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Research article

Multiple periodic solutions of differential delay systems with $2k$ lags

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Abstract: The quantity of $2(2k + 1)$-periodic solutions to a specific differential delay system with $2k$ lags is studied and resolved by variational methods. Several results are revealed and two examples are given to illustrate the application of the main results.

Keywords: periodic solution; delay; differential system; variation
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1. Introduction

Delay differential equations naturally appear in many fields of science and engineering. Such equations have been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration and cardiac arrhythemias. In recent years, the existence of positive periodic solutions for delay differential equations has been received considerable attention. J. Kaplan and J. Yorke [10] gave conditions for the existence of 4− or 6− periodic solutions of equation

$$x'(t) = -\sum_{i=1}^{n} f(x(t-i)), \quad x \in \mathbb{R} \tag{1.1}$$

while $n = 1$ and $n = 2$, respectively. After then, lots of results are achieved for this equation in general cases [1–12]. W. Ge [4–6] proved several existence theorems of periodic solutions for (1.1) in the general case by use of the fixed-point theorem in cone. After transforming (1.1) into a Hamiltonian system and applying a theorem given by Mawhin and Willem [13], J.Li and X. He [11,12] proved a theorem for the existence of several periodic solutions under the condition $xf(x) > 0$ ($xf(x) < 0$) for

$x \neq 0$. G. Fei [1,2] studied the multiplicity of periodic solutions of (1.1) and proved theorems for the cases $n = 2k - 1$ and $n = 2k$ by using the index theory. In his work, the way to construct the functional for the second case is quite different from that for the first case.

Applying S^1 index theory Z. Guo and J. Yu [8] gave a theorem for the multiple periodic solutions of the delay-differential system with one lag in the form

$$x'(t) = -\nabla F(x(t-l)), x \in \mathbb{R}^N, F \in C^0(\mathbb{R}^N, \mathbb{R}^N). \tag{1.2}$$

And by applying the same index theory, in [9], they obtained results on the multiplicity of $2(n+1)$-periodic solutions to the delay differential system

$$x'(t) = -\sum_{t=1}^{n} \nabla F(x(t-i)), x \in \mathbb{R}^N, F \in C^0(\mathbb{R}^N, \mathbb{R}^N). \tag{1.3}$$

when $n = 2k - 1$. After then B. Zheng and Z. Guo studied (1.3) while $n = 2k$ and gave two criteria for the multiplicity of $2(2k+1)$-periodic solutions. The theorems given in both[9] and [14] contain a condition to calculate the following function

$$\Psi(A, B) = \sum_{j=1}^{\infty} \Psi_j(A, B), \quad n = 2k - 1,$$

$$\Psi(A, B) = \sum_{j=1, j \notin \Gamma}^{\infty} \Psi_j(A, B), \quad \Gamma \subset \mathbb{N}^+, \quad n = 2k,$$

for symmetric matrices A and B, respectively. This is a hard task to be fulfilled since it is not definite for the problem how large the j must be to ensure $\Psi_j(A, B) = 0$.

In [7], we researched the same problem for system (1.3) when $n = 2k-1$ and by constructing a new functional we gave criteria for the multiplicity of periodic solutions only based on the eigenvalues of symmetric matrices A and B.

In this paper we are to study the multiplicity of $2(2k+1)$-periodic solutions for system (1.3) when $n = 2k$, i.e.,

$$x'(t) = -\sum_{i=1}^{2k} \nabla F(x(t-i)), x \in \mathbb{R}^N, F \in C^0(\mathbb{R}^N, \mathbb{R}^N). \tag{1.4}$$

with the conditions $F \in C^1(\mathbb{R}^N, \mathbb{R}), \nabla F(-x) = -F(x), F(0) = 0$,

$$x(t-k-1) = -x(t). \tag{1.5}$$

At the same time, assume there are real symmetric matrices A_∞ and A_0 such that

$$\nabla F(x) = A_\infty x + o(|x|) \text{ as } |x| \to \infty, \nabla F(x) = A_0 x + o(|x|) \text{ as } |x| \to 0.$$

In section 2, some notations and variation structure for system (1.4) are introduced. Meanwhile, a lemma about the relation between the periodic solutions of system(1.4) and the critical points of the functional Φ are also proved in section 2. Section 3 presents and proves the main results of this paper. Before the proof of the main results some lemmas about the calculation of the differential of functional Φ are proven. Two examples are given in section 4 to illustrate the applications of the main results.

2. $2(2k+1)$-periodic solution of (1.4) and variation structure

Firstly, we consider a linear space which consists of all the $2(2k+1)$-periodic functions satisfying (1.5) in $\mathbb{R}^N$, i.e.

$$H^{\frac{1}{2}} = \{ \quad x(t) = \sum_{i=1}^{\infty}(a_i \cos \tfrac{(2i-1)\pi}{2k+1}t + b_i \sin \tfrac{(2i-1)\pi}{2k+1}t) : a_i, b_i \in \mathbb{R}^N, \\ \sum_{i=1}^{\infty}(2i-1)(|a_i|^2 + |b_i|^2) < \infty\}$$

Since our goal is to find the $2(2k+1)$-periodic solutions for system (1.4) with condition (1.5), we discuss now the special requirement for a solution of system (1.4) in $H^{\frac{1}{2}}$. After then, we are able to choose a suitable Hilbert space for our problem. Assume $\sum_{i=m}^{n} a_i = 0$ if $n < m$.

If $x(t)$ is a $2(2k+1)$-periodic solution of (1.4) satisfying (1.5), then, $x'(t-l) = \sum_{i=0}^{l-1} \nabla F(x(t-i)) - \sum_{i=l+1}^{2k} \nabla F(x(t-i)),\ l = 0, 1, \cdots, 2k$. Hence,

$$\sum_{l=0}^{2k}(-1)^l x'(t-l) = 0. \tag{2.1}$$

From the equation above, one has $\sum_{l=0}^{2k}(-1)^l x(t-l) = c$. However, $c = \sum_{l=0}^{2k}(-1)^l x(t-l-2k-1) = -\sum_{l=0}^{2k}(-1)^l x(t-l) = -c$, which means $c = 0$. Then,

$$\sum_{l=0}^{2k}(-1)^l x(t-l) = 0. \tag{2.2}$$

By Fourier's expansion theory and (1.5), it holds that

$$\begin{aligned} x(t) &= \sum_{i=1}^{\infty}(a_i \cos \tfrac{(2i-1)\pi}{2k+1}t + b_i \sin \tfrac{(2i-1)\pi}{2k+1}t) \\ &= \sum_{l=0}^{\infty}\sum_{i=1}^{2k+1}(a_{l(2k+1)+i} \cos \tfrac{(2l(2k+1)+2i-1)\pi}{2k+1}t + b_{l(2k+1)+i} \sin \tfrac{(2l(2k+1)+2i-1)\pi}{2k+1}t). \end{aligned}$$

Substituting the above expression into (2.2) we have

$$a_{l(2k+1)+k+1} = b_{l(2k+1)+k+1} = 0. \tag{2.3}$$

Then suppose

$$X = \left\{ x \in H^{\frac{1}{2}} : x(t-2k-1) = -x(t), \sum_{i=0}^{2k}(-1)^i x(t-i) = 0 \right\}.$$

For $x \in X$, define $P : X \to X$:

$$x \mapsto \sum_{l=0}^{\infty}(\sum_{i=1}^{k} + \sum_{i=k+2}^{2k+1})(2l(2k+1)+2i-1)(a_{l(2k+1)+i} \cos \tfrac{(2l(2k+1)+2i-1)\pi}{2k+1}t \\ + b_{l(2k+1)+i} \sin \tfrac{(2l(2k+1)+2i-1)\pi}{2k+1}t),$$

and denote

$$\langle x,y\rangle_X=\int_0^{2(2k+1)}(Px(t),y(t))dt,\ \langle x,y\rangle=\int_0^{2(2k+1)}(x(t),y(t))dt,x,y\in X.$$

