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Research article

The stability of bifurcating solutions for a prey-predator model with population flux by attractive transition

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Abstract: This paper investigates the stability of bifurcating solutions for a prey-predator model with population flux by attractive transition. Applying spectral analysis and the principle of exchange of stability, we obtain that the bifurcating solutions are stable/unstable under some certain conditions.

Keywords: spectral analysis; stability
Mathematics Subject Classification: 35B32, 35B35

1. Introduction

In this paper, we investigate the following Lotka-Volterra prey-predator model with population flux by attractive transition.

$$\begin{cases} u_t = d_1\Delta u + u(m_1 - u - cv), & x \in \Omega,\ t \in (0, T), \\ v_t = \nabla \cdot [d_2\nabla v + \alpha u^2\nabla(\frac{v}{u})] + v(m_2 + bu - v), & x \in \Omega,\ t \in (0, T), \\ u = v = 0, & x \in \partial\Omega,\ t \in (0, T), \\ u(\cdot, 0) = u_0 \geq 0,\ \ v(\cdot, 0) = v_0 \geq 0, & x \in \Omega. \end{cases} \tag{1.1}$$

Where u and v are the population densities of the prey and predator. d_1 and d_2 are the random diffusion coefficients. m_1 represents the growth rate of the prey population. d_1, d_2 and m_1 are positive constants and m_2 is a real constant which can be negative. m_2 represents the mortality rate while it is negative and it represents the increasing rate of the predator population while it is positive. b and c are positive constants which describe the rate of increase of the predator and the rate of decrease of the prey due to the predation respectively. $J := -\alpha u^2\nabla(\frac{v}{u}) = \alpha(-u\nabla v + v\nabla u)$ represents the population flux of the predator based on a biodiffusion in order that the transition probability of each individual of the predator depends on conditions at the point of arrival (see [2]). The nonnegative constant α is a magnitude of such a population flux by attractive transition.

If $\alpha = 0$ holds, then system (1.1) is reduced to the classical Lotka-Volterra prey-predator model. Some prey-predator models with the linear diffusion terms have been extensively studied by many mathematicians, see [3–7]. Kadota and Kuto [8] investigated a prey-predator system with cross diffusion of quasilinear fractional type. They discussed the local and global bifurcation solutions and obtained a sufficient condition for the existence of positive steady state solutions. Xu and Guo [10] considered the same model as in [8]. They studied the bifurcation steady states which bifurcated from the semitrivial solution with different bifurcating parameter and they obtained the stability of the local bifurcating solutions. Kuto [11] investigated a Lotka-Volterra prey-predator system with cross-diffusion in a spatially heterogeneous environment. Kuto obtained the global bifurcation branch of positive stationary solutions and the bifurcation branch could form a bounded fish-hook curve. Djilali [12] studied the influence of the nonlocal interspecific competition of the prey population on the dynamics of the diffusive predator-prey model with prey social behavior (i.e. herd behavior). It was proved that the turning patterns occur in the presence of the nonlocal competition and can not be found in the original system. Djilali [13] investigated a predator-prey model with social behavior (i.e. herd behavior). The existence of Hopf bifurcation and Turing driven instability were proved. By calculating the normal form, on the center of the manifold associated to the Hopf bifurcation points, the stability of periodic solution was proved. Djilali and Bentout [14] studied the same model as in [13]. They proved the non-existence of a non-constant steady state solution for some values of the diffusion coefficients. They also proved the existence of the non-constant steady state solution under a suitable condision on the diffusion coefficients by applying the Leray-Schauder degree theory.

In the following, we list the local bifurcation results and some preliminary results obtained in [1], which will be used in this paper. The corresponding steady state problem of (1.1) is as follows

$$\begin{cases} d_1\Delta u + u(m_1 - u - cv) = 0, & x \in \Omega, \\ \nabla \cdot [d_2\nabla v + \alpha u^2 \nabla(\frac{v}{u})] + v(m_2 + bu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

It is easy to see that the second equation of (1.2) can be written as

$$d_2\Delta v + \alpha(u\Delta v - v\Delta u) + v(m_2 + bu - v) = 0, x \in \Omega. \tag{1.3}$$

Substituting the first equation of (1.2) into (1.3), we get

$$\Delta v + \frac{v}{d_2 + \alpha u}(\frac{\alpha u}{d_1}(m_1 - u - cv) + m_2 + bu - v) = 0, x \in \Omega. \tag{1.4}$$

Together with (1.3) and (1.4), system (1.2) can be written as

$$\begin{cases} d_1\Delta u + u(m_1 - u - cv) = 0, & x \in \Omega, \\ \Delta v + \frac{v}{d_2+\alpha u}(\frac{\alpha u}{d_1}(m_1 - u - cv) + m_2 + bu - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

For any fixed $m_1 > d_1\lambda_1$(which λ_1 represents the least eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$), system (1.5) has a couple of sets of semitrivial solutions with parameter m_2 which can be denoted as follows

$$\Gamma_u := \{(\theta_{d_1,m_1}, 0, m_2) \in X \times \mathbf{R}\}, \Gamma_v := \{(0, \theta_{d_2,m_2}, m_2) \in X \times (d_2\lambda_1, \infty)\}.$$

For the following equation

$$\begin{cases} -\Delta\phi + q(x)\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

Assume $q(x) \in C(\overline{\Omega})$ holds, let $\lambda_1(q)$ be the least eigenvalue of (1.6), then $q \to \lambda_1(q) : C(\overline{\Omega}) \to \mathbf{R}$ is increasing, i.e. if $q_1(x) \le q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ in Ω, then $\lambda_1(q_1) < \lambda_1(q_2)$.

