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Research article

On Bounds of fractional integral operators containing Mittag-Leffler functions for generalized exponentially convex functions

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Abstract: Recently, a generalization of convex function called exponentially $(\alpha, h-m)$-convex function has been introduced. This generalization of convexity is used to obtain upper bounds of fractional integral operators involving Mittag-Leffler (ML) functions. Moreover, the upper bounds of left and right integrals lead to their boundedness and continuity. A modulus inequality is established for differentiable functions. The Hadamard type inequality is proved which shows upper and lower bounds of sum of left and right sided fractional integral operators.

Keywords: convex function; exponentially $(\alpha, h-m)$-convex function; Mittag-Leffler function; generalized fractional integral operators
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1. Introduction

Convexity is one of the fascinating and natural concepts, it is beneficial in optimization theory, theory of inequalities, numerical analysis, economics and in other subjects of pure and applied mathematics. Convex functions are defined in different ways due to their interesting graphical shapes in euclidean space. A convex function defined on an interval of real line is always continuous in the interior points, but need not be differentiable. Although, it has left and right increasing derivatives at each interior point. The derivative of a differentiable convex function is always an increasing function. A twice differentiable convex function has downward concavity. In analytical forms it is defined in several ways the classical one is given in the following definition.

Definition 1. *A function $\phi : M \subseteq \mathbb{R} \to \mathbb{R}$, where M is convex set, is said to be convex function, if the following inequality holds:*

$$\phi(tu + (1 - t)v) \leq t\phi(u) + (1 - t)\phi(v), \tag{1.1}$$

for all $u, v \in M$ and $t \in [0, 1]$.

The inequality (1.1) motivates the reader to extend, refine, generalize the notion of convexity. The authors have analyzed this inequality to introduce several new notions, for example m-convex function, s-convex function, h-convex function, p-convex function and many others. In [1], the notion of exponential convex function is introduced.

Definition 2. *A function $\phi : M \subseteq \mathbb{R} \to \mathbb{R}$, where M is an interval, is said to be exponentially convex function, if we have the following inequality:*

$$\phi(tu + (1 - t)v) \leq t\frac{\phi(u)}{e^{\sigma u}} + (1 - t)\frac{\phi(v)}{e^{\sigma v}},$$

for all $u, v \in M$, $t \in [0, 1]$ and $\sigma \in \mathbb{R}$.

In [2], the notion of h-convex function is introduced as follows:

Definition 3. *Let $h : N \supset [0, 1] \to \mathbb{R}$ be a non-negative function. A function $\phi : M \to \mathbb{R}$ is said to be h-convex function, if the following inequality holds:*

$$\phi(tu + (1 - t)v) \leq h(t)\phi(u) + h(1 - t)\phi(v),$$

for all $u, v \in M$ and $t \in [0, 1]$, where M and N are intervals in $\mathbb{R}$.

In [3], the following definition of $(h - m)$-convex function is given.

Definition 4. *Let $N \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : N \to \mathbb{R}$ be a non-negative function. We say that $\phi : [0, b] \to \mathbb{R}$ is $(h - m)$-convex function, if ϕ is non-negative and for all $u, v \in [0, b], m \in [0, 1]$ and $t \in (0, 1)$, one has*

$$\phi(tu + m(1 - t)v) \leq h(t)\phi(u) + mh(1 - t)\phi(v). \tag{1.2}$$

In [4], the following definition of (α, m)-convex function is given.

Definition 5. *A function $\phi : [0, b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, m)-convex function, where $(\alpha, m) \in [0, 1]^2$ and $b > 0$, if for every $u,\ v \in [0, b]$ and $t \in [0, 1]$ we have*

$$\phi(tu + m(1 - t)v) \leq t^{\alpha}\phi(u) + m(1 - t^{\alpha})\phi(v).$$

In [5], the definition of (s, m)-convex function is given.

