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Research article

On the sixth power mean of one kind two-term exponential sums weighted by Legendre's symbol modulo p

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Abstract: The main purpose of this article is using the elementary methods and the properties of the character sums of the polynomials to study the calculating problem of one kind sixth power mean of the two-term exponential sums weighted by Legendre's symbol modulo p, an odd prime, and give an interesting calculating formula for it.

Keywords: the two-term exponential sums; the sixth power mean; elementary method; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let $q \geq 3$ be a fixed integer. For any integer $k \geq 2$ and m with $(m, q) = 1$, we define the two-term exponential sums $C(m, k; q)$ as follows:

$$C(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right),$$

where as usual, $e(y) = e^{2\pi iy}$ and $i^2 = -1$.

Since this kind sums play an very important role in the study of analytic number theory, so many number theorists and scholars had studied the various properties of $C(m, k; q)$, and obtained a series of meaningful research results, we do not want to enumerate here, interested readers can refer to references [2–7,9–11,13–15]. Note that $|C(m, k; q)|$ is a multiplicative function of q, so people often only consider case that $q = p$ or p^r, where p is an odd prime, and $r \geq 2$ is a positive integer.

For example, H. Zhang and W. P. Zhang [13] proved that for any odd prime p, one has

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if } 3 | p-1, \end{cases}$$

where n represents any integer with $(n, p) = 1$.

L. Chen and X. Wang [3] studied the calculating problem of the fourth power mean of $G(m, 4; p)$, and proved the following conclusion:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p^3 & \text{if } p = 12k+11; \\ 2p^2(p-2) & \text{if } p = 12k+7; \\ 2p\left(p^2-4p-2\alpha^2\right) & \text{if } p = 24k+5; \\ 2p\left(p^2-6p-2\alpha^2\right) & \text{if } p = 24k+13; \\ 2p\left(p^2-10p-2\alpha^2\right) & \text{if } p = 24k+1; \\ 2p\left(p^2-8p-2\alpha^2\right) & \text{if } p = 24k+17, \end{cases}$$

where $\alpha = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$, and $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, and $a \cdot \overline{a} \equiv 1 \bmod p$.

Z. Y. Chen and W. P. Zhang [6] proved that for any prime p with $p \equiv 5 \bmod 8$, one has the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right|^2 = 3p^3 - 3p^2 + 2p^{\frac{3}{2}}\alpha - 3p,$$

where $\alpha = \alpha(p)$ is the same as defined in the above.

Very recently, J. Zhang and W. P. Zhang [14] studied the fourth power mean of the two-term exponential sums weighted by Legendre's symbol modulo an odd prime p, and proved that for any odd prime p, one has the identity

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 = \begin{cases} p^2(\delta-3) & \text{if } p \equiv 1 \bmod 6; \\ p^2(\delta+3) & \text{if } p \equiv -1 \bmod 6, \end{cases} \tag{1.1}$$

where $\delta = \sum_{d=1}^{p-1}\left(\frac{d-1+\overline{d}}{p}\right)$ is an integer which satisfies the estimate $|\delta| \le 2\sqrt{p}$.

The main purpose of this paper as a generalization of (3.1), and study the calculating problem of the $2h$-th power mean of the two-term exponential sums

$$G(h, p) = \sum_{m=1}^{p-1}\left(\frac{m}{p}\right) \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^{2h}, \tag{1.2}$$

where p is an odd prime, and $h \ge 3$ is an integer.

It is clear that J. Zhang and W. P. Zhang [14] proved an identity for $G(2, p)$. But for $h \ge 3$, it seems that none had studied it before, at least we have not seen such a result at present. We think this content is meaningful for further research. Because it can solve the problem of calculating the $2h$-th power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{m^2a^3+a}{p}\right)\right|^{2h} = \sum_{m=1}^{p-1}\left[1+\left(\frac{m}{p}\right)\right] \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^{2h}.$$

In other words, we shall deal with the $2h$-th power mean problem involving the sums of quadratic residues modulo p. This will provide some new ideas and methods for us to study the power mean problem on special sets. Of course, the problem we are studying in here is much more difficult than that in [14], because we are going to do the sixth power mean, some congruence equations involved more variables, this can lead to the computational difficulties.

