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Research article

Approximate iterative sequences for positive solutions of a Hadamard type fractional differential system involving Hadamard type fractional derivatives

Yaohong Li[1], Jiafa Xu[2,*] and Honglin Luo[2]

[1] School of Mathematics and Statistics, Suzhou University, Suzhou 234000, China
[2] School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

* **Correspondence:** Email: 20150028@cqnu.edu.cn.

Abstract: In this paper, we focus on a class of Hadamard type fractional differential system involving Hadamard type fractional derivatives on an infinite interval. By utilizing the monotone iterative technique and Banach's contraction mapping principle, some explicit monotone iterative sequences for approximating the extreme positive solutions and the unique positive solution for the system are constructed.

Keywords: monotone iterative technique; fractional differential equation; Hadamard type fractional derivative; infinite interval
Mathematics Subject Classification: 34A05, 34B18, 26A33

1. Introduction

Fractional derivative extends the classical integer order derivative to an arbitrary order case. Fractional order differential equations can better describe various phenomenon than integer order differential equations in many complex and widespread fields of engineering and science such as biology, physics, finance, electrical circuits, signal processing, control theory, and diffusion processes, there has been a rapid growth in the number of fractional differential equations from both theoretical and applied perspectives, see [1–18] and references cited therein.

Note that most of the results on the current works are based on Riemann-Liouville type and Caputo type fractional differential equations in the past ten years. Hadamard type fractional derivative is first introduced in 1892 [19], which contains logarithmic function of arbitrary exponent in the kernel of integral appearing in its definition. Hadamard type integrals arise in the formulation of many problems in mechanics such as in fracture analysis. For details and applications of Hadamard type fractional derivative and integral, see [3, 20–22]. Recently, more and more scholars pay special attention to

Hadamard type fractional differential equations on the finite interval [23–28]. For example, by applying Leray-Schauder's alternative and Banach's contraction principle, Ahmad and Ntouyas [29] established the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with a fully Hadamard type integral boundary conditions:

$$\begin{cases} {}^{H}D^{\alpha}u(t) = f(t,u(t),v(t)),\ 1 < t < e,\ 1 < \alpha \le 2, \\ {}^{H}D^{\beta}v(t) = g(t,v(t),u(t)),\ 1 < t < e,\ 1 < \beta \le 2, \\ u(1) = 0,\ u(e) = {}^{H}I^{r}u(\sigma_1) = \dfrac{1}{\Gamma(r)}\displaystyle\int_1^{\sigma_1} (\log\sigma_1 - \log s)^{r-1}\, u(s)\frac{\mathrm{d}s}{s}, \\ v(1) = 0,\ v(e) = {}^{H}I^{r}u(\sigma_2) = \dfrac{1}{\Gamma(r)}\displaystyle\int_1^{\sigma_2} (\log\sigma_2 - \log s)^{r-1}\, v(s)\frac{\mathrm{d}s}{s}, \end{cases} \tag{1.1}$$

where $\gamma > 0, 1 < \sigma_1, \sigma_2 < e, {}^{H}D^{(\cdot)}$ are the Hadamard type fractional derivative and ${}^{H}I^{r}$ is the Hadamard type fractional integral of order r, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R}$ are given continuous functions.

In [30] by means of comparison principle and the monotone iterative technique combined with the method of upper and lower solutions, Yang investigated the extremal iterative solutions for the following coupled system of nonlinear Hadamard type fractional differential equations:

$$\begin{cases} ({}^{H}D^{\alpha}_{a^+}x)(t) = f(t,x(t),y(t)),\ 0 < \alpha \le 1,\ a < t \le b, \\ ({}^{H}D^{\alpha}_{a^+}y)(t) = g(t,x(t),y(t)),\ 0 < \alpha \le 1,\ a < t \le b, \\ ({}^{H}J^{1-\alpha}_{a^+}x)(a^+) = x^*,\ ({}^{H}J^{1-\alpha}_{a^+}y)(a^+) = y^*, \end{cases} \tag{1.2}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), {}^{H}D^{\alpha}_{a^+}$ and ${}^{H}J^{\alpha}_{a^+}$ are the left-sided Hadamard type fractional derivative and Hadamard type fractional integral of order α, respectively.

On the other hand, some authors have also focused on the existence of solutions for Hadamard type fractional differential equations on the infinite intervals, see [31–36] and the references quoted therein. In another study [37], by applying standard fixed point theorems, Tariboon et al. obtained the existence of positive solutions of the Hadamard type fractional differential system with coupled integral boundary conditions:

$$\begin{cases} {}^{H}D^{p}x(t) + f(t,x(t),y(t)) = 0,\ 1 < p \le 2, t \in [1,+\infty), \\ {}^{H}D^{q}y(t) + g(t,x(t),y(t)) = 0,\ 1 < q \le 2, t \in [1,+\infty), \\ x(0) = 0,\ {}^{H}D^{p-1}x(+\infty) = \sum_{i=1}^{m} \lambda_i {}^{H}I^{\alpha_i}y(\eta_{)}, \\ y(0) = 0,\ {}^{H}D^{q-1}y(+\infty) = \sum_{j=1}^{n} \sigma_j {}^{H}I^{\beta_j}x(\xi), \end{cases} \tag{1.3}$$

In [38] Zhang and Liu focused on a class of Hadamard type fractional differential equation with nonlocal boundary conditions on an infinite interval:

$$\begin{cases} {}^HD^{\alpha}_{1^+}x(t) + a(t)f(t, x(t)) = 0,\ 2 < \alpha \le 3, t \in (1, +\infty), \\ x(0) = x'(0) = 0,\ {}^HD^{\alpha-1}_{1^+}x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^HI^{\beta_i}_{1^+}x(\eta) + b\sum_{j=1}^{n} \sigma_j x(\xi_j), \end{cases} \tag{1.4}$$

where ${}^HD^{\alpha}_{1^+}, {}^HI^{\beta_i}_{1^+}$ are the Hadamard type fractional derivative of order α and the Hadamard type fractional integral of order $\beta_i > 0\ \ (i = 1, 2, 3, \cdots, m), 1 < \eta < \xi_1 < \xi_2 < \cdots < \xi_n$. $b, \alpha_i, \sigma_j \ge 0\ \ (i = 1, 2, 3, \cdots, m; j = 1, 2, 3, \cdots, n)$ are given constants satisfy certain prior conditions. By using various fixed point methods, the authors not only obtained the existence and uniqueness of solutions, but also the iterative sequences of approximate solutions.

Motivated by the mentioned results above, a nature and meaningful question is if we know the existence of solution for the following Hadamard type fractional differential system (1.5), how can we seek it? This idea lead us to develop the research of approximate sequences of positive solutions for the following Hadamard type fractional differential system with Hadamard type fractional integral boundary conditions:

$$\begin{cases} {}^HD^{\alpha_1}u(t) + f_1(t, u(t), v(t), {}^HD^{\alpha_1-1}u(t), {}^HD^{\alpha_2-1}v(t)) = 0,\ 1 < \alpha_1 \le 2, t \in J, \\ {}^HD^{\alpha_2}v(t) + f_2(t, v(t), u(t), {}^HD^{\alpha_1-1}u(t), {}^HD^{\alpha_2-1}v(t)) = 0,\ 1 < \alpha_2 \le 2, t \in J, \\ u(0) = 0, {}^HD^{\alpha_1-1}u(+\infty) = \sum_{i=1}^{m_1} \lambda_{1i} {}^HI^{\beta_{1i}}u(\eta_1), \eta_1 \in J, \\ v(0) = 0, {}^HD^{\alpha_2-1}v(+\infty) = \sum_{i=1}^{m_2} \lambda_{2i} {}^HI^{\beta_{2i}}v(\eta_2), \eta_2 \in J \end{cases} \tag{1.5}$$

where $J = [1, +\infty), {}^HD^{\alpha_j}, {}^HI^{\beta_{ji}}$ are the common Hadamard type fractional derivative of order α_j and the Hadamard type fractional integral of order $\beta_{ji} > 0, f_j \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+), \lambda_{ji} > 0$ are given constants and satisfy $\Omega_j = \Gamma(\alpha_j) - \sum_{i=1}^{m_j} \frac{\lambda_{ji}\Gamma(\alpha_j)}{\Gamma(\alpha_j+\beta_{ji})}(\log \eta_j)^{\alpha_j+\beta_{ji}-1} > 0, j = 1, 2; i = 1, 2, \cdots, m_j, m_j \in N^+$.

In this paper, we emphasize that the nonlinearity terms f_j of the system (1.5) involve multiple unknown functions and the lower-order Hadamard type fractional derivative of multiple unknown functions. By utilizing the monotone iterative method, we establish some explicit monotone iterative sequences for approximating the extreme positive solutions and the unique positive solution, which are more valuable and interesting than just constructing the existence of solutions. Further we extend the iterative methods that are often used in a single equation to the system which is different from [31, 34, 38–42]. Finally we give some examples to verify the application of main results.

2. Preliminaries

First we recall some Hadamard type fractional calculus definitions and lemmas that are helpful to the proof of main results.

Definition 2.1 (see [1]). The Hadamard type fractional derivative of order q for a integrable function $g : [1, \infty) \to \mathbb{R}$ is given by

$${}^HD^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{ds}{s}, n - 1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 (see [1]). The Hadamard type fractional integral of order q for a integrable function g is given by

$$ {}^{H}I^{q} g(t) = \frac{1}{\Gamma(q)} \int_1^t (\log t - \log s)^{q-1}\, g(s) \frac{\mathrm{d}s}{s}, q > 0, $$

provided the integral exists.