Define

$$\|x\|_X=\sqrt{\langle x,x\rangle_X},\qquad \|x\|_2=\sqrt{\langle x,x\rangle}=\sqrt{\langle P^{-1}x,x\rangle_X}.$$

It is obvious that $(X,\|\cdot\|_X)$ is a Hilbert space and $P:X\to X$ is an invertible and self-adjoint operator. Define

$$Z=\{z(t)\}=\{(z_1(t),z_2(t),\cdots,z_{2k}(t))\}=\{(x(t),x(t-1),\cdots,x(t-2k+1)):x\in X\}\subset X^{2k}.$$

Therefore $z_1(t)=x(t)\in X$. Furthermore, let

$$X(i)=\left\{a\cos\frac{(2i-1)\pi}{2k+1}t+b\sin\frac{(2i-1)\pi}{2k+1}t:a,b\in\mathbb{R}^N\right\},$$

$$Z(i)=\{(x(t),x(t-1),\cdots,x(t-2k+1)):x(t)\in X(i)\}.$$

Then,

$$Z=\sum_{\substack{i=1\\ i\neq l(2k+1)+k+1}}^{\infty}Z(i)=\sum_{l=0}^{\infty}\sum_{i=1}^{k}Z(l(2k+1)+i).$$

Denote $H(z)=\sum\limits_{i=0}^{2k-1}F(x(t-i))+F(\sum\limits_{i=0}^{2k-1}(-1)^{i+1}x(t-i))$, then,

$$\begin{aligned}\nabla H(z)&=(\tfrac{\partial H(z)}{\partial z_1},\tfrac{\partial H(z)}{\partial z_2},\cdots,\tfrac{\partial H(z)}{\partial z_{2k}})\\&=(\nabla F(x(t))-\nabla F(x(t-2k)),\nabla F(x(t-1))+\nabla F(x(t-2k)),\cdots,\\&\quad\nabla F(x(t-1))-(-1)^i\nabla F(x(t-2k)),\cdots,\\&\quad\nabla F(x(t-2k+1))+\nabla F(x(t-2k)))\end{aligned}$$

where $x(t-2k)=\sum\limits_{i=0}^{2k-1}(-1)^{i+1}x(t-i),\ z_l=x(t-l+1),l=1,2,\cdots,2k$, and from (2.2),

$$z_{2k+1}=\sum_{i=0}^{2k-1}(-1)^{i+1}x(t-i)=\sum_{i=1}^{2k}(-1)^iz_i.$$

Define $\Phi:Z\to\mathbb{R}$ as a functional of z:

$$\Phi(z)=\frac{1}{2}\langle Lz,z\rangle+G(z)\tag{2.4}$$

where

$$\begin{aligned} Lz = \; & P^{-1}\left(\sum_{i=1}^{2k-1}(-1)^{i+1}x'(t-i),\cdots,\sum_{i=0}^{l-1}(-1)^{i+l}x'(t-i)+\sum_{i=l+1}^{2k-1}(-1)^{i+l+1}x'(t-i),\cdots,\right.\\ & \left.\sum_{i=0}^{2k-2}(-1)^{i+2k-1}x'(t-i)\right),\\ G(z) = \; & -\int_0^{2(2k+1)}(\sum_{i=0}^{2k-1}F(x(t-i))+F(\sum_{i=0}^{2k-1}(-1)^{i+1}x(t-i)))dt,\\ \langle z,y\rangle = \; & \int_0^{2(2k+1)}(\hat{P}z,y)dt,\ z,y\in Z, \end{aligned}$$

$\hat{P}=(P,P,\cdots,P):Z\to Z$. Then

$$||z|| = \langle z,z\rangle = 2k\langle x,x\rangle_X = 2k||x||_X^2, \tag{2.5}$$

$$\begin{aligned} \Phi(z) \; = \; & \frac{1}{2}\int_0^{2(2k+1)}\sum_{l=0}^{2k-1}(\sum_{i=0}^{l-1}(-1)^{i+l}x'(t-i)+\sum_{i=l+1}^{2k-1}(-1)^{i+l+1}x'(t-i),x(t-l))dt\\ & -\int_0^{2(2k+1)}[\sum_{i=0}^{2k-1}F(x(t-i))+F(\sum_{i=0}^{2k-1}(-1)^{i+1}x(t-i))]dt\\ = \; & \frac{1}{2}\sum_{0\le i<l\le 2k-1}\int_0^{2(2k+1)}(-1)^{i+l+1}[(x(t-i),x'(t-l))-(x'(t-i),x(t-l))]dt\\ & -\int_0^{2(2k+1)}[\sum_{i=0}^{2k-1}F(x(t-i))+F(\sum_{i=0}^{2k-1}(-1)^{i+1}x(t-i))]dt. \end{aligned}$$

By Mawhin-Willem's theorem [13, Theorem 1.4], Φ is continuously differentiable and

$$\begin{aligned} \langle\Phi'(z),v\rangle = \; & \langle Lu,v\rangle_Z-\int_0^{2(2k+1)}(\sum_{l=0}^{2k-1}(\nabla F(x(t-l))-(-1)^l\nabla F(x(t-2k)),y(t-l)))dt\\ = \; & \sum_{l=0}^{2k-1}\int_0^{2(2k+1)}(\sum_{i=0}^{l-1}(-1)^{i+l}x'(t-i)+\sum_{i=l+1}^{2k-1}(-1)^{i+l+1}x'(t-i)-\nabla F(x(t-l))\\ & -(-1)^{l+1}\nabla F(x(t-2k)),y(t-l)))dt \end{aligned}$$

where $z(t)=(x(t),x(t-1),\cdots,x(t-2k+1))$, $v(t)=(y(t),y(t-1),\cdots,y(t-2k+1))$.

Denote $\Phi'(z)=(\Phi_1'(z),\Phi_2'(z),\cdots,\Phi_{2k}'(z))$, where

$$\begin{aligned} \Phi_{l+1}'(z) = \; & P^{-1}(\sum_{i=0}^{l-1}(-1)^{i+l}x'(t-i)+\sum_{i=l}^{2k-1}(-1)^{i+l+1}x'(t-i)-\nabla F(x(t-l))\\ & +(-1)^l\nabla F(x(t-2k))),\ l=0,1,\cdots,2k-1. \end{aligned} \tag{2.6}$$

If a point $z\in Z$ satisfies $\Phi'(z)=0$, then z is called a critical point of Φ. The following lemma reveals the relation between the $2(2k+1)$-periodic solutions of system (1.4) and the critical points of Φ. Based on this lemma we can discuss the multiplicity of critical points of Φ instead of discussing that of $2(2k+1)$-periodic solutions of system (1.4). In the following, denote $x(t)$ as the first N components of $z(t)$ in Z and set $\sum_{i=0}^{-1}x'(t-i)=0$.

Lemma 2.1 If $x\in X$, $z=(x(t),x(t-1),\cdots,x(t-2k+1))\in Z$, then the following propositions are equivalent to each other.

1) $x(t)$ is one of the solutions of system (1.4),

2) $z(t)$ satisfies $\Phi_1'(z) = 0$, i.e.

$$\sum_{i=1}^{2k-1}(-1)^{i+1}x'(t-i) - \nabla F(x(t)) + \nabla F(x(t-2k)) = 0, \tag{2.7}$$

3) $z(t)$ is one of the critical points of Φ on Z, i.e., $z(t)$ satisfies

$$\sum_{i=0}^{l-1}(-1)^{i+l}x'(t-i) + \sum_{i=l+1}^{2k-1}(-1)^{i+l+1}x'(t-i) - \nabla F(x(t-l)) + (-1)^l\nabla F(x(t-2k)) = 0, \qquad l = 0, 1, \cdots, 2k-1 \qquad (2.8-l)$$

Proof. 1)⇒ 2).

Suppose $x(t)$ is a solution of system (1.4).