Define an operator $F : X \times \mathbf{R} \to Y$ by

$$F(u, v, m_2) = \begin{pmatrix} d_1\Delta u + u(m_1 - u - cv) \\ \Delta v + \frac{v}{d_2+\alpha u}(\frac{\alpha u}{d_1}(m_1 - u - cv) + m_2 + bu - v) \end{pmatrix}. \tag{1.7}$$

Then solving system (1.2) is equivalent to solving the equation $F(u, v, m_2) = 0$.

It is easy to compute that

$$\begin{aligned} &F_{(u,v)}(\theta_{d_1,m_1}, 0, m_2)\begin{pmatrix} u \\ v \end{pmatrix} \\ =& \begin{pmatrix} d_1\Delta u + (m_1 - 2\theta_{d_1,m_1})u - c\theta_{d_1,m_1}v \\ \Delta v + \frac{v}{d_2+\alpha\theta_{d_1,m_1}}(\frac{\alpha}{d_1}\theta_{d_1,m_1}(m_1 - \theta_{d_1,m_1}) + m_2 + b\theta_{d_1,m_1}) \end{pmatrix}. \end{aligned} \tag{1.8}$$

According to the result obtained in [1], we have

$$Ker\{F_{(u,v)}(\theta_{d_1,m_1}, 0, m_2)\} = span\{\phi^*, \psi^*\}, \tag{1.9}$$

where ψ^* satisfies the following equation

$$\begin{cases} -\Delta\psi^* - \frac{\psi^*}{d_2+\alpha\theta_{d_1,m_1}}(\frac{\alpha}{d_1}\theta_{d_1,m_1}(m_1 - \theta_{d_1,m_1}) + m_2 + b\theta_{d_1,m_1}) = 0, & x \in \Omega, \\ \psi^* = 0, & x \in \partial\Omega, \end{cases} \tag{1.10}$$

with $\|\psi^*\|_\infty = 1$, $m_2 = f(m_1)$, $\lim\limits_{m_1\to d_1\lambda_1} f(m_1) = d_2\lambda_1$, $\lim\limits_{m_1\to\infty} f(m_1) = -\infty$ and

$$\phi^* = [-d_1\Delta + 2\theta_{d_1,m_1} - m_1]^{-1}(-c\theta_{d_1,m_1}\psi^*). \tag{1.11}$$

In the following, we restate the existence of bifurcating solutions which bifurcate from $(\theta_{d_1,m_1}, 0, f(m_1))$ obtained in [1].

Lemma 1.1 (Proposition 4.4 in [1]) *Let $m_1 \in (d_1\lambda_1, \infty)$ be given arbitrarily. Positive solutions of (1.2) bifurcate from Γ_u as $m_2 = f(m_1)$. There exists a neighborhood $\mathcal{N}_1$ of $(u, v, m_2) = (\theta_{d_1,m_1}, 0, f(m_1)) \in X \times \mathbf{R}$ such that $F^{-1}(0) \cap \mathcal{N}_1$ consists of the union of $\Gamma_u \cap \mathcal{N}_1$ and the local curve*

$$\begin{bmatrix} u \\ v \\ m_2 \end{bmatrix}(s) = \begin{bmatrix} \theta_{d_1,m_1} \\ 0 \\ f(m_1) \end{bmatrix} + \begin{bmatrix} s(\phi^* + \widetilde{u}(s)) \\ s(\psi^* + \widetilde{v}(s)) \\ \mu(s) \end{bmatrix}, \quad s \in (-\delta, \delta), \tag{1.12}$$

with some $\delta > 0$. Here $(\widetilde{u}, \widetilde{v}, \mu)(s) \in X \times \mathbf{R}$ is continuous differentiable for $s \in (-\delta, \delta)$ satisfying $\int_\Omega \psi^* \widetilde{v}(s) = 0$ for all $s \in (-\delta, \delta)$ and $(\widetilde{u}, \widetilde{v}, \mu)(0) = (0, 0, 0)$ and

$$\mu'(0) = \frac{\int_\Omega \frac{(\psi^*)^2}{d_2+\alpha\theta_{d_1,m_1}} \left(\frac{\alpha^2\theta^2_{d_1,m_1} + \alpha\{d_1 f(m_1) - d_2(m_1 - 2\theta_{d_1,m_1})\} - bd_1d_2}{d_2+\alpha\theta_{d_1,m_1}} \phi^* + (\alpha c\theta_{d_1,m_1} + d_1)\psi^* \right)}{d_1 \int_\Omega \frac{(\psi^*)^2}{d_2+\alpha\theta_{d_1,m_1}}}. \tag{1.13}$$

All positive solutions contained in $F^{-1}(0) \cap \mathcal{N}_1$ can be expressed as

$$C_u^+ := \{(u, v, m_2)(s) : 0 < s < \delta\}.$$

In the following, we rewrite the existence of bifurcating solutions which bifurcate from $(0, \theta_{d_2,g(m_1)}, g(m_1))$ obtained in [1], first we give some preliminary results which has been obtained in [1].