Definition 6. *A function $\phi : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex function, where $(s, m) \in [0, 1]^2$ and $b > 0$, if for every $u,\ v \in [0, b]$ and $t \in [0, 1]$ we have*

$$\phi(tu + m(1 - t)v) \leq t^{s}\phi(u) + m(1 - t)^{s}\phi(v).$$

Farid et al. in [6] unified the all above definitions in a single notion called $(\alpha, h-m)$-convex function.

Definition 7. *Let $N \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : N \to \mathbb{R}$ be a non-negative function. We say that $\phi : [0,b] \to \mathbb{R}$ is a $(\alpha, h-m)$-convex function, if ϕ is non-negative and for all $u, v \in [0,b], (\alpha, m) \in [0,1]^2$ and $t \in (0,1)$, one has*

$$\phi(tu + m(1-t)v) \leq h(t^{\alpha})\phi(u) + mh(1-t^{\alpha})\phi(v). \tag{1.3}$$

A further generalization namely exponentially $(\alpha, h-m)$-convex function is given in [7].

Definition 8. *Let $N \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : N \to \mathbb{R}$ be a non-negative function. We say that $\phi : [0,b] \to \mathbb{R}$ is an exponentially $(\alpha, h-m)$-convex function, if ϕ is non-negative and for all $u, v \in [0,b], (\alpha, m) \in [0,1]^2$, $t \in (0,1)$ and $\sigma \in \mathbb{R}$, one has*

$$\phi(tu + m(1-t)v) \leq h(t^{\alpha})\frac{\phi(u)}{e^{\sigma u}} + mh(1-t^{\alpha})\frac{\phi(v)}{e^{\sigma v}}. \tag{1.4}$$

The above definition of exponentially $(\alpha, h-m)$-convex function unifies the definitions of convex, exponentially convex, m-convex, exponentially m-convex, s-convex, exponentially s-convex, h-convex, exponentially h-convex, $(h-m)$-convex, exponentially $(h-m)$-convex, (s,m)-convex, exponentially (s,m)-convex, (α,m)-convex, exponentially (α,m)-convex functions in a single inequality. The aim of this paper is to study the extended generalized fractional integral operators involving Mittag-Leffler (ML) functions for exponentially $(\alpha, h-m)$-convex function. By using definition of exponentially $(\alpha, h-m)$-convex function, bounds of these fractional integral operators are obtained. The results will hold at the same time for all convex functions explained in above.
The well-known Mittag-Leffler function $E_{\xi}(.)$ for one parameter is defined as follows [8]:

$$E_{\xi}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\xi n + 1)},$$

where $t, \xi \in \mathbb{C}$, $\Re(\xi) > 0$ and $\Gamma(.)$ is the gamma function. It is a natural extension of exponential, hyperbolic and trigonometric functions. This function and its extensions appear as solution of fractional integral equations and fractional differential equations. It was further explored by Wiman, Pollard, Humbert, Agarwal and Feller, see [9]. For its generalizations and extensions by various authors, we refer the reader to [9–13].
The following extended Mittag-Leffler function is introduced by Andrić et al. in [14]:

Definition 9. *Let $\mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)$ is defined by:*

$$E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.5}$$

where β_p is defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$$

and $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

A derivative formula of the extended generalized Mittag-Leffler function is given in following lemma.

Lemma 1. *[14] If* $m \in \mathbb{N}, \omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$ *and* $0 < k < \delta + \Re(\mu)$*, then*

$$\left(\frac{d}{dt}\right)^m [t^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega t^\mu; p)] = t^{\xi-m-1} E_{\mu,\xi-m,l}^{\gamma,\delta,k,c}(\omega t^\mu; p) \quad \Re(\xi) > m. \tag{1.6}$$

Remark 1. *The extended Mittag-Leffler function (1.5) produces the related functions defined in [11–13, 15, 16], see [17, Remark 1.3].*

Next we give the definition of the generalized fractional integral operator containing the extended generalized Mittag-Leffler function (1.5).