By the periodicity of $x(t)$ and (1.5), one has

$$x'(t-l) = \sum_{i=0}^{l-1}\nabla F(x(t-i)) - \sum_{i=l+1}^{2k-1}\nabla F(x(t-i)) - \nabla F(x(t-2k)) \qquad (2.9-l)$$

$l = 1, 2, \cdots, 2k-1$, $x(t-2k) = \sum_{i=0}^{2k-1}(-1)^{i+1}x(t-i)$. Multiplying $(-1)^{l+1}$ to the equalities $(2.9-l)$ and summing them up, we have

$$\sum_{l=1}^{2k-1}(-1)^{l+1}x'(t-l) = \nabla F(x(t)) - \nabla F(x(t-2k)),$$

which means (2.7) holds.

2)⇒ 3).

System (2.7) is the same as $(2.8-0)$.

From (2.7) one has

$$\sum_{i=1}^{2k-1}(-1)^{i+1}x'(t-i-1) - \nabla F(x(t-1)) + \nabla F(x(t-2k-1)) = 0,$$

i.e.,

$$\begin{aligned} 0 &= \sum_{i=2}^{2k-1}(-1)^i x'(t-i) + x'(t-2k) - \nabla F(x(t-1)) - \nabla F(x(t)) \\ &= \sum_{i=2}^{2k-1}(-1)^i x'(t-i) + \sum_{i=0}^{2k-1}(-1)^{i+1}x'(t-i) - \nabla F(x(t-1)) - \nabla F(x(t)). \end{aligned} \tag{2.10}$$

Then (2.7) and (2.10) yield

$$-x'(t) + \sum_{i=2}^{2k-1}(-1)^i x'(t-i) - \nabla F(x(t-1)) - \nabla F(x(t-2k)) = 0,$$

which means $(2.8-1)$.

Suppose $(2.8-l)$ holds. Then,

$$\begin{aligned} 0 = &\sum_{i=1}^{l}(-1)^{i+l+1}x'(t-i) + \sum_{i=l+2}^{2k-1}(-1)^{i+l+2}x'(t-i) + (-1)^l x'(t-2k) \\ &-\nabla F(x(t-l-1)) + (-1)^{l+1}\nabla F(x(t)) \\ = &\sum_{i=1}^{l}(-1)^{i+l+1}x'(t-i) + \sum_{i=l+2}^{2k-1}(-1)^{i+l+2}x'(t-i) + (-1)^{l+1}\sum_{i=0}^{2k-1}(-1)^i x'(t-i) \\ &-\nabla F(x(t-l-1)) + (-1)^{l+1}\nabla F(x(t)). \end{aligned} \tag{2.11}$$

Multiplying (2.7)with $(-1)^{l+1}$ and add it to (2.11), one can obtain $(2.8-(l+1))$.

3) $\Rightarrow$ 1).

Summing the equalities from $(2.8-1)$ to $(2.8-(2k-1))$, one has

$$-x'(t)=\sum_{i=1}^{2k}\nabla F(x(t-i)),$$

which implies $x(t)$ is a solution of system (1.4).

Denote $\hat{P}_i : Z \to Z(i)$, $Z(i) = \{(x(t), x(t-1), \cdots, x(t-2k+1)) : x(t) = a_i\cos\frac{(2i-1)\pi}{2k+1}t + b_i\sin\frac{(2i-1)\pi}{2k+1}t, a_i, b_i \in \mathbb{R}^N\}$, $i \neq (2l+1)(2k+1)$; $Z((2l+1)(2k+1)) = 0$, $\quad l \in \mathbb{N}^+ \cup \{0\}$. Furthermore, let $P_i = \sum_{j=1}^{i}\hat{P}_j : Z \to \sum_{j=1}^{i} Z(i)$, $\quad i = 1, 2, \cdots$. Then, one has $LP_i = P_iL$.

Let $-G'(z) = (P^{-1}(\nabla F(x(t)) - \nabla F(x(t-2k))), P^{-1}(\nabla F(x(t-1)) + \nabla F(x(t-2k))), \cdots,$ $P^{-1}(\nabla F(x(t-l)) + (-1)^{l+1}\nabla F(x(t-2k))), \cdots, P^{-1}(\nabla F(x(t-2k+1)) - \nabla F(x(t-2k))))$. Then $\Phi'(z) = Lz + G'(z)$. L is a bounded self-adjoint operator and $G'(z) : Z \to X^{2k}$ is compact since $(G'z)(t)$ is differentiable with respect to t.

Denote $B_\infty, B_0 \in \mathbb{R}^{2kN\times 2kN}$ as

$$B_\infty=(\frac{2k+1}{2k}A_\infty,\cdots,\frac{2k+1}{2k}A_\infty), B_0=(\frac{2k+1}{2k}A_0,\cdots,\frac{2k+1}{2k}A_0)$$

and for $z(t) = (x(t), x(t-1), \cdots, x(t-2k+1))$,

$$\begin{aligned} B_\infty z &= (\tfrac{2k+1}{2k}A_\infty x(t), \tfrac{2k+1}{2k}A_\infty x(t-1), \cdots, \tfrac{2k+1}{2k}A_\infty x(t-2k+1)),\\ B_0 z &= (\tfrac{2k+1}{2k}A_0 x(t), \tfrac{2k+1}{2k}A_0 x(t-1), \cdots, \tfrac{2k+1}{2k}A_0 x(t-2k+1)). \end{aligned} \tag{2.12}$$

Define $\delta z(t) = T_{2kN}z(t) : Z \to Z$, where

$$T_{2kN}=\begin{bmatrix} I & -I & \cdots & I & -I\\ I & O & \cdots & O & O\\ O & I & \cdots & O & O\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ O & O & \cdots & I & O \end{bmatrix} \tag{2.13}$$

is a $2kN\times 2kN$ matrix, I is a $N\times N$ unit matrix and O is a $N\times N$ zero matrix.

From (1.5) and (2.2), one has $\delta z(t) = z(t+1)$ and $\delta^{2k+1}z(t) = -z(t)$. Then, $\{1, \delta, \delta^2, \cdots, \delta^{4k+1}\}$ is a Lie group with $\delta^{4k+2} = 1$, where 1 stands for the identity transform in Z.

Now we have

$$\Phi(\delta z)=\Phi(z),$$

$$\Phi'(\delta z)=\delta\Phi'(z),$$

i.e., with respect to the Lie group $\{1, \delta, \delta^2, \cdots, \delta^{4k+1}\}$, Φ is invariant and Φ' is $\delta-$equivariant.

In order to prove our results we shall apply the following lemma.

Lemma 2.2 [2, Lemma 2.4]

Assume there are two S^1-invariant linear subspaces, Z^+ and $Z^- \subset Z$, and $r > 0$ such that

(a) $Z^+ \cup Z^-$ is a closed and finite codimension in Z,

(b) $\hat{L}(Z^-) \subset Z^-$ with $\hat{L} = L - P^{-1}B_\infty$ or $\hat{L} = L - P^{-1}B_0$,
(c) there exists $c_\infty \in \mathbb{R}$ such that

$$\Phi(z) \le c_\infty < \Phi(0),\ \forall z \in Z^- \cap S_r = \{z \in Z^- : \|z\| = r\},$$

(d) there exists $c_0 \in \mathbb{R}$ such that

$$\inf_{z\in Z^+} \Phi(z) \ge c_0,$$

(e) Φ satisfies $(P.S)_c$-condition for $c_0 \le c \le c_\infty$.

Then Φ possesses at least $m = \frac{1}{2}[\dim(Z^-\cap Z^+) - \mathrm{codim}_Z(Z^-\cup Z^+)]$ different critical orbits in $\Phi^{-1}([c_0, c_\infty])$ provided $m > 0$.

Remark 2.1 $(P.S)_c$-condition in (e) can be replaced by (P.S)-condition since (P.S)-condition implies $(P.S)_c$-condition for each $c \in \mathbb{R}$.