2. Several lemmas

In this section, we will give several necessary lemmas. Of course, the proofs of some lemmas need the knowledge of elementary and analytic number theory. In particular, the properties of the quadratic residues and the Legendre's symbol modulo p. All these can be found in references [1,8,12], and we do not repeat them. First we have

Lemma 1. Let $p > 3$ be an odd prime. Then we have the identity

$$\sum_{\substack{a=1 \\ a^3+b^3+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(ab)\chi_2(a+b+1)$$

$$= \chi_2(3)\cdot\delta(p) - \chi_2(-1) + \chi_2(-1)\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right),$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p.

Proof. Note that $\chi_2(b) = \chi_2\left(\overline{b}\right)$ and $\chi_2(b-1) = \chi_2\left(\overline{b}(1-\overline{b})\right)$, from the properties of the complete residue system modulo p we have

$$\sum_{\substack{a=1 \\ a^3+b^3+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(ab)\chi_2(a+b+1) = \sum_{\substack{a=1 \\ a^3+(b-1)^3+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} \chi_2(a(b-1))\chi_2(a+b)$$

$$= \sum_{\substack{a=1 \\ a^3b^3+b^3-3b^2+3b\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(ab(b-1))\chi_2(ab+b)$$

$$= \sum_{\substack{a=1 \\ a^3+1-3\overline{b}+3\overline{b}^2\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(a)\chi_2(a+1)\chi_2(b-1)$$

$$= \sum_{\substack{a=1 \\ \overline{3}\left(a^3+1\right)\equiv\overline{b}\left(1-\overline{b}\right) \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(a)\chi_2(a+1)\chi_2\left(\overline{b}(1-\overline{b})\right)$$

$$= \sum_{\substack{a=1 \\ 4\cdot\overline{3}\left(a^3+1\right)\equiv 1-(1-2b)^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(a)\chi_2(a+1)\chi_2\left(1-(1-2b)^2\right)$$

$$= \sum_{a=1}^{p-1}\chi_2(3)\chi_2(a)\chi_2(a+1)\chi_2\left(a^3+1\right)\left[1+\chi_2\left(1-4\cdot\overline{3}\left(a^3+1\right)\right)\right]$$

$$
\begin{aligned}
&= \chi_2(3)\sum_{a=1}^{p-2}\chi_2(a)\chi_2\left(a^2-a+1\right)+\sum_{a=1}^{p-1}\chi_2(a)\chi_2(a+1)\chi_2(a^3+1)\chi_2\left(-1-4a^3\right)\\
&= \chi_2(3)\sum_{a=1}^{p-2}\chi_2(a)\chi_2\left(a^2-a+1\right)+\sum_{a=1}^{p-1}\chi_2^8(a)\chi_2(\overline{a}+1)\chi_2(\overline{a}^3+1)\chi_2\left(-4-\overline{a}^3\right)\\
&= \chi_2(3)\sum_{a=1}^{p-2}\chi_2(a)\chi_2\left(a^2-a+1\right)+\sum_{a=1}^{p-1}\chi_2(a+1)\chi_2(a^3+1)\chi_2\left(-a^3-4\right)\\
&= \chi_2(3)\sum_{a=1}^{p-2}\chi_2\left(a-1+\overline{a}\right)+\chi_2(-1)\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right)\\
&= \chi_2(3)\cdot\delta(p)-\chi_2(-1)+\chi_2(-1)\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right).
\end{aligned}
$$

This proves Lemma 1.

Lemma 2. Let p be an odd prime, then we have the identity

$$
\begin{aligned}
&\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(1+b^3+c^3-d^3-e^3\right)\chi_2\left(1+b+c-d-e\right)\\
&= \chi_2(3)\cdot 2\cdot p^2\cdot \sum_{\substack{b=1\\ 4+b^3\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2\left(b+1\right)+p^2+\chi_2(3)\cdot 2\cdot p\\
&\quad +\chi_2(-1)\cdot p\cdot\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_2(bc)\chi_2\left(b^3+c^3+4\right)\chi_2(b+c+1).
\end{aligned}
$$