Definition 10. *[14] Let* $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$, $\delta > 0$ *and* $0 < k \leq \delta + \Re(\mu)$*. Let* $f \in L_1[a, b]$ *and* $x \in [a, b]$*. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by:*

$$\left(\epsilon_{\mu,\xi,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x; p) = \int_a^x (x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-t)^\mu; p) f(t) dt, \tag{1.7}$$

and

$$\left(\epsilon_{\mu,\xi,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x; p) = \int_x^b (t-x)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu; p) f(t) dt. \tag{1.8}$$

For application and related results involving Mittag-Leffler function, see [18, 19].

Remark 2. *The operators (1.7) and (1.8) produce in particular several kinds of known fractional integral operators, see [17, Remark 1.4]*

The classical Riemann-Liouville fractional integral operator is defined as follows:

Definition 11. *[16] Let* $f \in L_1[a, b]$*. Then Riemann-Liouville fractional integral operators of order* $\xi \in \mathbb{C}$, $\Re(\xi) > 0$ *are defined as follows:*

$$I_{a^+}^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_a^x (x-t)^{\xi-1} f(t) dt, x > a, \tag{1.9}$$

$$I_{b^-}^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_x^b (t-x)^{\xi-1} f(t) dt, x < b. \tag{1.10}$$

It can be noted that $\left(\epsilon_{\mu,\xi,l,0,a^+}^{\gamma,\delta,k,c} f\right)(x; 0) = I_{a^+}^\xi f(x)$ and $\left(\epsilon_{\mu,\xi,l,0,b^-}^{\gamma,\delta,k,c} f\right)(x; 0) = I_{b^-}^\xi f(x)$. From fractional integral operators (1.7) and (1.8) we can write:

$$J_{\xi,a^+}(x; p) := \left(\epsilon_{\mu,\xi,l,\omega,a^+}^{\gamma,\delta,k,c} 1\right)(x; p) = (x-a)^\xi E_{\mu,\xi+1,l}^{\gamma,\delta,k,c}(w(x-a)^\mu; p), \tag{1.11}$$

$$J_{\eta,b^-}(x; p) := \left(\epsilon_{\mu,\eta,l,\omega,b^-}^{\gamma,\delta,k,c} 1\right)(x; p) = (b-x)^\eta E_{\mu,\eta+1,l}^{\gamma,\delta,k,c}(w(b-x)^\mu; p). \tag{1.12}$$

In the upcoming section we compute the bounds of fractional integral operators involving extended Mittag-Leffler (ML) functions for exponentially $(\alpha, h - m)$-convex functions. The continuity of the fractional integrals is proved. Furthermore, the bounds of these operators are presented in the form of the Hadamard type inequality. A modulus inequality is established for differentiable functions whose derivatives in absolute are exponentially $(\alpha, h - m)$-convex. Many well-known results are deduced from given results.

2. Main results

Theorem 1. *Let $\phi : [u, v] \longrightarrow \mathbb{R}$, $u < mv$, be a real valued function. If f is positive and exponentially $(\alpha, h - m)$-convex, $m \in (0, 1]$, then there exist $\sigma, \tau \in \mathbb{R}$ such that for $\xi, \eta \geq 1$, the following fractional integral inequality holds:*

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,v^-}\phi\right)(x;p) \qquad (2.1)\\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(\frac{\phi(u)}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right)\\
&+ (v-x)J_{\eta-1,v^-}(x;p)\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\tau\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right), \quad x \in [u, v].
\end{aligned}$$

Proof. Let $x \in [u, v]$, for $t \in [u, x)$ and $\xi \geq 1$. Then the following inequality holds:

$$(x-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-t)^\mu;p) \leq (x-u)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-u)^\mu;p). \qquad (2.2)$$