Lemma 2.1 (see [1,32]). If $a, \alpha, \beta > 0$, then

$$ ({}^{H}D_a^{\alpha} (\log t - \log a)^{\beta - 1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\log x - \log a)^{\beta - \alpha - 1}. $$

Lemma 2.2 Let $h_j \in C[1, \infty)$ with $0 < \int_1^{\infty} h_j(s) \frac{\mathrm{d}s}{s} < \infty$ and $\Omega_j > 0, j = 1, 2$, then the following Hadamard type fractional differential system with Hadamard type fractional integral boundary conditions

$$ \begin{cases} {}^{H}D^{\alpha_1} u(t) + h_1(t) = 0, 1 < \alpha_1 \le 2, t \in J, \\ {}^{H}D^{\alpha_2} v(t) + h_2(t) = 0, 1 < \alpha_2 \le 2, t \in J, \\ u(0) = 0, {}^{H}D^{\alpha_1 - 1} u(+\infty) = \sum_{i=1}^{m_1} \lambda_{1i} {}^{H}I^{\beta_{1i}} u(\eta_1), \\ v(0) = 0, {}^{H}D^{\alpha_2 - 1} v(+\infty) = \sum_{i=1}^{m_2} \lambda_{2i} {}^{H}I^{\beta_{2i}} v(\eta_2), \end{cases} \tag{2.1} $$

has a unique solution:

$$ \begin{cases} u(t) = \displaystyle\int_1^{+\infty} G_1(t, s) h_1(s) \frac{\mathrm{d}s}{s}, \\ v(t) = \displaystyle\int_1^{+\infty} G_2(t, s) h_2(s) \frac{\mathrm{d}s}{s}, \end{cases} \tag{2.2} $$

where

$$ G_j(t, s) = g_j(t, s) + \sum_{i=1}^{m_j} \frac{\lambda_{ji} (\log t)^{\alpha_j - 1}}{\Omega_j \Gamma(\alpha_j + \beta_{ji})} g_{ji}(\eta_j, s), j = 1, 2, \tag{2.3} $$

and

$$ g_j(t, s) = \frac{1}{\Gamma(\alpha_j)} \begin{cases} (\log t)^{\alpha_j - 1} - (\log t - \log s)^{\alpha_j - 1}, 1 \le s \le t < +\infty, \\ (\log t)^{\alpha_j - 1}, 1 \le t \le s < +\infty, \end{cases} \tag{2.4} $$

$$ g_{ji}(\eta_j, s) = \begin{cases} (\log \eta_j)^{\alpha_j + \beta_{ji} - 1} - (\log \eta_j - \log s)^{\alpha_j + \beta_{ji} - 1}, 1 \le s \le \eta_j < +\infty, \\ (\log \eta_j)^{\alpha_j + \beta_{ji} - 1}, 1 \le \eta_j \le s < +\infty. \end{cases} \tag{2.5} $$

Proof. Utilizing Lemmas 2.5 of [32], we can derive directly the above results.

Remark 2.1 Applying definition 2.1 of Hadamard type fractional derivative and Lemma 2.1, from (2.2), (2.3), (2.4) and (2.5), by a simple computation, one can obtain

$$ \begin{cases} {}^{H}D^{\alpha_1 - 1} u(t) = \displaystyle\int_1^{+\infty} G_1^*(t, s) h_1(s) \frac{\mathrm{d}s}{s}, \\ {}^{H}D^{\alpha_2 - 1} v(t) = \displaystyle\int_1^{+\infty} G_2^*(t, s) h_2(s) \frac{\mathrm{d}s}{s}, \end{cases} $$

where

$$G_j^*(t,s) = k(t,s) + \sum_{i=1}^{m_j} \frac{\lambda_{ji}\Gamma(\alpha_j)}{\Omega_j\Gamma(\alpha_j+\beta_{ji})} g_{ji}(\eta_j,s),\ j=1,2, \tag{2.6}$$

and

$$k(t,s) = \begin{cases} 0, 1 \le s \le t < +\infty, \\ 1, 1 \le t \le s < +\infty. \end{cases} \tag{2.7}$$

For convenience, we introduce the following notations:

$$\Lambda_j = \frac{1}{\Gamma(\alpha_j)} + \sum_{i=1}^{m_j} \frac{\lambda_{ji}\Gamma(\alpha_j)}{\Omega_j\Gamma(\alpha_j+\beta_{ji})}(\log\eta_j)^{\alpha_j+\beta_{ji}-1},\ \Xi_j = 1 + \sum_{i=1}^{m_j} \frac{\lambda_{ji}\Gamma(\alpha_j)}{\Omega_j\Gamma(\alpha_j+\beta_{ji})}(\log\eta_j)^{\alpha_j+\beta_{ji}-1},\ j=1,2.$$

Lemma 2.3 (see [32]). The Green's function $G_j(t,s)$ defined by (2.3) has the following properties:
(A1): $G_j(t,s) \ge 0$ and $G_j(t,s)$ are continuous for all $(t,s) \in J \times J$, $j = 1,2$;
(A2): $\frac{G_j(t,s)}{1+(\log t)^{\alpha_j}} \le \Lambda_j$ for all $(t,s) \in J \times J$, $j = 1,2$.

Remark 2.2 The Green's function $G_j(t,s)$ and $G_j^*(t,s)$ defined by (2.3) and (2.6) still have the following properties:
(B1): $G_j(t,s) \le \Lambda_j(\log t)^{\alpha_j-1}$ for $(t,s) \in J \times J$, $j = 1,2$;
(B2): $0 \le G_j^*(t,s) \le \Xi_j$ for $(t,s) \in J \times J$, $j = 1,2$.

Proof. From (2.4) and (2.5), it is obvious that

$$g_j(t,s) \le \frac{(\log t)^{\alpha_j-1}}{\Gamma(\alpha_j)},\ g_{ji}(\eta_j,s) \le (\log\eta_j)^{\alpha_j+\beta_{ji}-1}, (t,s) \in J \times J,$$

then

$$G_j(t,s) \le \Lambda_j(\log t)^{\alpha_j-1}, (t,s) \in J \times J,$$

so (B1) holds. And from (2.6) and (2.7), it is easy to that (B2) holds.

Lemma 2.4 (see [32,33]). Let $U \subset X$ be a bounded set. Then U is a relatively compact in X if the following conditions hold:

(i) For any $u \in U$, $\frac{u(t)}{1+(\log t)^{\alpha-1}}$ and ${}^H D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J;

(ii) For any $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 1$ such that $|\frac{u(t_1)}{1+(\log t_1)^{\alpha-1}} - \frac{u(t_2)}{1+(\log t_2)^{\alpha-1}}| < \varepsilon$ and $|{}^H D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)| < \varepsilon$ for any $t_1, t_2 \ge C$ and $u \in U$.

Next we present some assumptions that will play an important role in subsequent discussion.

(C1) $\Omega_j = \Gamma(\alpha_j) - \sum_{i=1}^{m_j} \frac{\lambda_{ji}\Gamma(\alpha_j)}{\Gamma(\alpha_j+\beta_{ji})}(\log\eta_j)^{\alpha_j+\beta_{ji}-1} > 0$ and $f_j(t,0,0,0,0) \not\equiv 0, \forall t \in J,\ j = 1,2$;

(C2) There exist some nonnegative integrable functions $a_{j0}(t), a_{jk}(t)$ defined on J and some constants $0 < \gamma_{jk} < 1$ satisfy

$$|f_j(t,u_1,u_2,u_3,u_4)| \le a_{j0}(t) + \sum_{k=1}^{4} a_{jk}(t)|u_k|^{\gamma_{jk}}, \forall t \in J,\ u_k \in \mathbb{R}, j = 1,2,\ k = 1,2,3,4,$$

and

$$\int_1^{+\infty} a_{j0}(t)\frac{\mathrm{d}t}{t} = a_{j0}^* < +\infty, \int_0^{+\infty} a_{j1}(t)\Big[1 + (\log t)^{\alpha_j-1}\Big]^{\gamma_{j1}}\frac{\mathrm{d}t}{t} = a_{j1}^* < +\infty,$$

$$\int_1^{+\infty} a_{j2}(t)\Big[1 + (\log t)^{\alpha_j-1}\Big]^{\gamma_{j2}}\frac{\mathrm{d}t}{t} = a_{j2}^* < +\infty, \int_1^{+\infty} a_{j3}(t)\frac{\mathrm{d}t}{t} = a_{j3}^* < +\infty,$$

$$\int_1^{+\infty} a_{j4}(t)\frac{\mathrm{d}t}{t} = a_{j4}^* < +\infty, j = 1, 2;$$

(C3) Functions f_j are nondecreasing with respect to the second, third, fourth and last variables on $J, j = 1, 2$;

(C4) There exist some nonnegative integrable functions $b_{jk}(t)(j = 1, 2,\ k = 1, 2, 3, 4)$ defined on J satisfy

$$|f_j(t, u_1, u_2, u_3, u_4) - f_j(t, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)| \le \sum_{k=1}^{4} b_{jk}(t)|u_k - \bar{u}_k|, \forall t \in J, u_k, \bar{u}_k \in \mathbb{R},$$

and

$$\int_1^{+\infty} b_{j1}(t)\Big[1 + (\log t)^{\alpha_j-1}\Big]\frac{\mathrm{d}t}{t} = b_{j1}^* < +\infty, \int_1^{+\infty} b_{j2}(t)\Big[1 + t^{\alpha_j-1}\Big]\frac{\mathrm{d}t}{t} = b_{j2}^* < +\infty,$$

$$\int_1^{+\infty} b_{j3}(t)\frac{\mathrm{d}t}{t} = b_{j3}^* < +\infty,\ \int_1^{+\infty} b_{j4}(t)\frac{\mathrm{d}t}{t} = b_{j4}^* < +\infty, \int_1^{+\infty} |f_j(t, 0, 0, 0, 0)|\frac{\mathrm{d}t}{t} = \varrho_j < +\infty.$$