Proof. From the properties of the complete residue system and quadratic residue modulo p we have

$$
\begin{aligned}
&\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(1+b^3+c^3-d^3-e^3\right)\chi_2\left(1+b+c-d-e\right)\\
&= \sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(1+(b+d)^3+(c+e)^3-d^3-e^3\right)\chi_2\left(1+b+c\right)\\
&= \sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(4+b^3+3b(2d+b)^2+c^3+3c(2e+c)^2\right)\chi_2\left(1+b+c\right)\\
&= \sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(4+b^3+3bd^2+c^3+3ce^2\right)\chi_2\left(b+c+1\right). \qquad (2.1)
\end{aligned}
$$

For any integer n, note that the identity

$$
\sum_{a=0}^{p-1}\left(\frac{a^2+n}{p}\right)=\begin{cases} p-1 & \text{if } p\mid n,\\ -1 & \text{if } (p,n)=1.\end{cases} \qquad (2.2)
$$

Combining (2.1), (2.2) and the properties of the quadratic residue modulo p we have

$$\begin{aligned}
&\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(1+b^3+c^3-d^3-e^3\right)\chi_2\left(1+b+c-d-e\right)\\
=\ &\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(4+b^3+3bd^2+c^3+3ce^2\right)\chi_2\left(b+c+1\right)\\
=\ &\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(4+b^3+3bd^2+c^3+3ce^2\right)\chi_2\left(b+c+1\right)\\
&+\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(4+c^3+3ce^2\right)\chi_2\left(c+1\right)\\
=\ &p\sum_{\substack{b=1\\4+b^3+c^3+3ce^2\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2(3b)\chi_2\left(b+c+1\right)-\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(3b\right)\chi_2\left(b+c+1\right)\\
&+p^2+p\left(p\sum_{\substack{c=1\\4+c^3\equiv 0 \bmod p}}^{p-1}\chi_2(3c)\chi_2(c+1)-\sum_{c=1}^{p-1}\chi_2(3c)\chi_2(c+1)\right)\\
=\ &p^2\sum_{\substack{b=1\\4+b^3\equiv 0 \bmod p}}^{p-1}\chi_2(3b)\chi_2\left(b+1\right)+p\sum_{\substack{b=1\\4+b^3+c^3+3ce^2\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\sum_{e=0}^{p-1}\chi_2(3b)\chi_2\left(b+c+1\right)\\
&+p^2+p^2\cdot\sum_{\substack{c=1\\4+c^3\equiv 0 \bmod p}}^{p-1}\chi_2(3c)\chi_2(c+1)+p\cdot\chi_2(3)\\
=\ &2\chi_2(3)\cdot p^2\sum_{\substack{b=1\\4+b^3\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2\left(b+1\right)+p^2+\chi_2(3)\cdot p\\
&+p\cdot\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(1+\chi_2\left(-3c\left(4+b^3+c^3\right)\right)\right)\chi_2(b+c+1)\chi_2(3b)\\
=\ &2\chi_2(3)\cdot p^2\sum_{\substack{b=1\\4+b^3\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2\left(b+1\right)+p^2+\chi_2(3)\cdot p\\
&-p\sum_{b=1}^{p-1}\chi_2(b+1)\chi_2(3b)+p\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_2(-bc)\chi_2\left(b^3+c^3+4\right)\chi_2(b+c+1)\\
=\ &\chi_2(3)\cdot 2\cdot p^2\cdot\sum_{\substack{b=1\\4+b^3\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2\left(b+1\right)+p^2+\chi_2(3)\cdot 2\cdot p
\end{aligned}$$

$$+\chi_2(-1)\cdot p\cdot\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_2(bc)\chi_2\left(b^3+c^3+4\right)\chi_2(b+c+1).$$

This proves Lemma 2.