Let $V = v - \theta_{d_2,m_2}$, $\widetilde{F}(u, V, m_2) := F(u, V + \theta_{d_2,m_2}, m_2)$, where F is defined by (1.7). Thus we have

$$\widetilde{F}_{(u,V)}(0, 0, m_2) \begin{pmatrix} u \\ V \end{pmatrix} = \begin{pmatrix} d_1\Delta u + (m_1 - c\theta_{d_2,m_2})u \\ \Delta V + d_2^{-1}\{h_{21}(x)u + (m_2 - 2\theta_{d_2,m_2})V\} \end{pmatrix}, \tag{1.14}$$

where

$$h_{21}(x) = [\alpha\{\frac{m_1}{d_1} - \frac{m_2}{d_2} - (\frac{c}{d_1} - \frac{1}{d_2})\theta_{d_2,m_2}\} + b]\theta_{d_2,m_2}. \tag{1.15}$$

$Ker(\widetilde{F}_{(u,V)}(0, 0, m_2)) = span\{\overline{\phi}, \overline{\psi}\}$, where $\overline{\phi}$ satisfies

$$\begin{cases} -\Delta\overline{\phi} + \frac{c}{d_1}\theta_{d_2,m_2}\overline{\phi} = \frac{m_1}{d_1}\overline{\phi}, x \in \Omega, \\ \overline{\phi} = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.16}$$

with $m_1 = d_1\lambda_1(\frac{c\theta_{d_2,m_2}}{d_1})(=: g^{-1}(m_2))$, $\|\overline{\phi}\|_\infty = 1$, where $g^{-1}(m_2)$ is continuously differentiable and monotone increasing for $m_2 > d_2\lambda_1$ such that

$$\lim_{m_2 \to d_2\lambda_1} g^{-1}(m_2) = d_1\lambda_1, \quad \lim_{m_2 \to \infty} g^{-1}(m_2) = \infty. \tag{1.17}$$

For convenience, we denote the inverse function of $m_1 = g^{-1}(m_2)$ by $m_2 = g(m_1)$.

$$\overline{\psi} = [-d_2\Delta + 2\theta_{d_2,g(m_1)} - g(m_1)]^{-1}(h_{21}\overline{\phi}). \tag{1.18}$$

Now we restate the existence of bifurcating solutions which bifurcate from

$(0, \theta_{d_2,g(m_1)}, g(m_1))$ obtained in [1].

Lemma 1.2 (Proposition 4.6 in [1]) *Let $m_1 \in (d_1\lambda_1, \infty)$ be given arbitrarily. Positive solutions of (1.2) bifurcate from Γ_v as $m_2 = g(m_1)$. There exists a neighborhood $\mathcal{N}_2$ of $(u, v, m_2) = (0, \theta_{d_2,g(m_1)}, g(m_1))) \in X \times \mathbf{R}$ such that $F^{-1}(0) \cap \mathcal{N}_2$ consists of the union of $\Gamma_v \cap \mathcal{N}_2$ and the local curve*

$$\begin{bmatrix} u \\ v \\ m_2 \end{bmatrix}(s) = \begin{bmatrix} 0 \\ \theta_{d_2,g(m_1)} \\ g(m_1) \end{bmatrix} + \begin{bmatrix} s(\overline{\phi} + \widetilde{u}(s)) \\ s(\overline{\psi} + \widetilde{v}(s)) \\ \mu(s) \end{bmatrix}, \quad s \in (-\delta, \delta), \tag{1.19}$$

with some $\delta > 0$. *Here* $(\widetilde{u}, \widetilde{v}, \mu)(s) \in X \times \mathbf{R}$ *is continuous differentiable for* $s \in (-\delta, \delta)$ *satisfying* $\int_\Omega \overline{\phi}\widetilde{u}(s) = 0$ *for all* $s \in (-\delta, \delta)$ *and* $(\widetilde{u}, \widetilde{v}, \mu)(0) = (0, 0, 0)$ *and*

$$\mu^{'}(0) = -\frac{\int_\Omega (\overline{\phi} + c\overline{\psi})\overline{\phi}^2}{c\int_\Omega \zeta(x)\overline{\phi}^2}, \text{ where } \zeta := \frac{\partial \theta_{d_2,m_2}}{\partial m_2}\,|_{m_2 = g(m_1)}\ (> 0). \tag{1.20}$$

All positive solutions contained in $F^{-1}(0) \cap \mathcal{N}_2$ *can be expressed as*

$$C_v^+ := \{(u, v, m_2)(s) : 0 < s < \delta\}.$$

Oeda and Kuto [1] gives the asymptotic behavior of positive solutions of (1.2) as $\alpha \to \infty$ which can be written as follows.