Remark 2.2 If $(P.S)_c$-condition in (e) is replaced by (P.S)-condition, (c) and (d) can be changed into
(c') there exists $c_\infty \in \mathbb{R}$ such that

$$\Phi(z) \ge c_\infty > \Phi(0),\ \forall z \in Z^- \cap S_r = \{z \in Z^- : \|z\| = r\},$$

(d') there exists $c_0 \in \mathbb{R}$ such that

$$\inf_{z\in Z^+} \Phi(z) \le c_0$$

if $\Phi(z)$ is replaced by $-\Phi(z)$.

Remark 2.3 When $Z = \bigoplus_{i=1}^{\infty} Z(i)$, $Z^+(i) = Z^+ \cap Z(i)$, $Z^-(i) = Z^- \cap Z(i)$, we have $\dim Z(i) = 2N$ and

$$\begin{aligned} m &= \frac{1}{2}\sum_{i=1}^{\infty}[\dim(Z^+(i) \cap Z^-(i)) - \mathrm{codim}_{Z(i)}(Z^+(i) + Z^-(i))] \\ &= \frac{1}{2}\sum_{i=1}^{\infty}[\dim(Z^+(i) \cap Z^-(i)) - (\dim Z(i) - \dim(Z^+(i) + Z^-(i)))] \\ &= \frac{1}{2}\sum_{i=1}^{\infty}[\dim Z^+(i) + dimZ^-(i) - 2N]. \end{aligned} \tag{2.14}$$

3. Main results

Suppose $\{\alpha_j : j = 1, 2, \cdots, N\}$ are the eigenvalues of A_∞ and $\{\beta_j : j = 1, 2, \cdots, N\}$ are the eigenvalues of A_0, $d_j, e_j \in \mathbb{R}^N$ are the eigenvectors of A_∞ and A_0 with respect to α_j and β_j, respectively. Assume $\{d_1, d_2, \cdots, d_N\}$ and $\{e_1, e_2, \cdots, e_N\}$ are two orthogonal bases of $\mathbb{R}^N$.

Lemma 3.1 If $z \in Z(i),\ i \ne l(k+1)$, then there is $\gamma = \frac{\pi}{2k+1}$ such that

$$\langle Lz, z\rangle = \frac{2k+1}{2k}(2i-1)\gamma \tan\frac{(2i-1)\gamma}{2}\langle z, z\rangle. \tag{3.1}$$

Proof. Denote $|\cdot|$ as the norm in $\mathbb{R}^N$, then

$$\langle z, z\rangle = 2k\int_0^{2(2k+1)} |x(t)|^2 dt = 2k(2k+1)(|a_i|^2 + |b_i|^2). \tag{3.2}$$

It follows that

$$
\begin{aligned}
&\langle Lz, z\rangle \\
&= \sum_{0\le m<l\le 2k-1}(-1)^{m+l+1}\int_0^{2(2k+1)}[(x(t-m), x'(t-l)) - (x'(t-m), x(t-l))]dt \\
&= 2\sum_{0\le m<l\le 2k-1}(-1)^{m+l+1}\int_0^{2(2k+1)}(x(t-m), x'(t-l))dt \\
&\overset{j=l-m}{=} 2\sum_{j=1}^{2k-1}\sum_{l=j}^{2k-1}(-1)^{2l-j+1}\int_0^{2(2k+1)}(x(t-l+j), x'(t-l))dt \\
&= 2\sum_{j=1}^{2k-1}(-1)^{j+1}(2k-j)\int_0^{2(2k+1)}(x(t+j), x'(t))dt \\
&= 2\sum_{j=1}^{2k-1}(2k-j)(-1)^{j+1}(2i-1)\gamma(2k+1)(|a|^2+|b|^2)\sin j(2i-1)\gamma \\
&= 2(2i-1)\gamma(2k+1)\sum_{j=1}^{2k-1}(2k-j)(-1)^{j+1}\sin j(2i-1)\gamma(|a|^2+|b|^2).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{j=1}^{2k-1}(2k-j)(-1)^{j+1}\sin j(2i-1)\gamma \\
= \;& \frac{1}{2\cos\frac{(2i-1)\gamma}{2}}\sum_{j=1}^{2k-1}(-1)^{j+1}(2k-j)[\sin\frac{(2j-1)(2i-1)\gamma}{2} + \sin\frac{(2j+1)(2i-1)\gamma}{2}] \\
= \;& \frac{1}{2\cos\frac{(2i-1)\gamma}{2}}[2k\sin\frac{(2i-1)\gamma}{2} + \sum_{j=1}^{2k}(-1)^j\sin\frac{(2j-1)(2i-1)\gamma}{2}] \\
= \;& k\tan\frac{(2i-1)\gamma}{2} + \frac{1}{2\cos\frac{(2i-1)\gamma}{2}}\sum_{j=1}^{2k}(-1)^j[\sin j(2i-1)\gamma\cos\frac{(2i-1)\gamma}{2} \\
& -\cos j(2i-1)\gamma\sin\frac{(2i-1)\gamma}{2}] \\
= \;& (k+\frac{1}{2})\tan\frac{(2i-1)\gamma}{2},
\end{aligned}
$$

one has

$$
\begin{aligned}
\langle Lz, z\rangle &= (2k+1)^2(2i-1)\gamma\tan\frac{(2i-1)\gamma}{2}(|a_i|^2+|b_i|^2) \\
&= \frac{2k+1}{2k}(2i-1)\gamma\tan\frac{(2i-1)\gamma}{2}\langle z, z\rangle.
\end{aligned}
$$

Lemma 3.1 is proved.

Denote $\mathbb{E}_j = \{\lambda e_j : \lambda\in\mathbb{R}\}$, $\mathbb{D}_j = \{\lambda d_j : \lambda\in\mathbb{R}\}$, $j = 1, 2, \cdots, N$, $l \ge 0$ and

$$
\text{ind }\alpha_j = \begin{cases}
\sum_{i=1}^{k}\sharp\{l : 0 < [2l(2k+1)+(2i-1)]\gamma\tan\frac{(2i-1)\gamma}{2} < \alpha_j\}, & \alpha_j > 0, \\
0, & \alpha_j = 0, \\
-\sum_{i=1}^{k}\sharp\{l : 0 < [(2l+1)(2k+1)+2(k+1-i)]\gamma\tan\frac{(2i-1)\gamma}{2} < -\alpha_j\}, & \alpha_j < 0,
\end{cases}
$$

$$\operatorname{ind}\beta_j=\begin{cases}\sum\limits_{i=1}^{k}\sharp\{l:0<[2l(2k+1)+(2i-1)]\gamma\tan\frac{(2i-1)\gamma}{2}<\beta_j\}, & \beta_j>0,\\ 0, & \beta_j=0,\\ -\sum\limits_{i=1}^{k}\sharp\{l:0<[(2l+1)(2k+1)+2(k+1-i)]\gamma\tan\frac{(2i-1)\gamma}{2}<-\beta_j\}, & \beta_j<0,\end{cases}$$

then $\mathbb{R}^N=\sum\limits_{j=1}^{N}\mathbb{D}_j=\sum\limits_{j=1}^{N}\mathbb{E}_j$. Suppose

$$\operatorname{ind}A_\infty=\sum_{j=1}^{N}\operatorname{ind}\alpha_j,\ \operatorname{ind}A_0=\sum_{j=1}^{N}\operatorname{ind}\beta_j, \tag{3.3}$$

and

$$X_{D,j}(i)=\{x(t)=a_i\cos\frac{(2i-1)\gamma}{2}t+b_i\sin\frac{(2i-1)\gamma}{2}t, a_i,b_i\in\mathbb{D}_j\},$$
$$X_{E,j}(i)=\{x(t)=a_i\cos\frac{(2i-1)\gamma}{2}t+b_i\sin\frac{(2i-1)\gamma}{2}t, a_i,b_i\in\mathbb{E}_j\},$$

therefore,

$$X(i)=\sum_{j=1}^{N}X_{D,j}(i)=\sum_{j=1}^{N}X_{E,j}(i). \tag{3.4}$$

Denote

$$\begin{aligned}x(t)=\ &\sum_{l=0}^{\infty}\sum_{i=1}^{k}[a_{l(2k+1)+i}\cos(2l(2k+1)+2i-1)\gamma t+b_{l(2k+1)+i}\sin(2l(2k+1)+2i-1)\gamma t\\ &+a_{l(2k+1)+k+1+i}\cos((2l+1)(2k+1)+2i)\gamma t\\ &+b_{l(2k+1)+k+1+i}\sin((2l+1)(2k+1)+2i)\gamma t].\end{aligned}$$