Lemma 3. Let p be an odd prime, then we have the identity

$$\begin{aligned}&\sum_{\substack{b=1\\ b^3+4c^3-4\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\chi_2(b)\chi_2(b+c-1)\\ =\;& -2+\chi_2(-3)\cdot\delta(p)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1).\end{aligned}$$

Proof. From the properties of the complete residue system modulo p we have

$$\begin{aligned}&\sum_{\substack{b=1\\ b^3+4c^3-4\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\chi_2(b)\chi_2(b+c-1)=\sum_{\substack{b=1\\ b^3+4(c+1)^3-4\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\chi_2(b)\chi_2(b+c)\\
=\;&\sum_{\substack{b=1\\ b^3+4c^3+12c^2+12c\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\chi_2(b)\chi_2(b+c)=\sum_{\substack{b=1\\ b^3+4+12c+12c^2\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\chi_2(b)\chi_2(b+1)\\
=\;&\sum_{\substack{b=1\\ b^3+1+3(2c+1)^2\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\chi_2(b)\chi_2(b+1)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1)\\
=\;&\sum_{\substack{b=1\\ b^3+1+3c^2\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\chi_2(b)\chi_2(b+1)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1)\\
=\;&\sum_{b=1}^{p-1}\left(1+\chi_2\left(-3b^3-3\right)\right)\chi_2(b)\chi_2(b+1)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1)\\
=\;&\sum_{b=1}^{p-1}\chi_2\left(1+\overline{b}\right)+\chi_2(-3)\sum_{b=1}^{p-2}\chi_2\left(b-1+\overline{b}\right)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1)\\
=\;&-2+\chi_2(-3)\cdot\delta(p)-\sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(b)\chi_2(b+1).\end{aligned}$$

This proves Lemma 3.

Lemma 4. Let p be a prime. Then we have the identity

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-1\right)\chi_2(a+b+c-d-e-1)$$

$$= \chi_2(-1)\left(4p^2-p\right)\cdot\delta(p)-\chi_2(3)\cdot\left(8p^2-p\right)-p^2$$
$$-\chi_2(-1)\cdot p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^3+b^3+4\right)\chi_2(ab)\chi_2(a+b+1)$$
$$-2\chi_2(3)\cdot p^2\cdot\sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(a)\chi_2(a+1)$$
$$+\chi_2(3)\cdot p^2\cdot\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right).$$

Proof. From the properties of the complete residue system modulo p we have

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-1\right)\chi_2(a+b+c-d-e-1)$$
$$= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left((a+d)^3+(b+e)^3+c^3-d^3-e^3-1\right)\chi_2(a+b+c-1)$$
$$= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+3a(2d+a)^2+b^3+3b(2e+b)^2+4c^3-4\right)$$
$$\times\chi_2(a+b+c-1)$$
$$= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+3ad^2+b^3+3be^2+4c^3-4\right)\chi_2(a+b+c-1)$$
$$= \sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+3ad^2+b^3+3be^2+4c^3-4\right)\chi_2(a+b+c-1)$$
$$+p\cdot\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(b^3+3be^2+4c^3-4\right)\chi_2(b+c-1). \tag{2.3}$$

From (2.2) and the properties of the quadratic residue modulo p we have

$$\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+3ad^2+b^3+3be^2+4c^3-4\right)\chi_2(a+b+c-1)$$
$$= p\cdot\sum_{\substack{a=1\\ a^3+b^3+3be^2+4c^3-4\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2(3a)\chi_2(a+b+c-1)$$
$$-\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2(3a)\chi_2(a+b+c-1)$$
$$= p\cdot\sum_{\substack{a=1\\ a^3+b^3+3be^2+4c^3-4\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2(3a)\chi_2(a+b+c-1)$$

$$+p^2 \cdot \sum_{\substack{a=1 \\ a^3+4c^3-4\equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \chi_2(3a)\chi_2(a+c-1)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\left(1+\chi_2\left(4-a^3-b^3-4c^3\right)\chi_2(3b)\right)\chi_2(3a)\chi_2(a+b+c-1)$$

$$+p^2 \cdot \sum_{\substack{a=1 \\ a^3+4c^3+12c^2+12c\equiv 0 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \chi_2(3a)\chi_2(a+c)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\chi_2\left(4-a^3-b^3-4c^3\right)\chi_2(ab)\chi_2(a+b+c-1)$$

$$+\chi_2(3)\cdot p^2 \cdot \sum_{\substack{a=1 \\ a^3+4+12\overline{c}+12\overline{c}^2\equiv 0 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \chi_2(a)\chi_2(a+1)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\chi_2\left(-a^3-b^3-4c^3-12c^2-12c\right)\chi_2(ab)\chi_2(a+b+c)$$