Lemma 1.3 (Theorem 2.2 in [1]) *Suppose that* $(m_1, m_2, d_1, d_2, b, c)$ *satisfies*

$$\begin{aligned} &m_1 > d_1\lambda_1,\\ &m_2 \neq \tfrac{d_2}{d_1}m_1 - (\tfrac{d_2}{d_1} + b)\tfrac{\|\theta_{d_1,m_1}\|_2^2}{\|\theta_{d_1,m_1}\|_1}(= f^\infty(m_1, d_1, d_2, b)),\\ &m_2 \neq \tfrac{d_2}{d_1}m_1 - (\tfrac{d_2}{d_1} - \tfrac{1}{c})\tfrac{\|\theta_{d_1,m_1}\|_2^2}{\|\theta_{d_1,m_1}\|_1}(= h(m_1, d_1, d_2, c)),\\ &m_2 \neq g(m_1, d_1, d_2, c). \end{aligned} \tag{1.21}$$

Let $\{(u_n, v_n)\}$ *be any sequence of positive solutions to (1.2) with* $\alpha = \alpha_n \to \infty$. *Then the following alternative holds true.*

(i) If $\{(\alpha_n\|u_n\|_\infty)\}$ *is unbounded, then*

$$f^\infty(m_1, d_1, d_2, b) < m_2 < h(m_1, d_1, d_2, c)$$

and

$$\lim_{n\to\infty}(u_n, v_n) = (1 - s, \frac{s}{c})\theta_{d_1,m_1} \ \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

passing to a subsequence, where $s \in (0, 1)$ *is defined by*

$$m_2 = (1 - s)f^\infty(m_1, d_1, d_2, b) + sh(m_1, d_1, d_2, c).$$

(ii) If $\{(\alpha_n\|u_n\|_\infty)\}$ *is bounded, then there exists* $(w, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ *such that*

$$\lim_{n\to\infty}(\alpha_n u_n, v_n) = (w, v) \ \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

passing to a subsequence, and moreover, (w, v) *is a positive solution to*

$$\begin{cases} d_1\Delta u + u(m_1 - cv) = 0, & x \in \Omega,\\ \nabla \cdot [d_2\nabla v + w^2\nabla(\frac{v}{w})] + v(m_2 - v) = 0, & x \in \Omega,\\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.22}$$

According to the first type (i), the coexistence steady state (u, v) of prey and predator can be approximated by a coexistence steady state of the competition model with equal conditions. In the second type (ii), the component of prey shrinks with order $O(\frac{1}{\alpha})$ when α is sufficiently large. The bifurcation structure of positive solutions of (1.22) will be discussed in a forthcoming paper [9].

In this paper, we study the stability of bifurcating solutions obtained in [1]. Applying spectral analysis and the principle of exchange of stability, we obtain that the bifurcating solutions are stable/unstable under some certain conditions. The plan of this paper is as follows. In section 2, we prove that the bifurcating solutions near $(\theta_{d_1,m_1}, 0, f(m_1))$ are locally asymptotically stable/unstable under some certain conditions. In section 3, we prove that the bifurcating steady states near $(0, \theta_{d_2,g(m_1)}, g(m_1))$ are locally asymptotically stable/unstable under some certain conditions. A conclusion section ends the paper.

2. The stability of bifurcating steady states near $(0, \theta_{d_2,g(m_1)}, g(m_1))$

We consider the stability of bifurcating solutions near $(0, \theta_{d_2,g(m_1)}, g(m_1))$ of the following system.

$$\begin{cases} u_t = d_1\Delta u + u(m_1 - u - cv), \quad (x,t) \in \Omega \times (0,+\infty), \\ v_t = \Delta v + \frac{v}{d_2+\alpha u}(\frac{\alpha u}{d_1}(m_1 - u - cv) + m_2 + bu - v), \quad (x,t) \in \Omega \times (0,+\infty), \\ u = v = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.1}$$

Denote

$$\begin{aligned} &\widetilde{f}(u(s), v(s), m_2) = \frac{d_2\alpha m_1 v(s)}{d_1(d_2+\alpha u(s))^2} - \frac{2d_2\alpha u(s)v(s)+\alpha^2u^2(s)v(s)}{d_1(d_2+\alpha u(s))^2} - \frac{\alpha d_2 cv^2(s)}{d_1(d_2+\alpha u(s))^2} \\ &-\frac{\alpha m_2 v(s)}{(d_2+\alpha u(s))^2} + \frac{bd_2v(s)}{(d_2+\alpha u(s))^2} + \frac{\alpha v^2(s)}{(d_2+\alpha u(s))^2}, \\ &\widetilde{g}(u(s), v(s), m_2) = \frac{\alpha m_1 u(s)}{d_1(d_2+\alpha u(s))} - \frac{\alpha u^2(s)}{d_1(d_2+\alpha u(s))} - \frac{2\alpha cu(s)v(s)}{d_1(d_2+\alpha u(s))} + \frac{m_2}{d_2+\alpha u(s)} \\ &+\frac{bu(s)}{d_2+\alpha u(s)} - \frac{2v(s)}{d_2+\alpha u(s)} \end{aligned} \tag{2.2}$$