3. Main results

In this paper, we will use two Banach spaces which are define by

$$X = \{u \in C(J, \mathbb{R}), {}^H D^{\alpha_1-1}u \in C(J, \mathbb{R})| \sup_{t\in J} \frac{|u(t)|}{1 + (\log t)^{\alpha_1-1}} < +\infty, \sup_{t\in J} |{}^H D^{\alpha_1-1}u(t)| < +\infty\}$$

equipped with the norm $\|u\|_X = \max\{\|u\|_1, \|{}^H D^{\alpha_1-1}u\|\}$, where $\|u\|_1 = \sup_{t\in J} \frac{|u(t)|}{1+(\log t)^{\alpha_1-1}}$ and $\|{}^H D^{\alpha_1-1}u\| = \sup_{t\in J} |{}^H D^{\alpha_1-1}u(t)|$, and

$$Y = \{v \in C(J, \mathbb{R}), {}^H D^{\alpha_2-1}v \in C(J, \mathbb{R})| \sup_{t\in J} \frac{|v(t)|}{1 + (\log t)^{\alpha_2-1}} < +\infty, \sup_{t\in J} |{}^H D^{\alpha_2-1}v(t)| < +\infty\}$$

equipped with the norm $\|u\|_Y = \max\{\|v\|_2, \|{}^H D^{\alpha_2-1}v\|\}$, where $\|v\|_2 = \sup_{t\in J} \frac{|v(t)|}{1+t^{\alpha_2-1}}$ and $\|{}^H D^{\alpha_2-1}v\| = \sup_{t\in J} |{}^H D^{\alpha_2-1}v(t)|$. Then the space $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces which can be shown similarly to Lemma 2.7 of the literature [32]. Moreover, the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space with the norm

$$\|\cdot\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}.$$

Lemma 3.1 If assumption (C2) holds, then for any $(u, v) \in X \times Y$,

$$\int_1^{+\infty} |f_i(t, u(t), v(t), {}^H D^{\alpha_1-1}u(t), {}^H D^{\alpha_2-1}v(t))|\frac{\mathrm{d}t}{t} \le a_{j0}^* + \sum_{k=1}^{4} a_{jk}^*\|(u, v)\|_{X\times Y}^{\gamma_{ik}},\ j = 1, 2.$$

Proof. For any $(u,v)\in X\times Y$, by assumption (C2), one can obtain

$$\begin{aligned}
&|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))|\\
\le& a_{j0}(t)+a_{j1}(t)|u(t)|^{\gamma_{j1}}+a_{j2}(t)|v(t)|^{\gamma_{j2}}+a_{j3}(t)|{}^HD^{\alpha_1-1}u(t))|^{\gamma_{j3}}+a_{j4}(t)|{}^HD^{\alpha_2-1}v(t))|^{\gamma_{j4}}\\
\le& a_{j0}(t)+a_{j1}(t))[1+(\log t)^{\alpha_1-1}]^{\gamma_{j1}}\frac{|u(t)|^{\gamma_{j1}}}{[1+(\log t)^{\alpha_1-1}]^{\gamma_{j1}}}\\
&+a_{j2}(t))[1+(\log t)^{\alpha_2-1}]^{\gamma_{j2}}\frac{|v(t)|^{\gamma_{j2}}}{[1+(\log t)^{\alpha_2-1}]^{\gamma_{j2}}}\\
&+a_{j3}(t)|{}^HD^{\alpha_1-1}u(t)|^{\gamma_{j3}}+a_{j4}(t)|{}^HD^{\alpha_2-1}v(t)|^{\gamma_{j4}}\\
\le& a_{j0}(t)+a_{j1}(t))[1+(\log t)^{\alpha_1-1}]^{\gamma_{j1}}\|u\|_X^{\gamma_{j1}}+a_{j2}(t))[1+(\log t)^{\alpha_2-1}]^{\gamma_{j2}}\|v\|_Y^{\gamma_{j2}}\\
&+a_{j3}(t)\|u\|_X^{\gamma_{j3}}+a_{j4}(t)\|v\|_Y^{\gamma_{j4}},\ \ j=1,2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_1^{+\infty}|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))|\frac{\mathrm{d}t}{t}\\
\le&\int_1^{+\infty}\Big(a_{j0}(t)+a_{j1}(t))[1+(\log t)^{\alpha_1-1}]^{\gamma_{j1}}\|u\|_X^{\gamma_{j1}}+a_{j2}(t))[1+(\log t)^{\alpha_2-1}]^{\gamma_{j2}}\|v\|_Y^{\gamma_{j2}}\\
&+a_{j3}(t)\|u\|_X^{\gamma_{j3}}+a_{j4}(t)\|v\|_Y^{\gamma_{j4}}\Big)\frac{\mathrm{d}t}{t}\\
\le& a_{j0}^*+a_{j1}^*\|u\|_X^{\gamma_{j1}}+a_{i2}^*\|v\|_Y^{\gamma_{j2}}+a_{j3}^*\|u\|_X^{\gamma_{j3}}+a_{j4}^*\|v\|_Y^{\gamma_{j4}}\\
\le& a_{j0}^*+\sum_{k=1}^4 a_{jk}^*\|(u,v)\|_{X\times Y}^{\gamma_{jk}},\ \ j=1,2.
\end{aligned}$$

Lemma 3.2 If assumption (C4) holds, then for any $(u,v)\in X\times Y$,

$$\int_1^{+\infty}|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))|\frac{\mathrm{d}t}{t}\le\sum_{k=1}^4 b_{jk}^*\|(u,v)\|_{X\times Y}+\varrho_j,\ \ j=1,2.$$

Proof. For any $(u,v)\in X\times Y$, by assumption (C4), one can obtain

$$\begin{aligned}
&|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))|\\
=&|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))-f_j(t,0,0,0,0)+f_i(t,0,0,0,0)|\\
\le&|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))-f_j(t,0,0,0,0)|+|f_j(t,0,0,0,0)|\\
\le& b_{j1}(t)[1+(\log t)^{\alpha_1-1}]\frac{|u(t)|}{[1+(\log t)^{\alpha_1-1}]}+b_{j2}(t)[1+(\log t)^{\alpha_2-1}]\frac{|v(t)|}{[1+(\log t)^{\alpha_2-1}]}\\
&+b_{j3}(t)|{}^HD^{\alpha_1-1}u(t)|+b_{j4}(t)|{}^HD^{\alpha_2-1}v(t)|+|f_j(t,0,0,0,0)|\\
\le& b_{j1}(t)[1+(\log t)^{\alpha_1-1}]\|u\|_X+b_{j2}(t)[1+(\log t)^{\alpha_2-1}]\|v\|_Y+b_{j3}(t)\|u\|_X+b_{j4}(t)\|v\|_Y\\
&+|f_j(t,0,0,0,0)|,\ \ j=1,2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_1^{+\infty}|f_j(t,u(t),v(t),{}^HD^{\alpha_1-1}u(t),{}^HD^{\alpha_2-1}v(t))|\frac{\mathrm{d}t}{t}\le& b_{j1}^*\|u\|_X+b_{j2}^*\|v\|_Y+b_{j3}^*\|u\|_X+b_{j4}^*\|v\|_Y+\varrho_j\\
\le&\sum_{k=1}^4 b_{jk}^*\|(u,v)\|_{X\times Y}+\varrho_j,\ \ j=1,2.
\end{aligned}$$

Define two cones $P_1 = \{u \in X | u(t) \geq 0, {}^H D^{\alpha_1 - 1} u(t) \geq 0, t \in J\}$ and $P_2 = \{v \in Y | v(t) \geq 0, {}^H D^{\alpha_2 - 1} v(t) \geq 0, t \in J\}$, then $P_1 \times P_2 \subset X \times Y$ is also a cone by $P_1 \times P_2 = \{(u, v) \in X \times Y | u(t) \geq 0, v(t) \geq 0, {}^H D^{\alpha_1 - 1} u(t) \geq 0, {}^H D^{\alpha_2 - 1} v(t) \geq 0, t \in J\}$.

From Lemma 2.2, we can know that the system (2.2) is equivalent to the following system of Hammerstein-type integral equations:

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_1^{+\infty} G_1(t, s) f_{1(u,v)}(s) \frac{\mathrm{d}s}{s} \\ \displaystyle\int_1^{+\infty} G_2(t, s) f_{2(u,v)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix} := \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix}, \ \text{ for } (u, v) \in P_1 \times P_2, t \in J, \tag{3.1}$$

and for convenience, we set

$$\begin{aligned} f_{1(u,v)}(s) &= f_1\big(s, u(s), v(s), {}^H D^{\alpha_1 - 1} u(s), {}^H D^{\alpha_2 - 1} v(s)\big), \\ f_{2(u,v)}(s) &= f_2\big(s, u(s), v(s), {}^H D^{\alpha_1 - 1} u(s), {}^H D^{\alpha_2 - 1} v(s)\big). \end{aligned}$$

Therefore one can define an operator $T : P_1 \times P_2 \to P_1 \times P_2$ as follows:

$$T(u, v)(t) = (T_1, T_2)(u, v)(t), \ \text{ for } (u, v) \in P_1 \times P_2, t \in J. \tag{3.2}$$

By Remark 2.1, one can also define

$$\begin{pmatrix} {}^H D^{\alpha_1 - 1} T_1(u, v)(t) \\ {}^H D^{\alpha_2 - 1} T_2(u, v)(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_1^{+\infty} G_1^*(t, s) f_{1(u,v)}(s) \frac{\mathrm{d}s}{s} \\ \displaystyle\int_1^{+\infty} G_2^*(t, s) f_{2(u,v)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix}, \ \text{ for } (u, v) \in P_1 \times P_2, t \in J. \tag{3.3}$$

Therefore, if $(u, v) \in P_1 \times P_2 / (0, 0)$ is a fixed point of the operator T, then (u, v) is a positive solution for the Hadamard type fractional differential system (1.5). It is obvious that the system (1.5) has a positive solution if and only if the operator equation $(u, v) = T(u, v)$ has a positive fixed point in $P_1 \times P_2$, where T is given as (3.2). Next we will directly consider the existence of fixed points of the operator T.

Lemma 3.3 If assumption (C1), (C2) and (C3) hold, then the operator $T : P_1 \times P_2 \to P_1 \times P_2$ is completely continuous.

Proof. Due to $G_j(t, s) \geq 0, G_j^*(t, s) \geq 0$ and $f_j \geq 0$, we have $T_j(u, v)(t) \geq 0, {}^H D^{\alpha_j - 1} T_j(u, v)(t) \geq 0$, for any $(u, v) \in P_1 \times P_2, \ t \in J, \ j = 1, 2$, so it is easy to know $T : P_1 \times P_2 \to P_1 \times P_2$.

Next we show in four steps that the operator $T : P_1 \times P_2 \to P_1 \times P_2$ is completely continuous.

