H}}^2+(\varpi_3\circ w)-(q+1)\int_{\mathbb{R}^n}\vartheta(x)\mathcal{G}(u,v,w)dx.\end{aligned} \tag{2.10}$$

Lemma 2.5. *Let* (u, v, w) *be the solution of problem (1.1). If*

$$\|u_0\|_{\mathcal{H}}^2+\|v_0\|_{\mathcal{H}}^2+\|w_0\|_{\mathcal{H}}^2-(q+1)\int_{\mathbb{R}^n}\vartheta(x)\mathcal{G}(u_0,v_0,w_0)dx>0. \tag{2.11}$$

Then under condition (3.1), the functional $\Pi(u,v,w)>0,\ \forall t>0.$

Proof. By (2.11) and continuity, there exists a time $t_1>0$ such that

$$\Pi(u,v,w)\geq 0,\forall t<t_1.$$

Let

$$Y=\{(u,v,w)\mid \Pi(u(t_0),v(t_0),w(t_0))=0,\ \Pi(u,v,w)>0,\forall t\in[0,t_0)\}. \tag{2.12}$$

Then, by (2.9), (2.10), we have for all $(u,v,w)\in Y$,

$$\begin{aligned}\Lambda(u,v,w) &= \frac{q-1}{2(q+1)}\Big[\left(1-\int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2+\left(1-\int_0^t \varpi_2(s)\,ds\right)\|v\|_{\mathcal{H}}^2\\ &+\left(1-\int_0^t \varpi_3(s)\,ds\right)\|w\|_{\mathcal{H}}^2\Big]\\ &+\frac{q-1}{2(q+1)}\Big[(\varpi_1\circ u)+(\varpi_2\circ v)+(\varpi_3\circ w)\Big]+\frac{1}{q+1}\Pi(u,v,w)\\ &\geq \frac{q-1}{2(q+1)}\Big[l\,\|u\|_{\mathcal{H}}^2+m\,\|v\|_{\mathcal{H}}^2+\nu\,\|w\|_{\mathcal{H}}^2+(\varpi_1\circ u)+(\varpi_2\circ v)+(\varpi_3\circ w)\Big].\end{aligned}$$

Owing to (2.3), it follows for $(u,v,w)\in Y$

$$\begin{aligned}l\,\|u\|_{\mathcal{H}}^2+m\,\|v\|_{\mathcal{H}}^2+\nu\,\|w\|_{\mathcal{H}}^2 &\leq \frac{2(q+1)}{q-1}\Lambda(u,v,w)\\ &\leq \frac{2(q+1)}{q-1}\mathcal{E}(t)\\ &\leq \frac{2(q+1)}{q-1}\mathcal{E}(0).\end{aligned} \tag{2.13}$$

By (1.18), (3.1) we have

$$(q+1)\int_{\mathbb{R}^n}\mathcal{G}(u(t_0),v(t_0),w(t_0))$$

$$
\begin{aligned}
&\leq \eta\Big(l\|u(t_0)\|_{\mathcal{H}}^2 + m\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2\Big)^{(q+1)/2}\\
&\leq \eta\Big(\frac{2(q+1)}{q-1}E(0)\Big)^{(q-1)/2}(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2)\\
&\leq \gamma(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2)\\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2\\
&+ \Big(1-\int_0^{t_0}\varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2\\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2\\
&+ \Big(1-\int_0^{t_0}\varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2\\
&+ (\varpi_1\circ u) + (\varpi_2\circ v) + (\varpi_3\circ w), \qquad (2.14)
\end{aligned}
$$

hence $\Pi(u(t_0), v(t_0), w(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u(t_0), v(t_0), w(t_0)) = 0$. Thus $\Pi(u, v, w) > 0,\ \forall t > 0$. □

3. Decay estimates

The decay rate for solution is given in the next Theorem

Theorem 3.1. *(Decay of solution) Let (1.2)–(1.17) and (1.11)–(1.14) hold. Under condition (2.1) and*

$$\gamma = \eta\Big(\frac{2(q+1)}{q-1}\mathcal{E}(0)\Big)^{(q-1)/2} < 1, \tag{3.1}$$

there exists $t_0 > 0$ depending only on $\varpi_1, \varpi_2, \varpi_3$, λ_1 and $\chi'(0)$ such that

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0)\exp\bigg(-\int_{t_0}^{t}\frac{\varpi(s)}{1-\varpi_0(t)}\bigg), \tag{3.2}$$

holds for all $t \geq t_0$.