$$+\chi_2(3)\cdot p^2 \cdot \sum_{\substack{a=1 \\ a^3+1+3(2c+1)^2\equiv 0 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \chi_2(a)\chi_2(a+1)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_2\left(-a^3-b^3-4c^3-12c^2-12c\right)\chi_2(ab)\chi_2(a+b+c)$$

$$+p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(-a^3-b^3\right)\chi_2(ab)\chi_2(a+b)$$

$$+\chi_2(3)\cdot p^2 \cdot \left(\sum_{\substack{a=1 \\ a^3+1+3c^2\equiv 0 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \chi_2(a)\chi_2(a+1) - \sum_{\substack{a=1 \\ a^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1) \right)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_2\left(-a^3-b^3-1-3(2c+1)^2\right)\chi_2(ab)\chi_2(a+b+1)$$

$$+p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(-a^3-1\right)\chi_2(a)\chi_2(a+1) - \chi_2(3)p^2 \sum_{\substack{a=1 \\ a^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1)$$

$$+\chi_2(3)\cdot p^2 \cdot \sum_{a=1}^{p-1}\left(1+\chi_2\left(-3a^3-3\right)\right)\chi_2(a)\chi_2(a+1)$$

$$= p \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\chi_2\left(-a^3-b^3-1-3c^2\right)\chi_2(ab)\chi_2(a+b+1)$$

$$-p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(-a^3-b^3-4\right)\chi_2(ab)\chi_2(a+b+1)$$
$$+p(p-1)\cdot\sum_{a=1}^{p-2}\chi_2(-1)\chi_2(a-1+\overline{a})-\chi_2(3)p^2\sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(a)\chi_2(a+1)$$
$$+\chi_2(3)\cdot p^2\cdot\sum_{a=1}^{p-1}\chi_2(a)\chi_2(a+1)+p^2\cdot\sum_{a=1}^{p-2}\chi_2(-1)\chi_2(a-1+\overline{a})$$
$$= p^2\cdot\sum_{\substack{a=1\\ a^3+b^3+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_2(-3ab)\chi_2(a+b+1)-p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(-3ab)\chi_2(a+b+1)$$
$$-p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(-a^3-b^3-4\right)\chi_2(ab)\chi_2(a+b+1)$$
$$+p(2p-1)\cdot\sum_{a=1}^{p-2}\chi_2(-1)\chi_2(a-1+\overline{a})+\chi_2(3)\cdot p^2\cdot\sum_{a=1}^{p-1}\chi_2(a)\chi_2(a+1)$$
$$-\chi_2(3)p^2\sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1}\chi_2(a)\chi_2(a+1). \tag{2.4}$$

It is easy to prove that

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(ab)\chi_2(a+b+1)=\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\chi_2(ab)\chi_2(a+b+1)$$
$$=\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\chi_2(a(b-1))\chi_2(a+b)=\chi_2(-1)(p-1)+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(a(b-1))\chi_2(a+1)$$
$$=\chi_2(-1)(p-1)-\chi_2(-1)\sum_{a=1}^{p-1}\chi_2(a)\chi_2(a+1)=\chi_2(-1)\cdot p. \tag{2.5}$$

Combining (2.4), (2.5) and Lemma 1 we can deduce that

$$\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+3ad^2+b^3+3be^2+4c^3-4\right)\chi_2(a+b+c-1)$$
$$=\chi_2(-1)\cdot(3p^2-p)\cdot\delta(p)-\chi_2(3)\cdot\left(5p^2-p\right)$$
$$-\chi_2(-1)\cdot p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^3+b^3+4\right)\chi_2(ab)\chi_2(a+b+1)$$
$$+\chi_2(3)\cdot p^2\cdot\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right)$$

$$-\chi_2(3)\cdot p^2\cdot \sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1). \tag{2.6}$$