Step 1 Take $U = \{(u, v) | (u, v) \in P_1 \times P_2, \|(u, v)\|_{X \times Y} \leq M\}$. For any $(u, v) \in U$, by Lemma 2.3, Lemma 3.1 and Remark 2.2, one can obtain

$$\begin{aligned} \|T_1(u, v)\|_1 &= \sup_{t \in J} \Big| \int_1^{+\infty} \frac{G_1(t, s)}{1 + (\log t)^{\alpha_1 - 1}} f_{1(u,v)}(s) \frac{\mathrm{d}s}{s} \Big| \leq \Lambda_1 \int_1^{+\infty} |f_{1(u,v)}(s)| \frac{\mathrm{d}s}{s} \\ &\leq \Lambda_1 \Big(a_{10}^* + \sum_{k=1}^{4} a_{1k}^* \|(u, v)\|_{X \times Y}^{\gamma_{1k}} \Big) < \infty \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|{}^{H}D^{\alpha_1-1}T_1(u,v)\| &= \sup_{t\in J}\Big|\int_1^{\infty} G_1^*(t,s)f_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\Big| \le \Xi_1\int_1^{+\infty}|f_{1(u,v)}(s)|\frac{\mathrm{d}s}{s} \\ &\le \Xi_1\Big(a_{10}^* + \sum_{k=1}^{4} a_{1k}^*\|(u,v)\|_{X\times Y}^{\gamma_{1k}}\Big) < \infty. \end{aligned} \tag{3.5}$$

Thus

$$\|T_1(u,v)\|_X \le \max\{\Lambda_1,\Xi_1\}\Big(a_{10}^* + \sum_{k=1}^{4} a_{1k}^* M^{\gamma_{1k}}\Big).$$

Similarly

$$\|T_2(u,v)\|_Y \le \max\{\Lambda_2,\Xi_2\}\Big(a_{20}^* + \sum_{k=1}^{4} a_{2k}^* M^{\gamma_{2k}}\Big).$$

Then

$$\begin{aligned} \|T(u,v)\|_{X\times Y} &= \max\Big\{\|T_1(u,v)\|_X, \|T_2(u,v)\|_Y\Big\} \\ &\le \max\{\Lambda_1,\Xi_1,\Lambda_2,\Xi_2\}\max\Big(a_{10}^* + \sum_{k=1}^{4} a_{1k}^* M^{\gamma_{1k}}, a_{20}^* + \sum_{k=1}^{4} a_{2k}^* M^{\gamma_{2k}}\Big) < \infty. \end{aligned}$$

which implies that TU is uniformly bounded for any $(u,v)\in U$.

Step 2 Let $I \subset J$ be any compact interval. Then, for all $t_1, t_2 \in I, t_2 > t_1$ and $(u,v)\in U$, we have

$$\begin{aligned} \Big|\frac{T_1(u,v)(t_2)}{1+(\log t_2)^{\alpha_1-1}} - \frac{T_1(u,v)(t_1)}{1+(\log t)_1^{\alpha_1-1}}\Big| &\le \Big|\int_1^{+\infty}\Big(\frac{G_1(t_2,s)}{1+(\log t_2)^{\alpha_1-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{\alpha_1-1}}\Big)f_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\Big| \\ &\le \int_1^{+\infty}\Big|\frac{G_1(t_2,s)}{1+(\log t_2)^{\alpha_1-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{\alpha_1-1}}\Big|\Big|f_{1(u,v)}(s)\Big|\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.6}$$

Noticing that $G_1(t,s)/1+(\log t)^{\alpha_1-1}$ is uniformly continuous for any $(t,s)\in I\times I$. Moreover the function $G_1(t,s)/1+(\log t)^{\alpha_1-1}$ is only associated with t for $s \ge t$, which implies that $G_1(t,s)/1+(\log t)^{\alpha_1-1}$ is uniformly continuous on $I\times(J\setminus I)$. That is, for all $s\in J$ and $t_1,t_2\in I, \forall\epsilon>0, \exists\delta(\epsilon)>0$ if $|t_1-t_2|<\delta$ such that

$$\Big|\frac{G_1(t_2,s)}{1+(\log t_2)^{\alpha_1-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{\alpha_1-1}}\Big| < \epsilon. \tag{3.7}$$

By Lemma 3.1, for all $(u,v)\in U$, we have

$$\int_1^{+\infty}|f_{1(u,v)}(s)|\frac{\mathrm{d}s}{s} \le a_{10}^* + \sum_{k=1}^{4} a_{1k}^* M^{\gamma_{1k}} < \infty. \tag{3.8}$$

For all $t_1,t_2\in I, t_2>t_1$ and $(u,v)\in U$, together (3.6), (3.7) and (3.8) mean that

$$\forall\epsilon>0, \exists\delta(\epsilon)>0 \textit{ such that if } |t_1-t_2|<\delta \textit{ then } \Big|\frac{T_1(u,v)(t_2)}{1+(\log t_2)^{\alpha_1-1}} - \frac{T_1(u,v)(t_1)}{1+(\log t)_1^{\alpha_1-1}}\Big| < \epsilon.$$

That is, $T_1(u,v)(t)/1+(\log t)^{\alpha_1-1}$ is equicontinuous on I.

Note that

$${}^{H}D^{\alpha_1-1}T_1(u,v)(t) = \int_1^{+\infty} G_1^*(t,s)f_{1(u,v)}(s)\mathrm{d}s$$

and function $G_1^*(t,s) \in C(J\times J)$ is independent of t, which implies that ${}^HD^{\alpha_1-1}T_1(u,v)(t)$ is equicontinuous on I.

In the same way, one can easily show that $T_2(u,v)(t)/1+(\log t)^{\alpha_2-1}$ and $D^{\alpha_2-1}T_2(u,v)(t)$ are equicontinuous. Hence T_1 and T_2 are equicontinuous on I. Then the operator T is equicontinuous for all $(u,v)\in U$ on any compact interval I of J.

Step 3 Now we prove the operator T is equiconvergent at $+\infty$. Due to

$$\begin{aligned}\lim_{t\to+\infty}\frac{G_j(t,s)}{1+(\log t)^{\alpha_j-1}} &= \frac{1}{\Gamma(\alpha_j)}+\sum_{i=1}^{m_j}\frac{\lambda_{ji}\Gamma(\alpha_j)}{\Omega_j\Gamma(\alpha_j+\beta_{ji})}g_{ji}(\eta_j,s)\\ &\le\frac{1}{\Gamma(\alpha_j)}+\sum_{i=1}^{m_j}\frac{\lambda_{ji}\Gamma(\alpha_j)}{\Omega_j\Gamma(\alpha_j+\beta_{ji})}(\log\eta_j)^{\alpha_j+\beta_{ji}-1}<+\infty,\ j=1,2,\end{aligned}$$

one can infer that for any $\epsilon>0$, there exists a constant $C=C(\epsilon)>0$, for any $t_1,t_2\ge C$ and $s\in J$, such that

$$\Big|\frac{G_j(t_2,s)}{1+(\log t_2)^{\alpha_j-1}}-\frac{G_j(t_1,s)}{1+(\log t_1)^{\alpha_j-1}}\Big|<\epsilon,\ j=1,2,$$

with the help of Lemma 3.1 and (3.6), which mean that $T_j(u,v)(t)/1+(\log t)^{\alpha_j-1}(j=1,2)$ are equiconvergent at $+\infty$. Meanwhile function $G_j^*(t,s)(j=1,2)$ are independent of t, one can easily show that ${}^HD^{\alpha_j-1}T_j(u,v)(t)(j=1,2)$ are equiconvergent at $+\infty$.

From **Step 1**, **Step 2** and **Step 3**, Lemma 2.4 holds. So the operator T is relatively compact in $P_1\times P_2$.

Step 4 Finally we prove that the operator $T: P_1\times P_2\to P_1\times P_2$ is continuous. Set $(u_n,v_n),(u,v)\in P_1\times P_2$ and $(u_n,v_n)\to(u,v)(n\to\infty)$. So $\|(u_n,v_n)\|_{X\times Y}<+\infty, \|(u,v)\|_{X\times Y}<+\infty$. Similar to (3.4) and (3.5), one has

$$\|T_1(u_n,v_n)\|_1=\sup_{t\in J}\Big|\int_0^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{\alpha_1-1}}f_{1(u_n,v_n)}(s)\frac{\mathrm{d}s}{s}\Big|\le\Lambda_1\Big[a_{10}^*+\sum_{k=1}^{4}a_{1k}^*\|(u_n,v_n)\|_{X\times Y}^{\gamma_{1k}}\Big],$$

and

$$\|{}^HD^{\alpha_1-1}T_1(u_n,v_n)\|=\sup_{t\in J}\Big|\int_0^{+\infty}G_1^*(t,s)f_{1(u_n,v_n)}(s)ds\Big|\le\Xi_1\Big[a_{10}^*+\sum_{k=1}^{4}a_{1k}^*\|(u_n,v_n)\|_{X\times Y}^{\gamma_{1k}}\Big].$$

Via the Lebesgue dominated convergence theorem and continuity of function f_1, we know

$$\lim_{n\to\infty}\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{\alpha_1-1}}f_{1(u_n,v_n)}(s)\frac{\mathrm{d}s}{s}=\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{\alpha_1-1}}f_{1(u,v)}(s)\frac{\mathrm{d}s}{s},$$

and

$$\lim_{n\to\infty}\int_1^{+\infty}G_1^*(t,s)f_{1(u_n,v_n)}(s)\frac{\mathrm{d}s}{s}=\int_1^{\infty}G_1^*(t,s)f_{1(u,v)}(s)\frac{\mathrm{d}s}{s}.$$

Again

$$\|T_1(u_n,v_n)-T_1(u,v)\|_1\le\sup_{t\in J}\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{\alpha_1-1}}\Big|f_{1(u_n,v_n)}(s)-f_{1(u,v)}(s)\Big|\frac{\mathrm{d}s}{s}\to0,\ n\to\infty,$$

and

$$\|{}^HD^{\alpha_1-1}T_1(u_n,v_n)-{}^HD^{\alpha_1-1}T_1(u,v)\|_1\le\sup_{t\in J}\int_1^{+\infty}K_1^*(t,s)\Big|f_{1(u_n,v_n)}(s)-f_{1(u,v)}(s)\Big|\frac{\mathrm{d}s}{s}\to 0,\ n\to\infty.$$

Therefore, as $n\to\infty$,

$$\|T_1(u_n,v_n)-T_1(u,v)\|_X=\max\{\|T_1(u_n,v_n)-T_1(u,v)\|_1,\|{}^HD^{\alpha_1-1}T_1(u_n,v_n)-{}^HD^{\alpha_1-1}T_1(u,v)\|\}\to 0.$$

This implies that the operator T_1 is continuous. At the same way, one can obtain than the operator T_2 is continuous. That is, the operator T is continuous.