Proof. (*Of Theorem 3.1.*) By (1.18) and (2.13), we have for $t \geq 0$

$$0 < l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \leq \frac{2(q+1)}{q-1}\mathcal{E}(t). \tag{3.3}$$

Let

$$I(t) = \frac{\varpi(t)}{1-\varpi_0(t)},$$

where ϖ and ϖ_0 defined in (1.12) and (1.13).
Noting that $\lim\limits_{t\to+\infty}\varpi(t) = 0$ by (1.11)–(1.13), we have

$$\lim_{t\to+\infty} I(t) = 0,\ \ I(t) > 0,\ \forall t \geq 0.$$

Then we take $t_0 > 0$ such that

$$0 < \frac{1}{2} I(t) < \chi'(0),$$

with (1.14) for all $t > t_0$. Due to (2.3), we have

$$\begin{aligned}
\mathcal{E}(t) &\le \frac{1}{2}\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta} + \|w_t\|^2_{L^2_\vartheta} \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\, ds\right) \|u\|^2_{\mathcal{H}} + \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\, ds\right) \|v\|^2_{\mathcal{H}} \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_3(s)\, ds\right) \|w\|^2_{\mathcal{H}} \\
&\le \frac{1}{2}\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta} + \|w_t\|^2_{L^2_\vartheta} \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \frac{1}{2}(1 - \varpi_0(t))[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}].
\end{aligned}$$

Then, by definition of $I(t)$, we have

$$\begin{aligned}
I(t)\mathcal{E}(t) &\le \frac{1}{2} I(t)\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta} + \|w_t\|^2_{L^2_\vartheta} \Big) \\
&+ \frac{1}{2}\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}] \\
&+ \frac{1}{2} I(t)[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)],
\end{aligned} \tag{3.4}$$

and Lemma 2.3, we have for all $t_1, t_2 \ge 0$

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
&\le -\frac{1}{2}\int_{t_1}^{t_2} \Big(\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}] \Big)\, dt \\
&+ \frac{1}{2}\int_{t_1}^{t_2} ((\varpi_1' \circ u) + (\varpi_2' \circ v) + (\varpi_3' \circ w))\, dt \\
&- \alpha \int_{t_1}^{t_2} \Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta} + \|w_t\|^2_{L^2_\vartheta} \Big) dt,
\end{aligned}$$

then,

$$\begin{aligned}
\mathcal{E}'(t) &\le -\frac{1}{2}\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}] \\
&+ \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v) + (\varpi_3' \circ w)] \\
&- \alpha\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta} + \|w_t\|^2_{L^2_\vartheta} \Big),
\end{aligned} \tag{3.5}$$

Finally, $\forall t \ge t_0$, we have

$$\mathcal{E}'(t) + I(t)\mathcal{E}(t)$$

$$
\begin{aligned}
&\le \Big(\frac{1}{2}I(t)-\alpha\Big)\Big(\|u_t\|^2_{L^2_\vartheta}+\|v_t\|^2_{L^2_\vartheta}+\|w_t\|^2_{L^2_\vartheta}\Big)\\
&+ \frac{1}{2}[(\varpi_1'\circ u)+(\varpi_2'\circ v)+(\varpi_3'\circ w)]\\
&+ \frac{1}{2}I(t)((\varpi_1\circ u)+(\varpi_2\circ v)+(\varpi_3\circ w)),
\end{aligned}
$$

and we can choose $t_0 > 0$ large enough such that

$$\frac{1}{2}I(t) < \alpha,$$

then

$$
\begin{aligned}
&\mathcal{E}'(t)+I(t)\mathcal{E}(t)\\
&\le \frac{1}{2}\int_0^t \{\varpi_1'(t-\tau)+I(t)\varpi_2(t-\tau)\}\,\|u(t)-u(\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \{\varpi_2'(t-\tau)+I(t)\varpi_2(t-\tau)\}\,\|v(t)-v(\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \{\varpi_3'(t-\tau)+I(t)\varpi_3(t-\tau)\}\,\|w(t)-w(\tau)\|^2_{\mathcal{H}}\,d\tau\\
&\le \frac{1}{2}\int_0^t \{\varpi_1'(\tau)+I(t)\varpi_1(\tau)\}\,\|u(t)-u(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \{\varpi_2'(\tau)+I(t)\varpi_2(\tau)\}\,\|v(t)-v(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \{\varpi_3'(\tau)+I(t)\varpi_3(\tau)\}\,\|w(t)-w(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&\le \frac{1}{2}\int_0^t \Big\{-\chi\Big(\varpi_1(\tau)\Big)+\chi'(0)\varpi_1(\tau)\Big\}\,\|u(t)-u(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \Big\{-\chi\Big(\varpi_2(\tau)\Big)+\chi'(0)\varpi_2(\tau)\Big\}\,\|v(t)-v(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&+ \frac{1}{2}\int_0^t \Big\{-\chi\Big(\varpi_3(\tau)\Big)+\chi'(0)\varpi_3(\tau)\Big\}\,\|w(t)-w(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\
&\le 0,
\end{aligned}
$$

by the convexity of χ and (1.14), we have

$$\chi(\xi) \ge \chi(0) + \chi'(0)\xi = \chi'(0)\xi.$$

Then

$$\mathcal{E}(t) \le \mathcal{E}(t_0)\exp\left(-\int_{t_0}^t I(s)ds\right),$$

which completes the proof. □

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Conflict of interest

The author agrees with the contents of the manuscript, and there is no conflict of interest among the author.

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