Linearizing (2.1) at $(u(s), v(s))$ defined by (1.19) and investigating the following eigenvalue problem, by (2.2), we have

$$\begin{cases} d_1\Delta u + m_1 u - 2u(s)u - cv(s)u - cu(s)v = \lambda u, \quad x \in \Omega, \\ \Delta v + \widetilde{f}(u(s), v(s), m_2)u + \widetilde{g}(u(s), v(s), m_2)v = \lambda v, \ \ x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.3}$$

According to (1.7), (2.3) can be rewritten as follows

$$F_{(u,v)}(u(s), v(s), m_2)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix}, \tag{2.4}$$

$$\begin{aligned} &F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1))\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} d_1\Delta u + m_1 u - c\theta_{d_2,g(m_1)}u \\ \Delta v + \widetilde{f}(0, \theta_{d_2,g(m_1)}, g(m_1))u + \widetilde{g}(0, \theta_{d_2,g(m_1)}, g(m_1))v \end{pmatrix} \\ &= \begin{pmatrix} d_1\Delta u + m_1 u - c\theta_{d_2,g(m_1)}u \\ \Delta v + d_2^{-1}\{h_{21}(x)u + (m_2 - 2\theta_{d_2,g(m_1)})v\}, \end{pmatrix}, \end{aligned} \tag{2.5}$$

where $h_{21}(x)$ is defined by (1.15).

It is easy to see that

$$Ker(F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))) = span\{\overline{\phi},\overline{\psi}\}, \tag{2.6}$$

where $\overline{\phi}$ and $\overline{\psi}$ are defined by (1.16) and (1.18).

According to Shi [16] (Theorem 2.1 and (4.5)), we can define the functional

$$l_1 : X \to \mathbb{R} by < [f,g], l_1 >:= \int_\Omega f\overline{\phi}dx. \tag{2.7}$$

Theorem 2.1. *For any fixed $m_1 \in (d_1\lambda_1,\infty)$, the bifurcating steady state $(u(s),v(s))$ defined by (1.19) of system (1.2) is locally asymptotically stable when $\mu^{'}(0) < 0$ defined by (1.20); the bifurcating steady state $(u(s),v(s))$ defined by (1.19) of system (1.2) is locally asymptotically unstable when $\mu^{'}(0) > 0$ defined by (1.20).*

Proof. First we show that 0 is the first eigenvalue of $F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))$.

From the above, we get that 0 is the eigenvalue of $F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))$. Therefore we will show that 0 is the first eigenvalue of $F_{(u,v)}(\theta_{d_1,m_1},0,f(m_1))$. Otherwise, there exists a positive eigenvalue $\widetilde{\lambda}_1$ of $F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))$ with the corresponding eigenfunction $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in X$ such that

$$F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \widetilde{\lambda}_1 u_1 \\ \widetilde{\lambda} v_1 \end{pmatrix}, \tag{2.8}$$

that is

$$\begin{cases} d_1\Delta u_1 + m_1u_1 - c\theta_{d_2,g(m_1)}u_1 = \widetilde{\lambda}_1 u_1, \\ \Delta v_1 + \widetilde{f}(0,\theta_{d_2,g(m_1)},g(m_1))u_1 + \widetilde{g}(0,\theta_{d_2,g(m_1)},g(m_1))v_1 = \widetilde{\lambda}_1 v_1, \end{cases} \tag{2.9}$$

where $\widetilde{f}$ and $\widetilde{g}$ are defined by (2.2)

Assume $u_1 = 0$ and $v_1 \neq 0$ hold, from the second equation of (2.9), we obtain

$$-d_2\Delta v_1 + 2\theta_{d_2,g(m_1)}v_1 - g(m_1)v_1 = -\widetilde{\lambda}_1 d_2 v_1. \tag{2.10}$$

Because of $\widetilde{\lambda}_1 d_2 > 0$ and in [17] (Lemma 2.1),it was proved that all the eigenvalues of the operator $(-d_2\Delta + 2\theta_{d_2,g(m_1)} - g(m_1))$ are strictly positive,which is in contradiction with (2.10), therefore we have $u_1 \neq 0$.

By virtue of (1.16) and the scalar elliptic equation theorem, 0 is the first eigenvalue of the first equation of (2.9), which contradicts $\widetilde{\lambda}_1$. Therefore we obtain that 0 is the first eigenvalue of $F_{(u,v)}(0,\theta_{d_2,g(m_1)},g(m_1))$ and the other eigenvalues are negative.