Summarize all of the above discussions, one can infer that the operator $T:P_1\times P_2\to P_1\times P_2$ is completely continuous. So the proof of Lemma 3.3 is completed.

For convenience, we set

$$\Upsilon=\max\Big\{\Lambda_1,\Lambda_2,\Xi_1,\Xi_2\Big\}.$$

Define a partial order over the product space:

$$\begin{pmatrix}u_1\\v_1\end{pmatrix}\ge\begin{pmatrix}u_2\\v_2\end{pmatrix}$$

if $u_1(t)\ge u_2(t), v_1(t)\ge v_2(t), {}^HD^{\alpha_1-1}u_1(t)\ge {}^HD^{\alpha_1-1}u_2(t), {}^HD^{\alpha_2-1}v_1(t)\ge {}^HD^{\alpha_2-1}v_2(t), t\in J.$

Theorem 3.1 If assumption (C1), (C2) and (C3) hold, then the system (1.5) exist two positive solutions (u^*,v^*) and (w^*,z^*) satisfying $0\le\|(u^*,v^*)\|_{X\times Y}\le R$ and $0\le\|(w^*,z^*)\|_{X\times Y}\le R$, where R is a positive preset constant. Moreover, there exist $\lim\limits_{n\to\infty}(u_n,v_n)=(u^*,v^*)$ and $\lim\limits_{n\to\infty}(w_n,z_n)=(w^*,z^*)$, where (u_n,v_n) and (w_n,z_n) are given by the following monotone iterative sequences

$$\begin{pmatrix}u_n(t)\\v_n(t)\end{pmatrix}=\begin{pmatrix}T_1(u_{n-1},v_{n-1})(t)\\T_2(u_{n-1},v_{n-1})(t)\end{pmatrix},n=1,2,\ldots,\ with\ \begin{pmatrix}u_0(t)\\v_0(t)\end{pmatrix}=\begin{pmatrix}R(\log t)^{\alpha_1-1}\\R(\log t)^{\alpha_2-1}\end{pmatrix}\tag{3.9}$$

and

$$\begin{pmatrix}w_n(t)\\z_n(t)\end{pmatrix}=\begin{pmatrix}T_1(w_{n-1},z_{n-1})(t)\\T_2(w_{n-1},z_{n-1})(t)\end{pmatrix},n=1,2,\ldots,\ with\ \begin{pmatrix}w_0(t)\\z_0(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.\tag{3.10}$$

In addition

$$\begin{pmatrix}w_0(t)\\z_0(t)\end{pmatrix}\le\begin{pmatrix}w_1(t)\\z_1(t)\end{pmatrix}\le\cdots\le\begin{pmatrix}w_n(t)\\z_n(t)\end{pmatrix}\le\cdots\le\begin{pmatrix}w^*\\z^*\end{pmatrix}\le\cdots\le\begin{pmatrix}u^*\\v^*\end{pmatrix}\le\cdots\le\begin{pmatrix}u_n(t)\\v_n(t)\end{pmatrix}$$

$$\le\cdots\le\begin{pmatrix}u_1(t)\\v_1(t)\end{pmatrix}\le\begin{pmatrix}u_0(t)\\v_0(t)\end{pmatrix}\tag{3.11}$$

and

$$\begin{pmatrix}{}^HD^{\alpha_1-1}w_0(t)\\{}^HD^{\alpha_2-1}z_0(t)\end{pmatrix}\le\begin{pmatrix}{}^HD^{\alpha_1-1}w_1(t)\\{}^HD^{\alpha_2-1}z_1(t)\end{pmatrix}\le\cdots\le\begin{pmatrix}{}^HD^{\alpha_1-1}w_n(t)\\{}^HD^{\alpha_2-1}z_n(t)\end{pmatrix}\le\cdots\le\begin{pmatrix}{}^HD^{\alpha_1-1}w^*\\{}^HD^{\alpha_2-1}z^*\end{pmatrix}\le\cdots\le$$

$$\begin{pmatrix}{}^HD^{\alpha_1-1}u^*\\{}^HD^{\alpha_2-1}v^*\end{pmatrix}\le\cdots\le\begin{pmatrix}{}^HD^{\alpha_1-1}u_n(t)\\{}^HD^{\alpha_2-1}v_n(t)\end{pmatrix}\le\cdots\le\begin{pmatrix}{}^HD^{\alpha_1-1}u_1(t)\\{}^HD^{\alpha_2-1}v_1(t)\end{pmatrix}\le\begin{pmatrix}{}^HD^{\alpha_1-1}u_0(t)\\{}^HD^{\alpha_2-1}v_0(t)\end{pmatrix}.\tag{3.12}$$

Proof. First, Lemma 3.3 means the fact that $T(P_1 \times P_2) \subset P_1 \times P_2$ for any $(u, v) \in P_1 \times P_2, t \in J$.

Next, for $0 \le \gamma_{1k}, \gamma_{2k} < 1 (k = 1, 2, 3, 4)$, set

$$R \ge \max\Big\{5a_{10}^*\Upsilon, 5a_{20}^*\Upsilon, (5\Upsilon a_{1k}^*)^{1/(1-\gamma_{1k})}, (5\Upsilon a_{2k}^*)^{1/(1-\gamma_{2k})}, k = 1, 2, 3, 4\Big\},$$

and $U_R = \{(u, v) \in P_1 \times P_2 : \|(u, v)\|_{X\times Y} \le R\}$. For any $(u, v) \in U_R$, similar to (3.4) and (3.5), one can obtain

$$\|T_1(u, v)\|_1 \le \Lambda_1\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^*\|(u, v)\|_{X\times Y}^{\gamma_{1k}}\Big] \le \Upsilon\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* R^{\gamma_{1k}}\Big] \le R$$

and

$$|{}^H D^{\alpha_1 - 1} T_1(u, v)\| \le \Xi_1\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^*\|(u, v)\|_{X\times Y}^{\gamma_{1k}}\Big] \le \Upsilon\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* R^{\gamma_{1k}}\Big] \le R.$$

This implies that $\|T_1(u, v)\|_X \le R$ for all $(u, v) \in U_R$. In the same way, $\|T_2(u, v)\|_Y \le R$. Consequently one has

$$\|T(u, v)\|_{X\times Y} = \max\Big\{\|T_1(u, v)\|_X, \|T_2(u, v)\|_Y\Big\} \le R.$$

That is, $T(U_R) \subset U_R$.

Via the complete continuity of the operator T, we present the sequences (u_n, v_n) and (w_n, z_n) by $(u_n, v_n) = T(u_{n-1}, v_{n-1}), (w_n, z_n) = T(w_{n-1}, z_{n-1})$ for $n = 1, 2, \cdots$. In virtue of (3.9) and (3.10), it is obvious that $(u_0(t), v_0(t)),\ (w_0(t), z_0(t)) \in U_R$.

Due to $T(U_R) \subset U_R$, it is easy to see that $(u_n, v_n), (w_n, z_n) \in T(U_R)$ for $n = 1, 2, \cdots$. Thus we just need to show that there exist (u^*, v^*) and (w^*, z^*) satisfying $\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*)$ and $\lim_{n\to\infty}(w_n, z_n) = (w^*, z^*)$, which are two monotone sequences for approximating positive solutions of the system (1.5).

For $t \in J, (u_n, v_n) \in U_R$, from Lemma 2.2 and (3.9), one has

$$\begin{aligned} u_1(t) = T_1(u_0, v_0)(t) = \int_1^{+\infty} G_1(t, s) f_{1(u_0,v_0)}(s)\frac{\mathrm{d}s}{s} &\le \Lambda_1\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* R^{\gamma_{1k}}\Big](\log t)^{\alpha_1 - 1} \\ &\le R(\log t)^{\alpha_1 - 1} = u_0(t) \end{aligned}$$

and

$$\begin{aligned} v_1(t) = T_2(u_0, v_0)(t) = \int_1^{+\infty} G_2(t, s) f_{2(u_0,v_0)}(s)\frac{\mathrm{d}s}{s} &\le \Lambda_2\Big[a_{20}^* + \sum_{k=1}^{4} a_{2k}^* R^{\gamma_{2k}}\Big](\log t)^{\alpha_2 - 1} \\ &\le R(\log t)^{\alpha_2 - 1} = v_0(t), \end{aligned}$$

that is

$$\begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} T_1(u_0, v_0)(t) \\ T_2(u_0, v_0)(t) \end{pmatrix} \le \begin{pmatrix} R(\log t)^{\alpha_1 - 1} \\ R(\log t)^{\alpha_2 - 1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}. \tag{3.13}$$

Next we consider the monotonicity of the Hadamard type fractional derivative of (u, v). By (3.13) we have

$$
\begin{aligned}
{}^{H}D^{\alpha_1-1}u_1(t) ={}& {}^{H}D^{\alpha_1-1}T_1(u_0,v_0)(t) = \int_1^{+\infty} G_1^*(t,s)f_{1(u_0,v_0)}(s)\frac{\mathrm{d}s}{s} \\
\le{}& \Xi_1\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* R^{\gamma_{1k}}\Big] \le R = {}^{H}D^{\alpha_1-1}u_0(t), \\
{}^{H}D^{\alpha_2-1}v_1(t) ={}& {}^{H}D^{\alpha_2-1}T_2(u_0,v_0)(t) = \int_1^{+\infty} G_2^*(t,s)f_{2(u_0,v_0)}(s)\frac{\mathrm{d}s}{s} \\
\le{}& \Xi_2\Big[a_{20}^* + \sum_{k=1}^{4} a_{2k}^* R^{\gamma_{2k}}\Big] \le R = {}^{H}D^{\alpha_2-1}v_0(t),
\end{aligned}
$$

that is

$$
\begin{pmatrix} {}^{H}D^{\alpha_1-1}u_1(t) \\ {}^{H}D^{\alpha_2-1}v_1(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(u_0,v_0)(t) \\ {}^{H}D^{\alpha_2-1}T_2(u_0,v_0)(t) \end{pmatrix} \le \begin{pmatrix} R \\ R \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}u_0(t) \\ {}^{H}D^{\alpha_2-1}v_0(t) \end{pmatrix} \tag{3.14}
$$