Using the definition of exponentially $(\alpha, h - m)$-convex function, for $\sigma \in \mathbb{R}$ we get

$$\phi(t) \leq h\left(\left(\frac{x-t}{x-u}\right)^\alpha\right)\frac{\phi(u)}{e^{\sigma u}} + mh\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\frac{\phi\left(\frac{x}{m}\right)}{e^{\sigma\frac{x}{m}}}. \qquad (2.3)$$

After multiplying (2.2) and (2.3) and then integrating over $[u, x]$, we have

$$\begin{aligned}
&\int_u^x (x-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-t)^\mu;p)\phi(t)dt\\
&\leq (x-u)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-u)^\mu;p)\left(\frac{\phi(u)}{e^{\sigma u}}\int_u^x h\left(\frac{x-t}{x-u}\right)^\alpha dt\right.\\
&\left.+m\frac{\phi\left(\frac{x}{m}\right)}{e^{\sigma\frac{x}{m}}}\int_u^x h\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)dt\right).
\end{aligned}$$

By using the definition of left integral operators, we get

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x;p) \qquad (2.4)\\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(\frac{\phi(u)}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned}$$

Similarly, on the other hand for $t \in (x, v]$ and $\eta \geq 1$, the following inequality holds:

$$(t-x)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(t-x)^\mu;p) \leq (v-x)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(v-x)^\mu;p). \qquad (2.5)$$

Again by using definition of exponentially $(\alpha, h - m)$-convexity of ϕ, for $\tau \in \mathbb{R}$ we have

$$\phi(t) \leq h\left(\frac{t-x}{v-x}\right)^\alpha\frac{\phi(v)}{e^{\tau v}} + mh\left(1-\left(\frac{t-x}{v-x}\right)^\alpha\right)\frac{\phi\left(\frac{x}{m}\right)}{e^{\tau\frac{x}{m}}}. \qquad (2.6)$$

Multiplying (2.5) with (2.6) and then integrating over $[x, v]$, we have

$$\begin{aligned}
&\int_u^x (t-x)^{\eta-1} E_{\mu,\eta,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu; p)\phi(t)dt \\
&\leq (v-x)^{\eta-1} E_{\mu,\eta,l}^{\gamma,\delta,k,c}(\omega(v-x)^\mu; p)\left(\frac{\phi(v)}{e^{\tau v}}\int_u^x h\left(\frac{t-x}{v-x}\right)^\alpha dt\right. \\
&\left.+m\frac{\phi\left(\frac{x}{m}\right)}{e^{\tau\frac{x}{m}}}\int_u^x h\left(1-\left(\frac{t-x}{v-x}\right)^\alpha\right)dt\right).
\end{aligned}$$

By using the definition of right integral operators, we get

$$\begin{aligned}
&\left(\epsilon_{\mu,\eta,l,\omega,u^+}^{\gamma,\delta,k,c}\phi\right)(x;p) \\
&\leq (v-x)J_{\eta-1,v^-}(x;p)\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\tau\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.7}$$

Sum of inequalities (2.4) and (2.7) gives the required inequality (2.1). □

Some particular results are stated in the following corollaries.

Corollary 1. *If we set $\xi = \eta$ in (2.1), then the following inequality is obtained:*

$$\begin{aligned}
&\left(\epsilon_{\mu,\xi,l,\omega,u^+}^{\gamma,\delta,k,c}\phi\right)(x;p) + \left(\epsilon_{\mu,\xi,l,\omega,v^-}^{\gamma,\delta,k,c}\phi\right)(x;p) \\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(\frac{\phi(u)}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right) \\
&+ (v-x)J_{\xi-1,v^-}(x;p)\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{x}{m}\right)}{e^{\tau\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right), \quad x \in [u,v].
\end{aligned} \tag{2.8}$$