For small $0 < s < \delta$, by Proposition I.7.2 in [15], there exist perturbed eigenvalue $\lambda(s)$ and continuous differential functions $\varphi_1(s),\varphi_2(s) \in X \cap Range(F_{u,v}(0,\theta_{d_2,g(m_1)},g(m_1)))$ satisfying

$$F(u(s),v(s),m_2(s))\begin{pmatrix} \overline{\phi}+\varphi_1(s) \\ \overline{\psi}+\varphi_2(s) \end{pmatrix} = \lambda(s)\begin{pmatrix} \overline{\phi}+\varphi_1(s) \\ \overline{\psi}+\varphi_2(s) \end{pmatrix}, \tag{2.11}$$

with $\lambda(0) = \varphi_1(0) = \varphi_2(0) = 0$.

Similarly, there exist perturbed eigenvalue $\lambda(m_2)$ and continuous differential functions $\varphi_1(m_2), \varphi_2(m_2) \in X \cap Range(F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)))$ satisfying

$$F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)) \begin{pmatrix} \overline{\phi} + \varphi_1(m_2) \\ \overline{\psi} + \varphi_2(m_2) \end{pmatrix} = \lambda(m_2) \begin{pmatrix} \overline{\phi} + \varphi_1(m_2) \\ \overline{\psi} + \varphi_2(m_2) \end{pmatrix}, \tag{2.12}$$

with $\lambda(m_2) = \varphi_1(m_2) = \varphi_2(m_2) = 0$.

Differentiating (2.12) with respect to m_2 at $m_2 = g(m_1)$ and together with $\lambda(m_2) = \varphi_1(m_2) = \varphi_2(m_2) = 0$, we obtain

$$\begin{aligned} &\tfrac{d}{dm_2} F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)) \begin{pmatrix} \overline{\phi} \\ \overline{\psi} \end{pmatrix} \\ &+ F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)) \begin{pmatrix} \varphi_1'(g(m_1)) \\ \varphi_2'(g(m_1)) \end{pmatrix} = \lambda'(g(m_1)) \begin{pmatrix} \overline{\phi} \\ \overline{\psi} \end{pmatrix}, \end{aligned} \tag{2.13}$$

where $\lambda^{'}(g(m_1)) = \frac{d}{dm_2}\lambda(m_2)|_{m_2=g(m_1)}$.

According to (2.7) and (2.13), we have

$$\langle \frac{d}{dm_2} F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)) \begin{pmatrix} \overline{\phi} \\ \overline{\psi} \end{pmatrix}, l_1 \rangle = \lambda'(g(m_1)) \parallel \overline{\phi} \parallel^2_{L^2(\Omega)} . \tag{2.14}$$

In virtue of (2.5), we have

$$\begin{aligned} &\tfrac{d}{dm_2} F_{(u,v)}(0, \theta_{d_2,g(m_1)}, g(m_1)) \begin{pmatrix} \overline{\phi} \\ \overline{\psi} \end{pmatrix} \\ = &\begin{pmatrix} -c \frac{\partial \theta_{d_2,m_2}}{\partial m_2} |_{m_2=g(m_1)} \\ -\frac{\alpha \theta_{d_2,m_2}}{d_2^2} \overline{\phi} + \frac{1}{d_2} \overline{\psi} + \frac{1}{d_2} \overline{\phi} \frac{\partial h_{21}(x)}{\partial \theta_{d_2,m_2}} \frac{\partial \theta_{d_2,m_2}}{\partial m_2} |_{m_2=g(m_1)} - \frac{2}{d_2} \overline{\psi} \frac{\partial \theta_{d_2,m_2}}{\partial m_2} |_{m_2=g(m_1)} \end{pmatrix}. \end{aligned} \tag{2.15}$$

According to (2.7), (2.14) and (2.15), we get

$$\begin{aligned} \lambda'(g(m_1)) \parallel \overline{\phi} \parallel^2_{L^2(\Omega)} = \int_\Omega -c \frac{\partial \theta_{d_2,m_2}}{\partial m_2} |_{m_2=g(m_1)} \overline{\phi} dx < 0. \\ \left(\frac{\partial \theta_{d_2,m_2}}{\partial m_2} |_{m_2=g(m_1)} > 0, c > 0, \overline{\phi} > 0 \right). \end{aligned} \tag{2.16}$$

Applying the formula I.7.40 in [15], we have

$$- \dot{\lambda}(0) = \dot{m}_2(0) \lambda'(g(m_1)), \tag{2.17}$$

where $\dot{\lambda}(s) = \frac{d}{ds}\lambda(s)$.

Using Lemma 1.2 and (2.17), we have

$$sgn(\dot{\lambda}(0)) = sgn(\dot{m}_2(0)) = sgn(\mu^{'}(0)), \tag{2.18}$$

where $\mu^{'}(0)$ is defined by (1.20).

By (2.18),when $\mu^{'}(0) < 0$ holds, then $\lambda(s) < 0$ for small $s > 0$,the bifurcating solution $(u(s), v(s))$ defined by (1.19) of system (1.2) is locally asymptotically stable.When $\mu^{'}(0) > 0$ holds,then $\lambda(s) > 0$ for small $s > 0$,the bifurcating solution $(u(s), v(s))$ defined by (1.19) of system (1.2) is locally asymptotically unstable. □

For system (2.1), let $\alpha = 100$, $b = 2, c = 10$, $m_1 = 300$, $m_2 = 20$, $\Omega = (0, 1)$, $t = 1000$ and $(u_0, v_0) = (0.01sin^2(\pi x), 0.01sin(\pi x))$ hold, which guarantee $\mu'(0) < 0$. We give the following simulation results which verify the stability of locally bifurcating steady states near $(0, \theta_{d_2,g(m_1)}, g(m_1))$, see Figure 1.