Then, by (3.13) and (3.14), for any $t \in J$, via the monotonicity conditions (C3) of functions $f_j (j = 1, 2)$, we do the second iteration

$$
\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} T_1(u_1,v_1)(t) \\ T_2(u_1,v_1)(t) \end{pmatrix} \le \begin{pmatrix} T_1(u_0,v_0)(t) \\ T_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},
$$

$$
\begin{pmatrix} {}^{H}D^{\alpha_1-1}u_2(t) \\ {}^{H}D^{\alpha_2-1}v_2(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(u_1,v_1)(t) \\ {}^{H}D^{\alpha_2-1}T_2(u_1,v_1)(t) \end{pmatrix} \le \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(u_0,v_0)(t) \\ {}^{H}D^{\alpha_2-1}T_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}u_1(t) \\ {}^{H}D^{\alpha_2-1}v_1(t) \end{pmatrix}.
$$

For $t \in J$, by method of induction, the sequences $\{(u_n, v_n)\}_{n=0}^{\infty}$ satisfy

$$
\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \quad \begin{pmatrix} {}^{H}D^{\alpha_1-1}u_{n+1}(t) \\ {}^{H}D^{\alpha_2-1}v_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} {}^{H}D^{\alpha_1-1}u_n(t) \\ {}^{H}D^{\alpha_2-1}v_n(t) \end{pmatrix}.
$$

With the help of iterative sequences $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$ and the complete continuity of the operator T, one can easily infer that $(u_n, v_n) \to (u^*, v^*)$ and $T(u^*, v^*) = (u^*, v^*)$.

For the sequences $\{(w_n, z_n)\}_{n=0}^{\infty}$, we employ a similar discussion. For $t \in J$, we have

$$
\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} T_1(w_0,z_0)(t) \\ T_2(w_0,z_0)(t) \end{pmatrix} = \begin{pmatrix} \int_1^{+\infty} G_1(t,s)f_{1(w_0,z_0)}(s)\frac{\mathrm{d}s}{s} \\ \int_1^{+\infty} G_2(t,s)f_{2(w_0,z_0)}(s)\frac{\mathrm{d}s}{s} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix},
$$

$$
\begin{pmatrix} {}^{H}D^{\alpha_1-1}w_1(t) \\ {}^{H}D^{\alpha_2-1}z_1(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(w_0,z_0)(t) \\ {}^{H}D^{\alpha_2-1}T_2(w_0,z_0)(t) \end{pmatrix} = \begin{pmatrix} \int_1^{+\infty} G_1^*(t,s)f_{1(w_0,z_0)}(s)\frac{\mathrm{d}s}{s} \\ \int_1^{+\infty} G_2^*(t,s)f_{1(w_0,z_0)}(s)\frac{\mathrm{d}s}{s} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}w_0(t) \\ {}^{H}D^{\alpha_2-1}z_0(t) \end{pmatrix}.
$$

Using the the monotonicity condition (C3) of functions f_j, one has

$$
\begin{pmatrix} w_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} T_1(w_1,z_1)(t) \\ T_2(w_1,z_1)(t) \end{pmatrix} \ge \begin{pmatrix} T_1(w_0,z_0)(t) \\ T_2(w_0,z_0)(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix},
$$

$$
\begin{pmatrix} {}^{H}D^{\alpha_1-1}w_2(t) \\ {}^{H}D^{\alpha_2-1}z_2(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(w_1,z_1)(t) \\ {}^{H}D^{\alpha_2-1}T_2(w_1,z_1)(t) \end{pmatrix} \ge \begin{pmatrix} {}^{H}D^{\alpha_1-1}T_1(w_0,z_0)(t) \\ {}^{H}D^{\alpha_2-1}T_2(w_0,z_0)(t) \end{pmatrix} = \begin{pmatrix} {}^{H}D^{\alpha_1-1}w_1(t) \\ {}^{H}D^{\alpha_2-1}z_1(t) \end{pmatrix}.
$$

Analogously, for $n = 0, 1, 2, \dots$ and $t \in J$, one has

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix}, \quad \begin{pmatrix} {}^H D^{\alpha_1-1} w_{n+1}(t) \\ {}^H D^{\alpha_2-1} z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} {}^H D^{\alpha_1-1} w_n(t) \\ {}^H D^{\alpha_2-1} z_n(t) \end{pmatrix}.$$

In virtue of the iterative sequences $(w_{n+1}, z_{n+1}) = T(w_n, z_n)$ and the complete continuity of the operator T, it is also easy to conclude that $(w_n, z_n) \to (w^*, z^*)$ and $T(w^*, z^*) = (w^*, z^*)$. Finally we demonstrate that (u^*, v^*) and (w^*, z^*) are the minimal and maximal positive solutions of the system (1.5). Suppose that $(\xi(t), \eta(t))$ is any positive solution of the Hadamard type fractional differential system (1.5), then $T(\xi(t), \eta(t)) = (\xi(t), \eta(t))$ and

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} Rt^{\alpha_1-1} \\ Rt^{\alpha_2-1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix},$$

$$\begin{pmatrix} {}^H D^{\alpha_1-1} w_0(t) \\ {}^H D^{\alpha_2-1} z_0(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} \xi(t) \\ {}^H D^{\alpha_2-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} u_0(t) \\ {}^H D^{\alpha_2-1} v_0(t) \end{pmatrix}.$$

Using the monotone conditions (C3) of the operator T, we obtain

$$\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} T_1(w_0, z_0)(t) \\ T_2(w_0, z_0)(t) \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} T_1(u_0, v_0)(t) \\ T_2(u_0, v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},$$

$$\begin{pmatrix} {}^H D^{\alpha_1-1} w_1(t) \\ {}^H D^{\alpha_2-1} z_1(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} \xi(t) \\ {}^H D^{\alpha_2-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} u_1(t) \\ {}^H D^{\alpha_2-1} v_1(t) \end{pmatrix}.$$

Repeating the above process, we have

$$\begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix},$$

$$\begin{pmatrix} {}^H D^{\alpha_1-1} w_n(t) \\ {}^H D^{\alpha_2-1} z_n(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} \xi(t) \\ {}^H D^{\alpha_2-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{\alpha_1-1} u_n(t) \\ {}^H D^{\alpha_2-1} v_n(t) \end{pmatrix},$$

which combine $\lim\limits_{n\to\infty}(w_n, z_n) = (w^*, z^*)$ and $\lim\limits_{n\to\infty}(u_n, u_n) = (u^*, v^*)$, we gain the results (3.11) and (3.12).

On the other hand, due to $f(t, 0, 0, 0, 0) \neq 0$ for all $t \in J$, we know that $(0, 0)$ isn't a solution of the Hadamard type fractional differential system (1.5). From (3.11) and (3.12), it is clear that (w^*, z^*) and (u^*, v^*) are two extreme positive solutions of the system (1.5), which can be constructed via limit of two monotone iterative sequences in (3.9) and (3.10).

Theorem 3.2 If assumption (C1), (C4) and

$$m = \Upsilon \max\Big\{\sum_{k=1}^{4} b_{1k}, \sum_{k=1}^{4} b_{2k}\Big\} < 1 \tag{3.15}$$

hold, then the system (1.5) has a unique positive solution (x, y) in $P_1 \times P_2$. Further there exists a iterative sequence (u_n, v_n) such that $\lim\limits_{n\to\infty}(u_n, v_n) = (x, y)$ is satisfied uniformly on any finite interval of J, where

$$\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} T_1(u_{n-1}, v_{n-1})(t) \\ T_2(u_{n-1}, v_{n-1})(t) \end{pmatrix}, n = 1, 2, \cdots . \tag{3.16}$$

Moreover there exists an error estimate for the approximation sequence

$$\|(u_n, v_n) - (x, y)\|_{X\times Y} = \frac{m^n}{1-m}\|(u_1, v_1) - (u_0, v_0)\|_{X\times Y}, n = 1, 2, \cdots . \tag{3.17}$$

Proof. Take

$$r \geq \Upsilon\varrho/(1-m),$$

where m is defined by (3.15) and $\varrho = \max\{\varrho_1, \varrho_2\}$, ϱ_j $(j = 1, 2)$ are defined by the aussumption (C4).

First we show that $TU_r \subset U_r$, where $U_r = \{(u, v) \in P_1 \times P_2, \|(u, v)\|_{X\times Y} \leq r\}$. For any $(u, v) \in U_r$, by Lemma 3.2 and Remark 2.2, we have

$$\|T_1(u, v)\|_1 \leq \Lambda_1\Big(\sum_{k=1}^{4} b_{1k}^* r + \varrho_1\Big)$$

and

$$\|{}^H D^{\alpha_1-1} T_1(u, v)\| \leq \Xi_1\Big(\sum_{k=1}^{4} b_{1k}^* r + \varrho_1\Big),$$

which implies

$$\|T_1(u, v)\|_X \leq \Upsilon\Big(\sum_{k=1}^{4} b_{1k}^* r + \varrho_1\Big) \leq mr + \Upsilon\varrho_1,\ \forall (u, v) \in U_r.$$

Similar

$$\|T_2(u, v)\|_Y \leq \Upsilon\Big(\sum_{k=1}^{4} b_{2k}^* r + \varrho_2\Big) \leq mr + \Upsilon\varrho_2,\ \forall (u, v) \in U_r.$$

So one has

$$\|T(u, v)\|_{X\times Y} \leq mr + \Upsilon\varrho \leq r, \forall (u, v) \in U_r.$$