Corollary 2. *Along with assumptions of Theorem 1, if $\phi \in L_\infty[u, v]$, then the following inequality is established:*

$$\begin{aligned}
&\left(\epsilon_{\mu,\xi,l,\omega,u^+}^{\gamma,\delta,k,c}\phi\right)(x;p) + \left(\epsilon_{\mu,\eta,l,\omega,v^-}^{\gamma,\delta,k,c}\phi\right)(x;p) \\
&\leq \|\phi\|_\infty\left(\left(\frac{(x-u)J_{\xi-1,u^+}(x;p)}{e^{\sigma u}} + \frac{(v-x)J_{\eta-1,v^-}(x;p)}{e^{\tau v}}\right)\int_0^1 h(\theta^\alpha)d\theta\right. \\
&\left.+m\left(\frac{(x-u)J_{\xi-1,u^+}(x;p)}{e^{\sigma\frac{x}{m}}} + \frac{(v-x)J_{\eta-1,v^-}(x;p)}{e^{\tau\frac{x}{m}}}\right)\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.9}$$

Corollary 3. *Further if $h \in L_\infty[u, v]$ and $\xi = \eta$ in (2.9), then we get the following result:*

$$\begin{aligned}
&\left(\epsilon_{\mu,\xi,l,\omega,u^+}^{\gamma,\delta,k,c}\phi\right)(x;p) + \left(\epsilon_{\mu,\eta,l,\omega,v^-}^{\gamma,\delta,k,c}\phi\right)(x;p) \\
&\leq \|\phi\|_\infty\|h\|_\infty\left((x-u)J_{\xi-1,u^+}(x;p)\left(\frac{1}{e^{\sigma u}} + \frac{m}{e^{\sigma\frac{x}{m}}}\right)\right. \\
&\left.+(v-x)J_{\eta-1,v^-}(x;p)\left(\frac{1}{e^{\tau v}} + \frac{m}{e^{\tau\frac{x}{m}}}\right)\right).
\end{aligned} \tag{2.10}$$

Remark 3. *(i) If we set $\alpha = m = 1$ and $h(t) = t$ in (2.1), then we obtain result for exponentially convex function.*
(ii) If we set $\alpha = m = 1$ and $h(t) = t$ in (2.8), then we obtain result for exponentially convex function.
(iii) If we say that $h(t) = t$ and $\sigma = \tau = 0$ in (2.1), then we obtain [20, Theorem 2.1].
(iv) If we set $h(t) = t$ and $\sigma = \tau = 0$ in (2.8), then we obtain [20, Corollary 2.1].
(v) If we set $\alpha = 1$ and $\sigma = \tau = 0$ in (2.1), then we obtain [21, Theorem 1].
(vi) If we set $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$ in (2.1), then we obtain [21, Corollary 1].
(vii) If we set $\alpha = 1$ and $\sigma = \tau = 0$ in (2.1) we obtain [22, Theorem 1].
(viii) If we set $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$ in (2.1) we obtain [23, Theorem 1].

Theorem 2. *With the assumptions of Theorem 1 if $\phi \in L_\infty[u, v]$, then operator defined in (1.7) and (1.8) are bounded and continuous.*

Proof. If $\phi \in L_\infty[u, v]$, then from (2.4) we have

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x; p)\right| \qquad (2.11)\\
&\leq \|\phi\|_\infty (x-u) J_{\xi-1,u^+}(x; p) \int_0^1 \left(\frac{1}{e^{\sigma u}} h(\theta^\alpha) + m\frac{1}{e^{\sigma\frac{x}{m}}} h(1-\theta^\alpha)\right) d\theta\\
&\leq \|\phi\|_\infty (v-u) J_{\xi-1,u^+}(v; p) \int_0^1 \left(\frac{1}{e^{\sigma u}} h(\theta^\alpha) + m\frac{1}{e^{\sigma\frac{u}{m}}} h(1-\theta^\alpha)\right) d\theta.
\end{aligned}$$