Similarly, applying Lemma 3 we have

$$\begin{aligned}
&\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(b^3+3be^2+4c^3-4\right)\chi_2(b+c-1)\\
=\ & \sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(b^3+3be^2+4c^3-4\right)\chi_2(b+c-1)+p\cdot\sum_{c=1}^{p-1}\chi_2(c^2+3c+3)\\
=\ & p\cdot \sum_{\substack{b=1\\ b^3+4c^3-4\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} \chi_2(3b)\chi_2(b+c-1)-\sum_{b=1}^{p-1}\sum_{c=0}^{p-1}\chi_2(3b)\chi_2(b+c-1)\\
&+p\cdot\sum_{c=0}^{p-1}\chi_2\left((2c+3)^2+3\right)-\chi_2(3)p\\
=\ & -3\chi_2(3)p-p+\chi_2(-1)p\delta(p)-\chi_2(3)p \sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(b)\chi_2(b+1).
\end{aligned} \tag{2.7}$$

Combining (2.3), (2.6) and (2.7) we may immediately deduce the identity

$$\begin{aligned}
&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-1\right)\chi_2(a+b+c-d-e-1)\\
=\ & \chi_2(-1)\cdot(3p^2-p)\cdot\delta(p)-\chi_2(3)\cdot\left(5p^2-p\right)\\
&-\chi_2(-1)\cdot p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^3+b^3+4\right)\chi_2(ab)\chi_2(a+b+1)\\
&+\chi_2(3)\cdot p^2\cdot\sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right)\\
&-\chi_2(3)\cdot p^2\cdot \sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1)\\
&-3\chi_2(3)p^2-p^2+\chi_2(-1)\cdot p^2\cdot\delta(p)-\chi_2(3)\cdot p^2\cdot \sum_{\substack{b=1\\ b^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(b)\chi_2(b+1)\\
=\ & \chi_2(-1)\left(4p^2-p\right)\cdot\delta(p)-\chi_2(3)\cdot\left(8p^2-p\right)-p^2\\
&-\chi_2(-1)\cdot p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^3+b^3+4\right)\chi_2(ab)\chi_2(a+b+1)
\end{aligned}$$

$$-2\chi_2(3)\cdot p^2\cdot \sum_{\substack{a=1\\ a^3+4\equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1)$$
$$+\chi_2(3)\cdot p^2\cdot \sum_{a=1}^{p-2}\chi_2\left(a^2-a+1\right)\chi_2\left(a^3+4\right).$$

This proves Lemma 4.

3. Results

In this paper, we will use the elementary methods and the properties of the character sums of the polynomials to study (1.2) with $h = 3$, and prove the following:

Theorem. Let $p > 3$ be an odd prime, then we have the identity

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^6 = \begin{cases} p^3\left(4\delta(p)+\beta(p)-10\right) & \text{if } p\equiv 1 \bmod 6;\\ p^3\left(4\delta(p)-\beta(p)+10\right) & \text{if } p\equiv -1 \bmod 6.\end{cases}$$

where $\delta(p) = \sum\limits_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right), \beta(p) = \sum\limits_{a=0}^{p-1}\left(\frac{(a^2-a+1)(a^3+4)}{p}\right).$

Proof. Applying several basic lemmas in section 2, we can easily complete the proof of our theorem. In fact, for any odd prime $p > 3$ we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)\cdot\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^6\\
=\ &\tau(\chi_2)\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\sum_{f=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-f^3\right)\\
&\times e\left(\frac{a+b+c-d-e-f}{p}\right)\\
=\ &\tau(\chi_2)\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3\right)e\left(\frac{a+b+c-d-e}{p}\right)\\
&+\tau^2(\chi_2)\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-1\right)\\
&\times\chi_2\left(a+b+c-d-e-1\right)\\
=\ &\tau^2(\chi_2)\cdot\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(1+b^3+c^3-d^3-e^3\right)\chi_2\left(1+b+c-d-e\right)\\
&+\tau^2(\chi_2)\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{e=0}^{p-1}\chi_2\left(a^3+b^3+c^3-d^3-e^3-1\right)\\
&\times\chi_2\left(a+b+c-d-e-1\right)+\sum_{m=1}^{p-1}\chi_2(m)\cdot\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4,
\end{aligned}\tag{3.1}$$