It follows from (3.1) and (3.2) that for $z_1(t)=x(t)$, one has

$$\begin{aligned}&\langle Lz,z\rangle\\ &=\gamma(2k+1)^2\sum_{l=0}^{\infty}\sum_{i=1}^{k}[(2l(2k+1)+2i-1)\tan\tfrac{(2l(2k+1)+2i-1)\gamma}{2}(|a_{l(2k+1)+i}|^2+|b_{l(2k+1)+i}|^2)\\ &\quad+(2l(2k+1)+2(k+1+i)-1)\tan\tfrac{(2l(2k+1)+2(k+1+i)-1)\gamma}{2}(|a_{l(2k+1)+k+1+i}|^2\\ &\quad+|b_{l(2k+1)+k+1+i}|^2)]\\ &=\gamma(2k+1)^2\sum_{l=0}^{\infty}\sum_{i=1}^{k}[(2l(2k+1)+2i-1)\tan\tfrac{(2i-1)\gamma}{2}(|a_{l(2k+1)+i}|^2+|b_{l(2k+1)+i}|^2)\\ &\quad-((2l+1)(2k+1)+2(k+1-i))\tan\tfrac{(2i-1)\gamma}{2}(|a_{l(2k+1)+2k+2-i}|^2+|b_{l(2k+1)+2k+2-i}|^2)].\end{aligned} \tag{3.5}$$

and

$$\langle P^{-1}A_\infty x,x\rangle=\int_0^{2(k+1)}(A_\infty x,x)dt=(2k+1)\alpha_j(|a_i|^2+|b_i|^2),\ x\in X_{D,j}(i),$$
$$\langle P^{-1}A_0 x,x\rangle=\int_0^{2(k+1)}(A_0 x,x)dt=(2k+1)\beta_j(|a_i|^2+|b_i|^2),\ x\in X_{E,j}(i),$$

and

$$\begin{aligned}&\langle\hat{P}^{-1}B_\infty z,z\rangle=\alpha_j(2k+1)^2(|a_i|^2+|b_i|^2),\quad x\in X_{D,j},\\ &\langle\hat{P}^{-1}B_0 z,z\rangle=\beta_j(2k+1)^2(|a_i|^2+|b_i|^2),\quad x\in X_{E,j}.\end{aligned} \tag{3.6}$$

Denote

$$Z^{-}_{\infty}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{\infty}z,z\rangle<0\},\ Z^{-}_{0}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{0}z,z\rangle<0\},$$

$$Z^{0}_{\infty}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{\infty}z,z\rangle=0\},\ Z^{0}_{0}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{0}z,z\rangle=0\},$$

$$Z^{+}_{\infty}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{\infty}z,z\rangle>0\},\ Z^{+}_{0}=\{z\in Z:\langle Lz-\hat{P}^{-1}B_{0}z,z\rangle>0\}.$$

Then

$$Z=Z^{-}_{\infty}\bigoplus Z^{0}_{\infty}\bigoplus Z^{+}_{\infty}=Z^{-}_{0}\bigoplus Z^{0}_{0}\bigoplus Z^{+}_{0}. \tag{3.7}$$

From (3.1) and Lemma 3.1 [7], we have the following lemmas.

Lemma 3.2 $\hat{L}(Z^{-}_{\infty})\subset Z^{-}_{\infty}$, $\hat{L}(Z^{0}_{\infty})\subset Z^{0}_{\infty}$, $\hat{L}(Z^{+}_{\infty})\subset Z^{+}_{\infty}$, $\hat{L}(Z^{-}_{0})\subset Z^{-}_{0}$, $\hat{L}(Z^{0}_{0})\subset Z^{0}_{0}$, $\hat{L}(Z^{+}_{0})\subset Z^{+}_{0}$, ($\hat{L}=L-\hat{P}^{-1}B_{\infty}$ or $\hat{L}=L-\hat{P}^{-1}B_{0}$).

Lemma 3.3 All the subspaces of Z,

$$Z^{-}_{0}+Z^{+}_{\infty},\ Z^{+}_{0}+Z^{-}_{\infty},\ Z^{+}_{0}+Z^{-}_{\infty}+Z^{0}_{0},\ Z^{-}_{0}+Z^{+}_{\infty}+Z^{0}_{0},\ Z^{+}_{0}+Z^{-}_{\infty}+Z^{0}_{\infty},$$

$$Z^{-}_{0}+Z^{+}_{\infty}+Z^{0}_{\infty},\ Z^{-}_{0}+Z^{+}_{\infty}+Z^{0}_{0}+Z^{0}_{\infty},\ Z^{+}_{0}+Z^{-}_{\infty}+Z^{0}_{0}+Z^{0}_{\infty},$$

are of finite codimensions in Z.

Denote $\Gamma^{+}_{\infty}=\{\alpha_j>0:$ there are $l\geq 0,\ i\in\{1,2,\cdots,k\}$ such that $\alpha_j=\gamma(2l(2k+1)+2i-1)\tan\frac{(2i-1)\gamma}{2}\}$, $\Gamma^{-}_{\infty}=\{\alpha_j<0:$ there are $l\geq 0,\ i\in\{1,2,\cdots,k\}$ such that $\alpha_j=-\gamma((2l+1)(2k+1)+2(k+1-i))\tan\frac{(2i-1)\gamma}{2}\}$, $\Gamma^{+}_{0}=\{\beta_j>0:$ there are $l\geq 0,\ i\in\{1,2,\cdots,k\}$ such that $\beta_j=\gamma(2l(2k+1)+2i-1)\tan\frac{(2i-1)\gamma}{2}\}$, $\Gamma^{-}_{0}=\{\beta_j<0:$ there are $l\geq 0,\ i\in\{1,2,\cdots,k\}$ such that $\beta_j=-\gamma((2l+1)(2k+1)+2(k+1-i))\tan\frac{(2i-1)\gamma}{2}\}$, $\Gamma_{\infty}=\Gamma^{+}_{\infty}\cup\Gamma^{-}_{\infty}$, $\Gamma_{0}=\Gamma^{+}_{0}\cup\Gamma^{-}_{0}$.

For $i\in\{1,2,\cdots,k\}$ and $l\geq 0$, denote

$\eta^{+}_{\infty}=\sharp\{(l,i):$ there is $\alpha_j\in\Gamma^{+}_{\infty}$ such that $\gamma(2l(2k+1)+2i-1)\tan\frac{(2i-1)\gamma}{2}=\alpha_j\}$,

$\eta^{-}_{\infty}=\sharp\{(l,i):$ there is $\alpha_j\in\Gamma^{-}_{\infty}$ such that $-\gamma((2l+1)(2k+1)+2(k+1-i))\tan\frac{(2i-1)\gamma}{2}=\alpha_j\}$,

$\eta^{+}_{0}=\sharp\{(l,i):$ there is $\beta_j\in\Gamma^{+}_{0}$ such that $\gamma(2l(2k+1)+2i-1)\tan\frac{(2i-1)\gamma}{2}=\beta_j\}$,

$\eta^{-}_{0}=\sharp\{(l,i):$ there is $\beta_j\in\Gamma^{-}_{0}$ such that $-\gamma((2l+1)(2k+1)+2(k+1-i))\tan\frac{(2i-1)\gamma}{2}=\beta_j\}$.