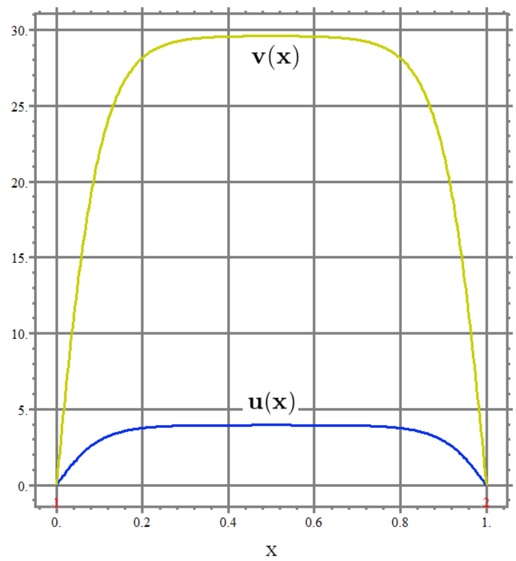

Figure 1. The stability of locally bifurcating steady states near $(0, \theta_{d_2,g(m_1)}, g(m_1))$.

3. The stability of bifurcating steady states near $(\theta_{d_1,m_1}, 0, f(m_1))$

In this section, we use the similar method in section 2 in order to investigate the stability of positive solutions bifurcating from $(\theta_{d_1,m_1}, 0, f(m_1))$.

We linearize (1.2) at $(u(s), v(s))$ defined by (1.12) and study the following eigenvalue problem

$$\begin{cases} d_1\Delta u + m_1 u - 2u(s)u - cv(s)u - cu(s)v = \sigma u, \quad x \in \Omega, \\ \Delta v + \widetilde{f}(u(s), v(s), m_2)u + \widetilde{g}(u(s), v(s), m_2)v = \sigma v, x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega. \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, \end{cases} \tag{3.1}$$

where $\widetilde{f}$ and $\widetilde{g}$ are defined by (2.2).

By (1.7), (3.1) can be rewritten by

$$F_{(u,v)}(u(s), v(s), m_2)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sigma u \\ \sigma v \end{pmatrix}, \tag{3.2}$$

$$F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1\Delta u + m_1 u - 2\theta_{d_1,m_1} u - c\theta_{d_1,m_1} v \\ \Delta v + \widetilde{g}(\theta_{d_1,m_1}, 0, f(m_1))v \end{pmatrix}, \tag{3.3}$$

where $\widetilde{g}$ is defined by (2.2).

Obviously, we have

$$N(F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))) = span\{\phi^*, \psi^*\}, \tag{3.4}$$

where ϕ^* and ψ^* are defined by (1.10) and (1.11).

According to Shi [16] (Theorem 2.1 and (4.5)), we can define the functional

$$l_2 : X \to \mathbb{R} by < [f, g], l_2 >:= \int_\Omega g\psi^* dx. \tag{3.5}$$

Theorem 3.2. *The bifurcating solution* $(u(s), v(s))$ *defined by (1.12) of system (1.2) is locally asymptotically stable when* $\mu'(0) > 0$ *holds defined by (1.13); the bifurcating steady state* $(u(s), v(s))$ *defined by (1.12) of system (1.2) is locally asymptotically unstable when* $\mu'(0) < 0$ *holds defined by (1.13).*

Proof. We first prove that 0 is the first eigenvalue of $F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))$.

From the above, we obtain that 0 is the eigenvalue of $F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))$. Then we will prove that 0 is the first eigenvalue of $F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))$. Otherwise, there exists a positive eigenvalue σ_1 of $F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))$ with the corresponding eigenfunction $\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \end{pmatrix} \in X$ such that

$$F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1)) \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \end{pmatrix} = \begin{pmatrix} \sigma_1 \widetilde{u}_1 \\ \sigma_1 \widetilde{v}_1 \end{pmatrix}, \tag{3.6}$$

that is

$$\begin{cases} d_1 \Delta \widetilde{u}_1 + m_1 \widetilde{u}_1 - 2\theta_{d_1,m_1} \widetilde{u}_1 - c\theta_{d_1,m_1} \widetilde{v}_1 = \sigma_1 \widetilde{u}_1, & x \in \Omega, \\ \Delta \widetilde{v}_1 + \widetilde{g}(\theta_{d_1,m_1}, 0, f(m_1)) \widetilde{v}_1 = \sigma_1 \widetilde{v}_1, & x \in \Omega, \\ \widetilde{u}_1 = \widetilde{v}_1 = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

where $\widetilde{g}$ is defined by (2.2).