where $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

Note that $\beta(p) = \sum_{a=0}^{p-1} \chi_2\left(a^2 - a + 1\right)\chi_2\left(a^3 + 4\right) = \sum_{a=1}^{p-2} \chi_2\left(a^2 - a + 1\right)\chi_2\left(a^3 + 4\right) + 2$, if $p \equiv 1 \bmod 12$, then $\tau^2(\chi_2) = p$ and $\chi_2(3) = \chi_2(-1) = 1$. From (1.1), (3.1), Lemma 2 and Lemma 4 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_2(m) \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right|^6 \\
= \;& \chi_2(3) \cdot 2 \cdot p^3 \cdot \sum_{\substack{b=1 \\ 4+b^3 \equiv 0 \bmod p}}^{p-1} \chi_2(b)\chi_2(b+1) + p^3 + \chi_2(3) \cdot 2 \cdot p^2 \\
&+\chi_2(-1) \cdot p^2 \cdot \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi_2(bc)\chi_2\left(b^3 + c^3 + 4\right)\chi_2(b + c + 1) \\
&+\chi_2(-1) \cdot \left(4p^3 - p^2\right) \cdot \delta(p) - \chi_2(3) \cdot \left(8p^3 - p^2\right) - p^3 \\
&-\chi_2(-1) \cdot p^2 \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi_2\left(a^3 + b^3 + 4\right)\chi_2(ab)\chi_2(a + b + 1) \\
&+\chi_2(3) \cdot p^3 \cdot \sum_{a=1}^{p-2} \chi_2\left(a^2 - a + 1\right)\chi_2\left(a^3 + 4\right) \\
&-2\chi_2(3) \cdot p^3 \cdot \sum_{\substack{a=1 \\ a^3+4 \equiv 0 \bmod p}}^{p-1} \chi_2(a)\chi_2(a+1) + p^2 \cdot (\delta(p) - 3) \\
= \;& p^3 \cdot (4\delta(p) + \beta(p) - 10). \qquad (3.2)
\end{aligned}$$

Similarly, if $p \equiv 5 \bmod 12$, then $\tau^2(\chi_2) = p$, $\chi_2(-1) = 1$, $\chi_2(3) = -1$. From (1.1), (3.1), Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1} \chi_2(m) \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right|^6 = p^3 \cdot (4\delta(p) - \beta(p) + 10). \qquad (3.3)$$

If $p \equiv 7 \bmod 12$, then $\tau^2(\chi_2) = -p$, $\chi_2(-1) = -1$, $\chi_2(3) = -1$. From (1.1), (3.1), Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1} \chi_2(m) \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right|^6 = p^3 \cdot (4\delta(p) + \beta(p) - 10). \qquad (3.4)$$

If $p \equiv 11 \bmod 12$, then $\tau^2(\chi_2) = -p$, $\chi_2(-1) = -1$, $\chi_2(3) = 1$. From (1.1), (3.1), Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1} \chi_2(m) \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right|^6 = p^3 \cdot (4\delta(p) - \beta(p) + 10). \qquad (3.5)$$

Combining (3.2), (3.3), (3.4) and (3.5) we have the identity

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^6 = \begin{cases} p^3\left(4\delta(p)+\beta(p)-10\right) & \text{if } p\equiv 1 \bmod 6; \\ p^3\left(4\delta(p)-\beta(p)+10\right) & \text{if } p\equiv -1 \bmod 6. \end{cases}$$

This completes the proof of our theorem.

It is clear that from the A. Weil's work [12] we have the estimate $|\delta(p)| \le 2\sqrt{p}$ and $|\beta(p)| \le 5\sqrt{p}$. So from this theorem we can also deduce the following:

Corollary. Let p be an odd prime, then we have the estimate

$$\left|\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\cdot\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^6\right| \le 13\cdot p^{\frac{7}{2}}\cdot\left(1+\frac{1}{\sqrt{p}}\right).$$

Some notes: It is clear that the trivial estimate of $G(3,p)$ is p^4. From our corollary we know that the estimate in our theorem is at most $p^{\frac{7}{2}}$. It saves a square root of p. This sharp estimate maybe have some good applications in some problems of analytic number theory. For example,

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{m^2\cdot a^3+a}{p}\right)\right|^6 = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{r\cdot m^2\cdot a^3+a}{p}\right)\right|^6 + O\left(p^{\frac{7}{2}}\right),$$

where r is any quadratic non-residue modulo p.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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