Therefore we have

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x; p)\right| \leq M\|\phi\|_\infty, \tag{2.12}$$

where $M = (v-u)J_{\xi-1,u^+}(v; p) \int_0^1 \left(\frac{1}{e^{\sigma u}} h(\theta^\alpha) + m\frac{1}{e^{\sigma\frac{u}{m}}} h(1-\theta^\alpha)\right) d\theta$. Also on the other hand from (2.7) we can obtain:

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,v^-}\phi\right)(x; p)\right| \leq K\|\phi\|_\infty, \tag{2.13}$$

where $K = (v-u)J_{\eta-1,u^+}(u; p) \int_0^1 \left(\frac{1}{e^{\tau v}} h(\theta^\alpha) + m\frac{1}{e^{\tau\frac{v}{m}}} h(1-\theta^\alpha)\right) d\theta$. Therefore $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x; p)$ and $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,u^+}\phi\right)(x; p)$ are bounded also these are linear, hence continuous. □

Theorem 3. *Let $\phi : [u, v] \longrightarrow \mathbb{R}$, $u < mv$, be a real valued function. If ϕ is differentiable and $|\phi'|$ is exponentially $(\alpha, h-m)$-convex, $m \in (0, 1]$, then there exist $\sigma, \tau \in \mathbb{R}$ such that for $\xi, \eta \geq 1$, the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:*

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x; p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(x; p) - \left(J_{\xi-1,u^+}(x; p)\phi(u) + J_{\eta-1,v^-}(x; p)\phi(v)\right)\right| \qquad (2.14)\\
&\leq (x-u)J_{\xi-1,u^+}(x; p)\left(\frac{|\phi'(u)|}{e^{\sigma u}} \int_0^1 h(\theta^\alpha) d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma\frac{x}{m}}} \int_0^1 h(1-\theta^\alpha) d\theta\right)\\
&+ (v-x)J_{\eta-1,v^-}(x; p)\left(\frac{|\phi'(v)|}{e^{\tau v}} \int_0^1 h(\theta^\alpha) d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\tau\frac{x}{m}}} \int_0^1 h(1-\theta^\alpha) d\theta\right), \quad x \in [u, v].
\end{aligned}$$

Proof. For $x \in [u, v]$ and $t \in [u, x)$, by using the definition of exponentially $(\alpha, h-m)$-convexity of $|\phi'|$, for $\sigma \in \mathbb{R}$ we have

$$|\phi'(t)| \leq h\left(\frac{x-t}{x-u}\right)^{\alpha} \frac{|\phi'(u)|}{e^{\sigma u}} + mh\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right) \frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma \frac{x}{m}}}. \tag{2.15}$$

From(2.15), we can write

$$\phi'(t) \leq h\left(\frac{x-t}{x-u}\right)^{\alpha} \frac{|\phi'(u)|}{e^{\sigma u}} + mh\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right) \frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma \frac{x}{m}}}. \tag{2.16}$$

Multiplication of (2.2) and (2.16), gives the following:

$$\begin{aligned}&(x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu}; p)\phi'(t)dt \leq (x-u)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu}; p) \\ &\left(h\left(\frac{x-t}{x-u}\right)^{\alpha} \frac{|\phi'(u)|}{e^{\sigma u}} + mh\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right) \frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma \frac{x}{m}}}\right).\end{aligned} \tag{2.17}$$

Now integrating over $[u, x]$, we get

$$\begin{aligned}&\int_u^x (x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu}; p)\phi'(t)dt \\ &\leq (x-u)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu}; p)\left(\frac{|\phi'(u)|}{e^{\sigma u}} \int_u^x h\left(\frac{x-t}{x-u}\right)^{\alpha} dt\right. \\ &\left.+m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma \frac{x}{m}}} \int_u^x h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right) dt\right) \\ &= (x-u)^{\xi} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu}; p)\left(\frac{|\phi'(u)|}{e^{\sigma u}} \int_0^1 h(\theta^{\alpha})d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma \frac{x}{m}}} \int_0^1 h(1-\theta^{\alpha})d\theta\right).\end{aligned} \tag{2.18}$$