Let $\mathbb{D}=\bigoplus\{\mathbb{D}_j:\alpha_j\in\Gamma_{\infty}\}$ and $\Pi:\mathbb{R}^N\to\mathbb{D}$ be an orthogonal projection with $\Pi\mathbb{R}^N=\mathbb{D}$. Assume

(A_1) $F\in C^1(\mathbb{R}^N,\mathbb{R})$ satisfies (1.4) and $F(-x)=F(x),\ F(0)=0$,

(A_2) there exist M and $r\in C^0(\mathbb{R}^+,\mathbb{R}^+)$ with $r(\infty)=+\infty$, $\frac{r(s)}{s}\to 0$ as $s\to\infty$, such that $|F(x)-\frac{1}{2}(A_{\infty}x,x)|\geq -M+r(|\,\Pi x\,|)$ whenever $x\in\cup\{\mathbb{D}_j:\alpha_j\in\Gamma_{\infty}\}$,

$(A^{\pm}_3)$ $\pm[F(x)-\frac{1}{2}(A_{\infty}x,x)]>0$, as $|x|\to\infty$,

$(A^{\pm}_4)$ $\pm[F(x)-\frac{1}{2}(A_{0}x,x)]>0, 0<|x|\ll 1$.

By a standard argument as the proof of Lemma 2.1 [13] and Lemma 3.3 [8], we have

Lemma 3.4 Assume (A_1) and (A_2) hold. Then, $\Phi(x)$ defined by (2.4) satisfies (P.S)-condition.

Lemma 3.5 Suppose (A_1) and (A_2) hold. Then there is $I>0$ such that

$$\begin{aligned} m &= \sum_{\substack{1\leq i<\infty\\ i\neq l(2k+1)+k+1}}[\dim(Z^{+}(i)\cap Z^{-}(i))-\mathrm{codim}_{Z(i)}(Z^{+}(i)+Z^{-}(i))]\\ &= \sum_{\substack{1\leq i\leq I\\ i\neq l(2k+1)+k+1}}[\dim(Z^{+}(i)\cap Z^{-}(i))-\mathrm{codim}_{Z(i)}(Z^{+}(i)+Z^{-}(i))], \end{aligned}$$

if

$$(Z^+, Z^-) \in \begin{array}{l}\{(Z_\infty^+, Z_0^-), (Z_0^+, Z_\infty^-), (Z_\infty^-, Z_0^+), (Z_0^-, Z_\infty^+), \\ (Z_\infty^+ + Z_\infty^0, Z_0^-), (Z_\infty^+, Z_0^- + Z_0^0), (Z_0^-, Z_\infty^+ + Z_\infty^0), (Z_0^- + Z_0^0, Z_\infty^+), \\ (Z_0^+, Z_\infty^- + Z_\infty^0), (Z_0^+ + Z_0^0, Z_\infty^-), (Z_\infty^- + Z_\infty^0, Z_0^+), (Z_\infty^-, Z_0^+ + Z_0^0), \\ (Z_\infty^+ + Z_\infty^0, Z_0^- + Z_0^0), (Z_0^+ + Z_0^0, Z_\infty^- + Z_\infty^0), (Z_\infty^- + Z_\infty^0, Z_0^+ + Z_0^0), \\ (Z_0^- + Z_0^0, Z_\infty^+ + Z_\infty^0)\}.\end{array}$$

Now we give the main results of this paper.

Theorem 3.1 Assume (A_1) and (A_2) hold. Then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_\infty^- - \eta_0^+, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_0^- - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-(2k+1)) = -x(t)$ provided $m > 0$.

Corollary 3.1 Suppose (A_1) and (A_2) hold.

i) If $\Gamma_\infty^+ \cup \Gamma_\infty^- = \emptyset$, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_0^+, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_0^-\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

ii) If $\Gamma_0^+ \cup \Gamma_0^- = \emptyset$, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_\infty^-, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

iii) If $\Gamma_0^+ \cup \Gamma_0^- = \Gamma_\infty^+ \cup \Gamma_\infty^- = \Gamma_0^+ \cup \Gamma_0^- = \emptyset$, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0), \text{ind}(A_0) - \text{ind}(A_\infty)\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

Corollary 3.1 can be directly obtained from Theorem 3.1.

Theorem 3.2 Suppose (A_1) and (A_2) hold.

i) If (A_3^+) holds, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) + \eta_\infty^+ - \eta_0^+, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_0^- - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

ii) If (A_3^-) holds, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_\infty^- - \eta_0^+, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_0^- + \eta_\infty^-\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

iii) If (A_4^+) holds, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_\infty^- - \eta_0^+, \text{ind}(A_0) - \text{ind}(A_\infty) + \eta_0^+ - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

iv) If (A_4^-) holds, then system (1.4) possesses at least

$$m = \max\{\text{ind}(A_\infty) - \text{ind}(A_0) - \eta_\infty^- + \eta_0^-, \text{ind}(A_0) - \text{ind}(A_\infty) - \eta_0^- - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

v) If (A_3^+), (A_4^-) hold, then system (1.4) possesses at least

$$m = \max\{\mathrm{ind}(A_\infty) - \mathrm{ind}(A_0) + \eta_\infty^+ + \eta_0^-, \mathrm{ind}(A_0) - \mathrm{ind}(A_\infty) - \eta_0^- - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

vi) If (A_3^-), (A_4^+) hold, then system (1.4) possesses at least

$$m = \max\{\mathrm{ind}(A_\infty) - \mathrm{ind}(A_0) - \eta_\infty^- - \eta_0^+, \mathrm{ind}(A_0) - \mathrm{ind}(A_\infty) + \eta_\infty^- + \eta_0^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

vii) If (A_3^+), (A_4^+) hold, then system (1.4) possesses at least

$$m = \max\{\mathrm{ind}(A_\infty) - \mathrm{ind}(A_0) + \eta_\infty^+ - \eta_0^-, \mathrm{ind}(A_0) - \mathrm{ind}(A_\infty) + \eta_0^+ - \eta_\infty^+\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

viii) If (A_3^-), (A_4^-) hold, then system (1.4) possesses at least

$$m = \max\{\mathrm{ind}(A_\infty) - \mathrm{ind}(A_0) - \eta_\infty^- + \eta_0^-, \mathrm{ind}(A_0) - \mathrm{ind}(A_\infty) + \eta_\infty^+ - \eta_0^-\}$$

different $2(2k+1)$-periodic orbits satisfying $x(t-2k-1) = -x(t)$ provided that $m > 0$.

Proof of Theorem 3.1.

Suppose $m = \mathrm{ind}(A_0) - \mathrm{ind}(A_\infty) - \eta_0^- - \eta_\infty^+ > 0$.

Let $Z^+ = Z_\infty^-$ and $Z^- = Z_0^+, \forall z \in Z^+$. There is $\sigma > 0$ such that

$$\frac{1}{2}\langle Lz - \hat{P}^{-1}B_\infty z, z\rangle \geq \sigma\|x\|_2^2.$$

Then,

$$|F(x) - \frac{1}{2}\langle P^{-1}A_x, x\rangle_X| \leq \frac{\sigma}{2(2k+1)}\|x\|_2^2 + M.$$

Therefore,

$$\begin{aligned}
\Phi(z) = &\ \tfrac{1}{2}\langle Lz, z\rangle + G(z)\\
= &\ \tfrac{1}{2}\langle Lz, z\rangle - \int_0^{2(2k+1)} \sum_{i=0}^{2k} F(x(t-i))dt\\
= &\ \tfrac{1}{2}\langle Lz, z\rangle - (2k+1)\int_0^{2(2k+1)} F(x(t))dt\\
= &\ \tfrac{1}{2}\langle Lz - \tfrac{1}{2}\hat{P}^{-1}B_\infty z, z\rangle - (2k+1)\int_0^{2(2k+1)}[F(x(t)) - \tfrac{1}{2}\langle P^{-1}Ax, x\rangle_X]dt\\
\geq &\ \tfrac{1}{2}\sigma\|x\|_2^2 - 2(2k+1)M \to +\infty, \quad \|x\|_2^2 \to \infty,
\end{aligned}$$

which implies that there is $c_0 \in \mathbb{R}$ such that

$$\Phi(z) \geq c_0, \quad z \in Z^+.$$

At the same time, there are $c_\infty \in \mathbb{R}^-$ and $\gamma > 0$ such that

$$\Phi(z) \leq c_\infty.$$

Suppose $c_0 < c_\infty$. Then conditions (c) and (d) in Lemma 2.2 are satisfied.