If $\widetilde{v}_1 = 0$ and $\widetilde{u}_1 \neq 0$ hold, the first equation of (3.7) implies

$$- d_1 \Delta \widetilde{u}_1 + 2\theta_{d_1,m_1} \widetilde{u}_1 - m_1 \widetilde{u}_1 = -\sigma_1 \widetilde{u}_1, \tag{3.8}$$

In [17] (Lemma 2.1), it was proved that all the eigenvalues of the operator $(-d_1\Delta + 2\theta_{d_1,m_1} - m_1)$ are strictly positive,which is in contradiction with (3.8), then $\widetilde{v}_1 \neq 0$.

According to (1.10) and the scalar elliptic equation theorem, 0 is the first eigenvalue of the second equation of (3.7), which contradicts σ_1. Then we have proved that 0 is the first eigenvalue of $F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))$ and the other eigenvalues are negative.

For small $0 < s < \delta$, by Proposition I.7.2 in [15], there exist perturbed eigenvalue $\sigma(s)$ and continuous differential functions $\omega_1(s), \omega_2(s) \in X \cap Range(F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1)))$ satisfying

$$F(u(s), v(s), m_2(s)) \begin{pmatrix} \phi^* + \omega_1(s) \\ \psi^* + \omega_2(s) \end{pmatrix} = \sigma(s) \begin{pmatrix} \phi^* + \omega_1(s) \\ \psi^* + \omega_2(s) \end{pmatrix}, \tag{3.9}$$

with $\sigma(0) = \omega_1(0) = \omega_2(0) = 0$.

Similarly, there exist perturbed eigenvalue $\sigma(m_2)$ and continuous differential functions $\omega_1(m_2), \omega_2(m_2) \in X \cap Range(F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1)))$ satisfying

$$F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} \phi^* + \omega_1(m_2) \\ \psi^* + \omega_2(m_2) \end{pmatrix} = \sigma(m_2)\begin{pmatrix} \phi^* + \omega_1(m_2) \\ \psi^* + \omega_2(m_2) \end{pmatrix}, \tag{3.10}$$

with $\sigma(m_2) = \omega_1(m_2) = \omega_2(m_2) = 0$.

Differentiating (3.10) with respect to m_2 at $m_2 = f(m_1)$ and together with $\sigma(m_2) = \omega_1(m_2) = \omega_2(m_2) = 0$, we have

$$\begin{aligned} &\tfrac{d}{dm_2}F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix} \\ &+F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} \omega_1'(f(m_1)) \\ \omega_2'(f(m_1)) \end{pmatrix} = \sigma'(f(m_1))\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix}, \end{aligned} \tag{3.11}$$

where $\sigma^{'}(f(m_1)) = \frac{d}{dm_2}\sigma(m_2)|_{m_2=f(m_1)}$.

Together with (3.5) and (3.11), we obtain

$$\langle\frac{d}{dm_2}F_{(u,v)}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix}, l_2\rangle = \sigma'(f(m_1)) \parallel \psi^* \parallel^2_{L^2(\Omega)} . \tag{3.12}$$

According to (1.8), we have

$$\frac{d}{dm_2}F_{u,v}(\theta_{d_1,m_1}, 0, f(m_1))\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\psi^*}{d_2+\alpha\theta_{d_1,m_1}} \end{pmatrix}. \tag{3.13}$$

Using (3.5), (3.12) and (3.13), we obtain

$$\sigma'(f(m_1)) \parallel \psi^* \parallel^2_{L^2(\Omega)} = \int_\Omega \frac{(\psi^*)^2}{d_2 + \alpha\theta_{d_1,m_1}}dx > 0. \tag{3.14}$$

It follows from the formula I.7.40 in [15] that

$$-\dot{\sigma}(0) = \dot{m}_2(0)\sigma'(f(m_1)), \tag{3.15}$$

where $\dot{\sigma}(s) = \frac{d}{ds}\sigma(s)$.

Together with Lemma 1.1 and (3.14), we obtain

$$sgn(\dot{\sigma}(0)) = -sgn(\dot{m}_2(0)) = -sgn(\mu^{'}(0)), \tag{3.16}$$

where $\mu^{'}(0)$ is defined by (1.13). According to (3.16),when $\mu^{'}(0) > 0$ holds, then $\lambda(s) < 0$ for small $s > 0$, the bifurcating solution $(u(s), v(s))$ defined by (1.12) of system (1.2) is locally asymptotically stable. When $\mu^{'}(0) < 0$ holds, then $\lambda(s) > 0$ for small $s > 0$, the bifurcating solution $(u(s), v(s))$ defined by (1.12) of system (1.2) is locally asymptotically unstable. □

4. Conclusions

In this paper, we have investigated the local stability of bifurcation steady states obtained in [1] for a prey-predator model with population flux by attractive transition. By applying spectral analysis and the principle of exchange of stability, we show the stability/unstability of the bifurcating solutions under some certain conditions. We give numerical simulation result (which satisfies $\mu^{'}(0) < 0$) in order to verify the local stability of bifurcation solutions near the bifurcating point $(0, \theta_{d_2,g(m_1)}, g(m_1))$.

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Conflict of interest

The authors declare no conflicts of interest.

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