The left hand side of (2.18) is computed as follows:

$$\int_u^x (x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu}; p)\phi'(t)dt, \tag{2.19}$$

substituting $x - t = r$, using the derivative property (1.6) of Mittag-Leffler function, we have

$$\begin{aligned}&\int_0^{x-u} r^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega r^{\mu}; p)\phi'(x-r)dr \\ &= (x-u)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu}; p)\phi(u) - \int_0^{x-u} r^{\xi-2} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega r^{\mu}; p)\phi(x-r)dr,\end{aligned}$$

now for $x - r = t$ in second term of the right hand side of the above equation and then using (1.7), we get

$$\int_0^{x-u} r^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega r^{\mu}; p)\phi'(x-r)dr$$

$$= (x-u)^{\xi-1} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-u)^{\mu};p)\phi(u) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x;p).$$

Therefore (2.18) becomes:

$$\begin{aligned}
&\left(J_{\xi-1,u^+}(x;p)\right)\phi(u) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x;p) \\
&\leq (x-u)J_{\alpha-1,u^+}(x;p)\left(\frac{|\phi'(u)|}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.20}$$

Again from (2.15) we can write

$$\phi'(t) \geq -\left(h\left(\frac{x-t}{x-u}\right)^\alpha \frac{|\phi'(u)|}{e^{\sigma u}} + mh\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma\frac{x}{m}}}\right). \tag{2.21}$$

Similarly as we did for (2.16), one can obtain:

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x;p) - J_{\xi-1,u^+}(x;p)\phi(u) \\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(\frac{|\phi'(u)|}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.22}$$

From (2.20) and (2.22), we get

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x;p) - J_{\xi-1,u^+}(x;p)\phi(u)\right| \\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(\frac{|\phi'(u)|}{e^{\sigma u}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\sigma\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.23}$$

Now for $x \in [u,v]$ and $t \in (x,v]$. Again by using exponentially $(\alpha, h-m)$-convexity of $|\phi'|$, for $\tau \in \mathbb{R}$ we have

$$|\phi'(t)| \leq h\left(\frac{t-x}{v-x}\right)^\alpha \frac{|\phi'(v)|}{e^{\tau v}} + mh\left(1-\left(\frac{t-x}{v-x}\right)^\alpha\right)\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\tau\frac{x}{m}}}. \tag{2.24}$$

Proceeding on the same lines as we did to get (2.23), the following inequality holds:

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(x;p) - J_{\eta-1,v^-}(x;p)\phi(v)\right| \\
&\leq (v-x)J_{\eta-1,v^-}(x;p)\left(\frac{|\phi'(v)|}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\left|\phi'\left(\frac{x}{m}\right)\right|}{e^{\tau\frac{x}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned} \tag{2.25}$$

From inequalities (2.23) and (2.25) via triangular inequality (2.14) is established. □

Corollary 4. *If we put $\xi = \eta$ in (2.14), then the following inequality is obtained:*

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,v^-}\phi\right)(x;p)\right. \tag{2.26}$$

$$
\begin{aligned}
&-\left(J_{\xi-1,u^+}(x;p)\frac{\phi(u)}{e^{\sigma u}}+J_{\xi-1,v^-}(x;p)\frac{\phi(v)}{e^{\tau v}}\right)\Bigg|\\
&\leq (x-u)J_{\xi-1,u^+}(x;p)\left(|\phi'(u)|\int_0^1 h(\theta^\alpha)d\theta+m\left|\phi'\left(\frac{x}{m}\right)\right|\int_0^1 h(1-\theta^\alpha)d\theta\right)\\
&+(v-x)J_{\xi-1,v^-}(x;p)\left(|\phi'(v)|\int_0^1 h(\theta^\alpha)d\theta+m\left|\phi'\left(\frac{x}{m}\right)\right|\int_0^1 h(1-\theta^\alpha)d\theta\right),\quad x\in[u,v].
\end{aligned}
$$