Since Lemmas 2.1 and 3.2–3.4 imply (P.S)-condition as well as conditions (a) and (b) in Lemma 2.2 hold under the requirement of Theorem 3.1, we only need to compute m.

From Lemma 3.5 we have

$$\begin{aligned} m &= \frac{1}{2}\sum_{\substack{1\le i\le I\\ i\ne l(2k+1)+k+1}}[\dim Z^+(i)\cap Z^-(i) - \mathrm{codim}_{Z(i)}(Z^+(i)+Z^-(i))] \\ &= \frac{1}{2}\sum_{\substack{1\le i\le I\\ i\ne l(2k+1)+k+1}}[\dim Z^+(i)+\dim Z^-(i)-\dim Z(i)] \\ &= \frac{1}{2}\sum_{\substack{1\le i\le I\\ i\ne l(2k+1)+k+1}}[\dim Z^+(i)+\dim Z^-(i)-2N]. \end{aligned} \tag{3.8}$$

Then,

$$\begin{aligned} m = & \frac{1}{2}\sum_{\substack{1\le i\le I\\ i\ne l(2k+1)+k+1}}[\dim(Z^+_\infty(i)+\dim Z^-_0(i))-2N] \\ = & \frac{1}{2}\sum_{\substack{l(2k+1)+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[\dim Z^+_\infty(l(2k+1)+i)+\dim Z^-_0(l(2k+1)+i)-2N] \\ & +\frac{1}{2}\sum_{\substack{l(2k+1)+k+1+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[\dim Z^+_\infty(l(2k+1)+k+1+i) \\ & +\dim Z^-_0(l(2k+1)+k+1+i)-2N] \\ = & \sum_{\substack{l(2k+1)+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[\dim Z^+_\infty(l(2k+1)+i)+\dim Z^-_0(l(2k+1)+i)-2N] \\ & +\frac{1}{2}\sum_{\substack{l(2k+1)+2(k+1)-i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[\dim Z^+_\infty(l(2k+1)+2(k+1)-i) \\ & +\dim Z^-_0(l(2k+1)+2(k+1)-i)-2N]. \end{aligned}$$

Denote $I_1 = \sharp\{l(2k+1)+i \le I : l\ge 0, i\in\{1,2,\cdots,k\}\}$, $I_2 = \sharp\{l(2k+1)+2(k+1)-i\le I : l\ge 0, i\in\{1,2,\cdots,k\}\}$, and $\mathrm{ind}A^+_\infty = \sum_{\alpha_j>0}\mathrm{ind}\alpha_j$, $\mathrm{ind}A^-_\infty = \sum_{\alpha_j<0}\mathrm{ind}\alpha_j$, $\mathrm{ind}A^+_0 = \sum_{\beta_j>0}\mathrm{ind}\beta_j$, $\mathrm{ind}A^-_0 = \sum_{\beta_j<0}\mathrm{ind}\beta_j$, then,

$$\mathrm{ind}A_\infty = \mathrm{ind}A^+_\infty + \mathrm{ind}A^-_\infty,\ \mathrm{ind}A_0 = \mathrm{ind}A^+_0 + \mathrm{ind}A^-_0.$$

Obviously,

$$\sum_{\substack{l(2k+1)+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}\dim Z^-_0(l(2k+1)+i) = 2\mathrm{ind}A^+_0,$$

$$\begin{aligned} &\sum_{\substack{l(2k+1)+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}\dim Z^+_\infty(l(2k+1)+i) \\ &= \sum_{\substack{l(2k+1)+i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[2N-\dim Z^-_\infty(l(2k+1)+i)-\dim Z^0_\infty(l(2k+1)+i)] \\ &= 2NI_1 - 2\mathrm{ind}A^+_\infty - 2\eta^+_\infty, \\ &\sum_{\substack{l(2k+1)+2(k+1)-i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}\dim Z^-_0(l(2k+1)+2(k+1)-i) \\ &= \sum_{\substack{l(2k+1)+2(k+1)-i\le I\\ i\in\{1,2,\cdots,k\},l\ge 0}}[2N-\dim Z^0_0-\dim Z^+_0] \\ &= 2NI_2 + 2\mathrm{ind}A^-_0 - 2\eta^-_0. \end{aligned}$$

Therefore,

$$\begin{aligned} m &= \mathrm{ind}A^+_0 + \mathrm{ind}A^-_0 - \mathrm{ind}A^+_\infty - \mathrm{ind}A^-_\infty - \eta^+_\infty - \eta^-_0 \\ &= \mathrm{ind}A_0 - \mathrm{ind}A_\infty - \eta^+_\infty - \eta^-_0. \end{aligned}$$

Theorem 3.1 is proved.

Proof of Theorem 3.2.

Since the proof of Theorem 3.2 for each case is similar, we prove the theorem only for case (ii).

Without loss of generality, suppose

$$\mathrm{ind}A_0 - \mathrm{ind}A_\infty - \eta_0^- + \eta_\infty^- > \max\{0, \mathrm{ind}A_\infty - \mathrm{ind}A_0 - \eta_\infty^- - \eta_0^+\},$$

and denote $Z^+ = Z_\infty^+ + Z_\infty^0,\ Z^- = Z_0^-$.

From (A_3^-), there is $M > 0$ such that

$$-[F(x) - \frac{1}{2}(A_\infty x, x)] > -M, \quad x \in \mathbb{R}^N.$$

Then, for $z \in Z_\infty^+ + Z_\infty^0$,

$$\begin{aligned}\Phi(z) &= \tfrac{1}{2}\langle Lz, z\rangle + G(z)\\ &= \tfrac{1}{2}\langle Lz - \tfrac{1}{2}\hat{P}^{-1}B_\infty z, z\rangle + G(z) + \tfrac{1}{2}\langle \hat{P}^{-1}B_\infty z, z\rangle\\ &\geq G(z) + \tfrac{1}{2}\langle \hat{P}^{-1}B_\infty z, z\rangle\\ &= -(2k+1)\int_0^{2(2k+1)}[F(x(t)) - \tfrac{1}{2}(A_\infty x, x)]dt\\ &> (2k+1)\int_0^{2(2k+1)}(-M)dt\\ &= -2M(2k+1)^2,\end{aligned}$$

which implies that condition (d) in Lemma 2.2 holds.

From (3.8), one has

$$\begin{aligned}m &= \tfrac{1}{2}\sum_{\substack{1\leq i\leq I\\ i\neq l+k+1}}[\dim Z^+(i) + \dim Z^-(i) - 2N]\\ &= \tfrac{1}{2}\sum_{\substack{1\leq i\leq I\\ i\neq l(2k+1)+k+1}}[\dim Z_\infty^+(i) + \dim Z_\infty^0(i) + \dim Z_0^-(i) - 2N]\\ &= \tfrac{1}{2}\sum_{\substack{1\leq i\leq I\\ i\neq l(2k+1)+k+1}}[\dim Z_\infty^+(i) + \dim Z_\infty^0(i) - 2N] + \tfrac{1}{2}\sum_{\substack{1\leq i\leq I\\ i\neq l(2k+1)+k+1}}\dim Z_\infty^0(i)\\ &= \mathrm{ind}A_0 - \mathrm{ind}A_\infty - \eta_\infty^+ - \eta_0^- + \tfrac{1}{2}\dim Z_\infty^0\\ &= \mathrm{ind}A_0 - \mathrm{ind}A_\infty - \eta_\infty^+ - \eta_0^- + \eta_\infty^+ + \eta_\infty^-\\ &= \mathrm{ind}A_0 - \mathrm{ind}A_\infty - \eta_0^- + \eta_\infty^-.\end{aligned}$$

Theorem 3.2 is proved.