Now we demonstrate that operator T is a contraction. For any $(u_1, v_1), (u_2, v_2) \in U_r$, by assumption (C4), we have

$$\begin{aligned}
&\|T_1(u_1, v_1) - T_1(u_2, v_2)\|_1 \\
&\leq \sup_{t\in J} \int_1^{+\infty} \frac{G_1(t, s)}{1 + (\log t)^{\alpha_1-1}}\Big|f_{1(u_1,v_1)}(s) - f_{1(u_2,v_2)}(s)\Big|\frac{\mathrm{d}s}{s} \\
&\leq \Lambda_1 \int_1^{+\infty} \Big[b_{11}(s)(1 + (\log s)^{\alpha_1-1})\frac{|u_1(s) - u_2(s)|}{1 + (\log s)^{\alpha_1-1}} + b_{12}(s)(1 + (\log s)^{\alpha_2-1})\frac{|v_1(s) - v_2(s)|}{1 + (\log s)^{\alpha_2-1}} \\
&\quad + b_{13}(s)|{}^H D^{\alpha_1-1} u_1(s) -{}^H D^{\alpha_1-1} u_2(s)|\Big] + b_{14}(s)|{}^H D^{\alpha_2-1} v_1(s) -{}^H D^{\alpha_2-1} v_2(s)|\Big]\frac{\mathrm{d}s}{s} \\
&\leq \Lambda_1 \sum_{k=1}^{4} b_{1k}^* \|(u_1, v_1) - (u_2, v_2)\|_{X\times Y}
\end{aligned}$$

and

$$\begin{aligned}
\|{}^H D^{\alpha_1-1} T_1(u_1, v_1) -{}^H D^{\alpha_1-1} T_1(u_2, v_2)\| &\leq \sup_{t\in J} \int_0^{+\infty} G_1^*(t, s)\Big|f_{1(u_1,v_1)}(s) - f_{1(u_2,v_2)}(s)\Big| ds \\
&\leq \Xi_1 \sum_{k=1}^{4} b_{1k}^* \|(u_1, v_1) - (u_2, v_2)\|_{X\times Y},
\end{aligned}$$

which implies

$$\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_X \le \Upsilon \sum_{k=1}^{4} b_{1k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}. \tag{3.18}$$

In the same way, one can obtain

$$\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_Y \le \Upsilon \sum_{k=2}^{4} b_{2k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}. \tag{3.19}$$

By (3.18) and (3.19), we gain

$$\|T(u_1,v_1)-T(u_2,v_2)\|_{X\times Y} \le m\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}, \forall (u_1,v_1),(u_2,v_2)\in U_r. \tag{3.20}$$

Due to $m<1$, then operator T is a contraction. With the help of the Banach fixed-point theorem, T has a unique fixed point (x,y) in U_r. That is, the system (1.5) has a unique positive solution (x,y).

Further, for any $(u_0,v_0)\in U_r$, $\|(u_n,v_n)-(x,y)\|_{X\times Y}\to 0$ as $n\to\infty$, where $u_n=T_1(u_{n-1},v_{n-1}), v_n=T_2(u_{n-1},v_{n-1}), n=1,2,\cdots$. From (3.20), we have

$$\|(u_n,v_n)-(u_{n-1},v_{n-1})\|_{X\times Y} \le m^{n-1}\|(u_1,v_1)-(u_0,v_0)\|_{X\times Y},$$

and

$$\begin{aligned}\|(u_n,v_n)-(u_j,v_j)\|_{X\times Y} &\le \|(u_n,v_n)-(u_{n-1},v_{n-1})\|_{X\times Y}+\|(u_{n-1},v_{n-1})-(u_{n-2},v_{n-2})\|_{X\times Y}\\ &\quad +\cdots+\|(u_{j+1},v_{j+1})-(u_j,v_j)\|_{X\times Y}\\ &\le \frac{m^n(1-m^{j-n})}{1-m}\|(u_1,v_1)-(u_0,v_0)\|_{X\times Y}.\end{aligned} \tag{3.21}$$

Taking $j\to+\infty$ on both sides of (3.21), one can obtain

$$\|(u_n,v_n)-(x,y)\|_{X\times Y} \le \frac{m^n}{1-m}\|u_1-u_0\|_{X\times Y}.$$

So the proof of Theorem 3.2 is completed.

4. Examples

Example 4.1 Consider the following Hadamard type fractional differential system

$$\begin{cases} -^{H}D^{1.8}u(t)=e^{-2t}+\dfrac{e^{-t}|u(t)|^{0.1}}{[1+(\log t)^{0.8}]^{0.1}}+\dfrac{e^{-2t}|v(t)|^{0.3}}{[1+(\log t)^{0.5}]^{0.3}}+\dfrac{t|^{H}D^{0.8}u(t)|^{0.2}}{2(4+t)^2}+\dfrac{t|^{H}D^{0.5}v(t)|^{0.4}}{5(1+t^2)},\\ -^{H}D^{1.5}v(t)=t^{-5}+\dfrac{e^{-3t}|u(t)|^{0.2}}{[1+(\log t)^{0.8}]^{0.2}}+\dfrac{e^{-4t}|v(t)|^{0.4}}{[1+(\log t)^{0.5}]^{0.4}}+\dfrac{t|^{H}D^{1.5}u(t)|^{0.2}}{5(1+t^2)}+\dfrac{t|^{H}D^{0.5}v(t)|^{0.6}}{(9+t)^2},\\ u(1)=0,\ ^{H}D^{0.8}u(+\infty)=0.2^{H}I^{1.8}u(2.5)+0.1^{H}I^{2.8}u(2.5),\\ v(1)=0,^{H}D^{0.5}v(+\infty)=0.1^{H}I^{1.5}u(1.5)+0.3^{H}I^{2.5}u(1.5)+2^{H}I^{3.5}u(1.5),\end{cases} \tag{4.1}$$

where $\alpha_1 = 1.8,\ \alpha_2 = 1.5,\ \beta_{11} = 1.8,\ \beta_{12} = 2.8,\ \lambda_{11} = 0.2,\ \lambda_{12} = 0.1,\ \beta_{21} = 1.5,\ \beta_{22} = 2.5,\ \beta_{23} = 3.5,\ \lambda_{21} = 0.1,\ \lambda_{22} = 0.3,\ \lambda_{23} = 2,\ \eta_1 = 2.5,\ \eta_2 = 1.5$ and

$$f_1(t,u_1,u_2,u_3,u_4) = e^{-2t} + \frac{e^{-t}|u_1|^{0.1}}{[1+(\log t)^{0.8}]^{0.1}} + \frac{e^{-2t}|u_2|^{0.3}}{[1+(\log t)^{0.5}]^{0.3}} + \frac{t|u_3|^{0.2}}{2(4+t)^2} + \frac{t|u_4|^{0.4}}{5(1+t^2)},$$

$$f_2(t,u_1,u_2,u_3,u_4) = t^{-5} + \frac{e^{-3t}|u_1|^{0.2}}{[1+(\log t)^{0.8}]^{0.2}} + \frac{e^{-4t}|u_2)|^{0.4}}{[1+(\log t)^{0.5}]^{0.4}} + \frac{t|u_3|^{0.2}}{5(1+t^2)} + \frac{t|u_4|^{0.6}}{(9+t)^2}.$$

Here $\gamma_{11} = 0.1,\ \gamma_{12} = 0.3,\ \gamma_{13} = 0.2,\ \gamma_{14} = 0.4,\ \gamma_{21} = 0.2,\ \gamma_{22} = 0.4,\ \gamma_{23} = 0.2,\ \gamma_{23} = 0.2,\ \gamma_{24} = 0.6$.

We find that $f_1(t,0,0,0,0) \not\equiv 0, f_1(t,0,0,0,0) \not\equiv 0$, for $\forall t \in J$ and $\Omega_1 = \Gamma(\alpha_1) - \sum_{i=1}^{2} \frac{\lambda_{1i}\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_{1i})}(\log \eta_1)^{\alpha_1+\beta_{1i}-1} \approx 0.885609 > 0,\ \ \Omega_2 = \Gamma(\alpha_2) - \sum_{i=1}^{3} \frac{\lambda_{2i}\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_{2i})}(\log \eta_2)^{\alpha_2+\beta_{2i}-1} \approx 0.872852 > 0$. So assumption (C1) holds.

Noting that

$$\begin{aligned}|f_1(t,u_1,u_2,u_3,u_4)| &\le e^{-2t} + \frac{e^{-t}|u_1|^{0.1}}{[1+(\log t)^{0.8}]^{0.1}} + \frac{e^{-2t}|u_2|^{0.3}}{[1+(\log t)^{0.5}]^{0.3}} + \frac{t|u_3|^{0.2}}{2(4+t)^2} + \frac{t|u_4|^{0.4}}{5(1+t^2)}\\ &= a_{10}(t) + a_{11}(t)|u_1|^{0.1} + a_{12}(t)|u_2|^{0.3} + a_{13}(t)|u_3|^{0.2} + a_{14}(t)|u_4|^{0.4},\\ |f_2(t,u_1,u_2,u_3,u_4)| &\le t^{-5} + \frac{e^{-3t}|u_1|^{0.2}}{[1+(\log t)^{0.8}]^{0.2}} + \frac{e^{-4t}|u_2)|^{0.4}}{[1+(\log t)^{0.5}]^{0.4}} + \frac{t|u_3|^{0.2}}{5(1+t^2)} + \frac{t|u_4|^{0.6}}{(9+t)^2}\\ &= a_{20}(t) + a_{21}(t)|u_1|^{0.2} + a_{22}(t)|u_2|^{0.2} + a_{23}(t)|u_3|^{0.2} + a_{24}(t)|u_4|^{0.6}\end{aligned}$$