4. Examples

Example 4.1 Consider the number of 6-periodic solutions of the following system

$$x'(t) = -\nabla F(x(t-1)) - \nabla F(x(t-2)) \tag{4.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = -\frac{\pi}{12\sqrt{3}}(9x_1^2 + 22x_1x_2 + 9x_2^2) - (x_1^{\frac{4}{3}} + 2x_2^{\frac{4}{3}}), \quad |x| \to \infty,$$

$$F(x) = \frac{\pi}{12\sqrt{3}}(29x_1^2 + 22\sqrt{3}x_1x_2 + 7x_2^2) + (x_1^2 + x_2^2)^{\frac{3}{2}}, \quad |x| \to 0.$$

Obviously, $F \in C^1(\mathbb{R}^2, \mathbb{R}),\ F(0) = 0,\ F(-x) = F(x)$,

$$\nabla F(x) = \begin{bmatrix} -\frac{3\pi}{2\sqrt{3}} & -\frac{11\pi}{6\sqrt{3}} \\ -\frac{11\pi}{6\sqrt{3}} & -\frac{3\pi}{2\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3}x_1^{\frac{1}{3}} \\ \frac{8}{3}x_2^{\frac{1}{3}} \end{bmatrix}, \quad |x| \to \infty$$

$$\nabla F(x) = \begin{bmatrix} \frac{29\pi}{6\sqrt{3}} & \frac{11\sqrt{3}\pi}{6\sqrt{3}} \\ \frac{11\sqrt{3}\pi}{6\sqrt{3}} & \frac{7\pi}{6\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3(x_1^2 + x_2^2)^2 x_1 \\ 3(x_1^2 + x_2^2)^{\frac{1}{2}} x_2 \end{bmatrix}, \quad |x| \to 0$$

and

$$A_\infty = \begin{bmatrix} -\frac{3\pi}{2\sqrt{3}} & -\frac{11\pi}{6\sqrt{3}} \\ -\frac{11\pi}{6\sqrt{3}} & -\frac{3\pi}{2\sqrt{3}} \end{bmatrix}, A_0 = \begin{bmatrix} -\frac{29\pi}{6\sqrt{3}} & \frac{11\pi}{6} \\ \frac{11\pi}{6} & -\frac{7\pi}{6\sqrt{3}} \end{bmatrix}.$$

The eigenvalues of A_∞ and A_0 are

$$\alpha_1 = \frac{\pi}{3\sqrt{3}},\ \alpha_2 = -\frac{10\pi}{3\sqrt{3}} \text{ and } \beta_1 = -\frac{\pi}{3\sqrt{3}},\ \beta_2 = \frac{10\pi}{3\sqrt{3}},$$

respectively. Since $k = 1$, one has

$$\begin{aligned} \operatorname{ind}\alpha_1 &= \sharp\{l \geq 0 :\ 0 < \tfrac{1}{3}(6l+1)\pi\tfrac{1}{\sqrt{3}} = \tfrac{\pi}{3\sqrt{3}}(6l+1) < \tfrac{\pi}{3\sqrt{3}}\} = 0, \\ \operatorname{ind}\alpha_2 &= -\sharp\{l \geq 0 :\ 0 < \tfrac{\pi}{3\sqrt{3}}(6l+5) < \tfrac{10\pi}{3\sqrt{3}}\} = -1, \\ \operatorname{ind}\beta_1 &= -\sharp\{l \geq 0 :\ 0 < \tfrac{\pi}{3\sqrt{3}}(6l+5) < \tfrac{\pi}{3\sqrt{3}}\} = 0, \\ \operatorname{ind}\beta_2 &= \sharp\{l \geq 0 :\ 0 < \tfrac{\pi}{3\sqrt{3}}(6l+1) < \tfrac{10\pi}{3\sqrt{3}}\} = 2. \end{aligned}$$

Therefore,

$$\operatorname{ind} A_\infty = -1, \operatorname{ind} A_0 = 2.$$

At the same time,

$$\eta_\infty^+ = 1,\ \eta_\infty^- = 0,\ \eta_0^+ = 0,\ \eta_0^- = 0,$$

and all the conditions of Theorem 3.2 (vi) are satisfied. Then system (4.1) has at least

$$m = \operatorname{ind} A_0 - \operatorname{ind} A_\infty + \eta_\infty^- + \eta_0^+ = 3$$

different 6-periodic orbits satisfying $x(t-3) = -x(t)$.

Example 4.2 Let $N = 2$. We discuss the multiplicity of 6-periodic solutions of system (4.1), where $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} F(x) = &\ (-\tfrac{25\pi}{6\sqrt{3}}x_1^2 + \tfrac{5\pi}{3}x_1x_2 - \tfrac{5\pi}{2\sqrt{3}}x_2^2 - x_1^4 - x_2^6)(1 - \varphi(|x|)) + \\ &(\tfrac{5\pi}{6\sqrt{3}}x_1^2 - \tfrac{5\pi}{\sqrt{3}}x_1x_2 + \tfrac{5\pi}{6\sqrt{3}}x_2^2 + x_1^{\frac{4}{3}} + x_2^{\frac{8}{5}})\varphi(|x|). \end{aligned}$$

Then

$$\nabla F(x) = A_0 x - \begin{bmatrix} 4x_1^3 \\ 6x_2^5 \end{bmatrix},\ |x| \to 0,$$

$$\nabla F(x) = A_\infty x + \begin{bmatrix} \frac{4}{3}x_1^{\frac{1}{3}} \\ \frac{8}{5}x_2^{\frac{3}{5}} \end{bmatrix},\ |x| \to \infty,$$

where

$$A_0 = \begin{bmatrix} -\frac{25\pi}{3\sqrt{3}} & \frac{5\pi}{3} \\ \frac{5\pi}{3} & -\frac{5\pi}{\sqrt{3}} \end{bmatrix},\ A_\infty = \begin{bmatrix} \frac{5\pi}{3\sqrt{3}} & -\frac{5\pi}{\sqrt{3}} \\ -\frac{5\pi}{\sqrt{3}} & \frac{5\pi}{3\sqrt{3}} \end{bmatrix}.$$

A_∞ and A_0 have their eigenvalues $\alpha_1 = -\frac{10\pi}{3\sqrt{3}}$, $\alpha_2 = \frac{20\pi}{3\sqrt{3}}$, and $\beta_1 = -\frac{10\pi}{\sqrt{3}}$, $\beta_2 = -\frac{10\pi}{3\sqrt{3}}$, respectively. So

$$\begin{aligned} \text{ind}\alpha_1 &= -\sharp\{l \geq 0 : 0 < \tfrac{\pi}{3\sqrt{3}}(6l+5) < \tfrac{10\pi}{3\sqrt{3}}\} = -1, \\ \text{ind}\alpha_2 &= \sharp\{l \geq 0 : 0 < \tfrac{\pi}{3\sqrt{3}}(6l+1) < \tfrac{20\pi}{3\sqrt{3}}\} = 4, \\ \text{ind}\beta_1 &= -\sharp\{l \geq 0 : 0 < \tfrac{\pi}{3\sqrt{3}}(6l+5) < \tfrac{10\pi}{\sqrt{3}}\} = -5, \\ \text{ind}\beta_2 &= -\sharp\{l \geq 0 : 0 < \tfrac{\pi}{3\sqrt{3}}(6l+5) < \tfrac{10\pi}{3\sqrt{3}}\} = -1, \end{aligned}$$

and then

$$\text{ind}A_\infty = 3,\ \text{ind}A_0 = -6.$$

On the other hand, we have

$$\eta_\infty^+ = \eta_\infty^- = \eta_0^+ = \eta_0^- = 0.$$

Therefore,

$$m = \text{ind}A_\infty - \text{ind}A_0 = 9,$$

and system (4.1) has at least 9 different 6-periodic orbits satisfying $x(t-3) = -x(t)$ by Theorem 3.1.

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Conflict of interest

The authors declare that there is no conflicts of interest regarding the publication of this paper.

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