and

$$\begin{aligned}a_{10}^* &= \int_1^{+\infty} a_{10}(t)\frac{dt}{t} = \int_1^{+\infty} e^{-2t}\frac{dt}{t} \le \int_1^{+\infty} e^{-2t}dt = \frac{1}{2e^2} < \infty,\\ a_{11}^* &= \int_1^{+\infty} a_{11}(t)[1+(\log t)^{0.8}]^{0.1}\frac{dt}{t} = \int_1^{+\infty} \frac{e^{-t}}{[1+(\log t)^{0.8}]^{0.1}}[1+(\log t)^{0.8}]^{0.1}\frac{dt}{t} \le \frac{1}{e} < \infty,\\ a_{12}^* &= \int_1^{+\infty} a_{12}(t)[1+(\log t)^{0.5}]^{0.3}\frac{dt}{t} = \int_1^{+\infty} \frac{e^{-2t}}{[1+(\log t)^{0.5}]^{0.3}}[1+(\log t)^{0.5}]^{0.3}\frac{dt}{t} \le \frac{1}{2e^2} < \infty,\\ a_{13}^* &= \int_1^{+\infty} a_{13}(t)\frac{dt}{t} = \int_1^{+\infty} \frac{t}{2(4+t)^2}\frac{dt}{t} = \frac{1}{10} < \infty,\\ a_{14}^* &= \int_1^{+\infty} a_{14}(t)\frac{dt}{t} = \int_1^{+\infty} \frac{t}{5(1+t^2)}\frac{dt}{t} = \frac{\pi}{10} < \infty,\\ a_{20}^* &= \int_1^{+\infty} a_{20}(t)\frac{dt}{t} = \int_1^{+\infty} t^{-5}\frac{dt}{t} = \frac{1}{5} < \infty,\end{aligned}$$

$$\begin{aligned}a_{21}^* &= \int_1^{+\infty} a_{11}(t)[1+(\log t)^{0.8}]^{0.2}\frac{dt}{t} = \int_1^{+\infty} \frac{e^{-3t}}{[1+(\log t)^{0.8}]^{0.2}}[1+(\log t)^{0.8}]^{0.2}\frac{dt}{t} \le \frac{1}{3e^3} < \infty,\\ a_{22}^* &= \int_1^{+\infty} a_{12}(t)[1+(\log t)^{0.5}]^{0.4}\frac{dt}{t} = \int_1^{+\infty} \frac{e^{-4t}}{[1+(\log t)^{0.5}]^{0.3}}[1+(\log t)^{0.5}]^{0.4}\frac{dt}{t} \le \frac{1}{4e^4} < \infty,\\ a_{23}^* &= \int_1^{+\infty} a_{13}(t)\frac{dt}{t} = \int_1^{+\infty} \frac{t}{5(1+t^2)}\frac{dt}{t} \le \frac{\pi}{10} < \infty,\\ a_{24}^* &= \int_1^{+\infty} a_{14}(t)\frac{dt}{t} = \int_1^{+\infty} \frac{t}{(9+t)^2}\frac{dt}{t} \le \frac{1}{10} < \infty,\end{aligned}$$

which imply that assumption (C2) holds.

From the expression of function f_j, we can infer that f_j is increasing respect to the variables $u_1, u_2, u_3, u_4, \forall t \in J, j = 1, 2$. Hence assumption (C3) is also satisfied . By Theorem 3.1, it follows that the system (4.1) have two pairs of positive solutions (u^*, v^*) and (w^*, z^*), which can be constructed via the limit of two explicit monotone iterative sequences in (3.11) and (3.12).

Example 4.2 Consider the following Hadamard type fractional differential system

$$\begin{cases} -{}^HD^{1.8}u(t) = e^{-2t} + \dfrac{e^{-t}|u(t)|}{[1+(\log t)^{0.8}]} + \dfrac{e^{-2t}|v(t)|^{0.3}}{[1+(\log t)^{0.5}]} + \dfrac{t|{}^HD^{0.8}u(t)|}{2(4+t)^2} + \dfrac{t|{}^HD^{0.5}v(t)|}{5(1+t^2)}, \\ -{}^HD^{1.5}v(t) = t^{-5} + \dfrac{e^{-3t}|u(t)|}{[1+(\log t)^{0.8}]} + \dfrac{e^{-4t}|v(t)|^{0.4}}{[1+(\log t)^{0.5}]} + \dfrac{t|{}^HD^{1.5}u(t)|}{5(1+t^2)} + \dfrac{t|{}^HD^{0.5}v(t)|}{(9+t)^2}, \\ u(1) = 0, \ {}^HD^{0.8}u(+\infty) = 0.2{}^HI^{1.8}u(2.5) + 0.1{}^HI^{2.8}u(2.5), \\ v(1) = 0, {}^HD^{0.5}v(+\infty) = 0.1{}^HI^{1.5}u(1.5) + 0.3{}^HI^{2.5}u(1.5) + 2{}^HI^{3.5}u(1.5), \end{cases} \tag{4.2}$$

where $\alpha_1 = 1.8$, $\alpha_2 = 1.5$, $\beta_{11} = 1.8$, $\beta_{12} = 2.8$, $\lambda_{11} = 0.2$, $\lambda_{12} = 0.1$, $\beta_{21} = 1.5$, $\beta_{22} = 2.5$, $\beta_{23} = 3.5$, $\lambda_{21} = 0.1$, $\lambda_{22} = 0.3$, $\lambda_{23} = 2$, $\eta_1 = 2.5$, $\eta_2 = 1.5$ and

$$f_1(t, u_1, u_2, u_3, u_4) = e^{-2t} + \frac{e^{-t}|u_1|}{1+(\log t)^{0.8}} + \frac{e^{-2t}|u_2|}{[1+(\log t)^{0.5}]} + \frac{t|u_3|}{2(4+t)^2} + \frac{t|u_4|}{5(1+t^2)},$$

$$f_2(t, u_1, u_2, u_3, u_4) = t^{-5} + \frac{e^{-3t}|u_1|}{1+(\log t)^{0.8}} + \frac{e^{-4t}|u_2)|}{[1+(\log t)^{0.5}]} + \frac{t|u_3|}{5(1+t^2)} + \frac{t|u_4|}{(9+t)^2}.$$

Same to example (4.1), it is easy to verify that assumption (C1) hods.

Observing that

$$\begin{aligned} &|f_1(t, u_1, u_2, u_3, u_4) - f_1(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)| \\ \le&\frac{e^{-t}}{1+(\log t)^{0.8}}|u_1 - \overline{u}_1| + \frac{e^{-2t}}{1+(\log t)^{0.5}}|u_2 - \overline{u}_2| + \frac{t}{2(4+t)^2}|u_3 - \overline{u}_3| + \frac{t}{5(1+t^2)}|u_4 - \overline{u}_4| \\ =&b_{11}(t)|u_1 - \overline{u}_1| + b_{12}(t)|u_2 - \overline{u}_2| + b_{13}(t)|u_3 - \overline{u}_3| + b_{14}(t)|u_4 - \overline{u}_4|, \\ &|f_2(t, u_1, u_2, u_3, u_4) - f_2(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)| \\ \le&\frac{e^{-3t}}{1+(\log t)^{0.8}}|u_1 - \overline{u}_1| + \frac{e^{-4t}}{1+(\log t)^{0.5}}|u_2 - \overline{u}_2| + \frac{t}{5(1+t^2)}|u_3 - \overline{u}_3| + \frac{t}{(9+t)^2}|u_4 - \overline{u}_4| \\ =&b_{21}(t)|u_1 - \overline{u}_1| + b_{22}(t)|u_2 - \overline{u}_2| + b_{23}(t)|u_3 - \overline{u}_3| + b_{24}(t)|u_4 - \overline{u}_4|, \end{aligned}$$

by a same computation as example (4.1), one can obtain

$$b_{11}^* = \int_1^{+\infty} b_{11}(t)[1 + (\log t)^{0.8}]\frac{\mathrm{d}t}{t} \le \frac{1}{e} < \infty,\ b_{12}^* = \int_1^{+\infty} b_{12}(t)[1 + (\log t)^{0.5}]\frac{\mathrm{d}t}{t} \le \frac{1}{2e^2} < \infty,$$
$$b_{13}^* = \int_1^{+\infty} b_{13}(t)\frac{\mathrm{d}t}{t} = \frac{1}{10} < \infty, b_{14}^* = \int_1^{+\infty} a_{14}(t)\frac{\mathrm{d}t}{t} = \frac{\pi}{10} < \infty,$$
$$b_{21}^* = \int_1^{+\infty} b_{21}(t)[1 + (\log t)^{0.8}]\frac{\mathrm{d}t}{t} \le \frac{1}{3e^3} < \infty, b_{22}^* = \int_1^{+\infty} b_{22}(t)[1 + (\log t)^{0.5}]\frac{\mathrm{d}t}{t} \le \frac{1}{4e^4} < \infty,$$
$$b_{23}^* = \int_1^{+\infty} b_{13}(t)\frac{\mathrm{d}t}{t} \le \frac{\pi}{10} < \infty, b_{24}^* = \int_1^{+\infty} b_{14}(t)\frac{\mathrm{d}t}{t} \le \frac{1}{10} < \infty,$$
$$\lambda_1 = \int_1^{+\infty} f_1(t, 0, 0, 0, 0)dt = \int_1^{+\infty} e^{-2t}\frac{\mathrm{d}t}{t} \le \frac{1}{2e^2} < \infty,$$
$$\lambda_2 = \int_1^{+\infty} f_2(t, 0, 0, 0, 0)dt = \int_1^{+\infty} t^{-5}\frac{\mathrm{d}t}{t} = \frac{1}{5} < \infty,$$

which show that assumption (C4) holds. By direct computation, one can obtain that $\Lambda_1 = 1.125161$, $\Lambda_2 = 1.143496$, $\Xi_1 = 1.051490$, $\Xi_2 = 1.015251$, $\Upsilon = 1.143496$,

$$m = \Upsilon \max\Big\{\sum_{k=1}^{4} b_{1k}^*, \sum_{k=1}^{4} b_{2k}^*\Big\} \le 1.143496 \times \max\Big\{0.849706, 0.435334\Big\} = 0.971635 < 1.$$

Hence all presupposed conditions of Theorem 3.2 are satisfied. Then the system (4.2) has a unique positive solution (x, y), which can be constructed via the limit of the iterative sequence in (3.16).

5. Conclusions

In this paper, we consider a class of Hadamard type fractional differential system. By the aid of monotone iterative technique and Banach's contraction mapping principle, under certain nonlinear and linear increasing conditions, we construct some explicit monotone iterative sequences for approximating the extreme positive solutions and the unique positive solutions. Our results generalize iterative solution of a single equation to the case of a system, and the nonlinear term contains Hadamard type fractional derivative which can be used more widely. Further work is still needed including discussions on iterative solution for Hadamard type fractional differential system with coupling integral condition and additional studies on iterative solution for impulsive Hadamard type fractional differential system.

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Conflict of interest

The authors declare no conflict of interest.

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