Remark 4. *(i) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.14), then we obtain result for exponentially convex function.*
(ii) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.26), then we obtain result for exponentially convex function.
(iii) If we take $h(t) = t$ and $\sigma = \tau = 0$ in (2.14), then we obtain [20, Theorem 2.2].
(iv) If we take $h(t) = t$ and $\sigma = \tau = 0$ in (2.26), then we obtain [20, Corollary 2.2].
(v) If we take $\alpha = 1$ and $\sigma = \tau = 0$ in (2.14), then we obtain [21, Theorem 2].
(vi) If we take $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$ in (2.14), then we obtain [21, Corollary 2].
(vii) If we take $\alpha = 1$ and $\sigma = \tau = 0$ in (2.14), then we obtain [22, Theorem 2].
(viii) If we take $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$in (2.14), then we obtain [23, Theorem 2].

It is easy to prove the next lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

Lemma 2. *Let $\phi : [u,v] \to \mathbb{R}$, $u < mv$, be exponentially $(\alpha, h-m)$-convex function. If $\frac{\phi\left(\frac{u+mv-x}{m}\right)}{e^{\sigma\left(\frac{u+mv-x}{m}\right)}} = \frac{\phi(x)}{e^{\sigma x}}$ and $m \in (0,1]$, then the following inequality holds:*

$$
\phi\left(\frac{u+mv}{2}\right)\leq\frac{\phi(x)}{e^{\sigma x}}\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right),\ x\in[u,v]. \tag{2.27}
$$

Proof. Since ϕ is exponentially $(\alpha, h-m)$-convex function, $\sigma \in \mathbb{R}$ we can write

$$
\phi\left(\frac{u+mv}{2}\right)\leq h\left(\frac{1}{2^\alpha}\right)\frac{\phi((1-t)u+mtv)}{e^{\sigma((1-t)u+mtv)}}+mh\left(1-\frac{1}{2^\alpha}\right)\frac{\phi\left(\frac{ta+m(1-t)v}{m}\right)}{e^{\sigma\left(\frac{ta+m(1-t)v}{m}\right)}}. \tag{2.28}
$$

Let $x = u(1-t) + mtv$, then we have

$$
\phi\left(\frac{u+mv}{2}\right)\leq h\left(\frac{1}{2^\alpha}\right)\frac{\phi(x)}{e^{\sigma x}}+mh\left(1-\frac{1}{2^\alpha}\right)\frac{\phi\left(\frac{u+mv-x}{m}\right)}{e^{\sigma\left(\frac{u+mv-x}{m}\right)}}. \tag{2.29}
$$

Hence by using the condition imposed on ϕ, we get the required inequality (2.27). □

Theorem 4. *Let $\phi : [u,v] \longrightarrow \mathbb{R}$, $u < mv$, be a real valued function. If ϕ is positive, exponentially $(\alpha, h-m)$-convex and $\frac{\phi\left(\frac{u+mv-x}{m}\right)}{e^{\sigma\left(\frac{u+mv-x}{m}\right)}} = \frac{\phi(x)}{e^{\sigma x}}$, $m \in (0,1]$, then there exist $\sigma, \tau \in \mathbb{R}$ such that for $\xi, \eta > 0$, the following fractional integral inequality holds:*

$$
\begin{aligned}
&\frac{e^{\sigma x}}{h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)}\phi\left(\frac{u+mv}{2}\right)\left[J_{\eta+1,v^-}(u;p)+J_{\xi+1,u^+}(v;p)\right] \qquad (2.30)\\
&\leq\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(u;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p)\\
&\leq\left[J_{\eta-1,v^-}(u;p)+J_{\xi-1,u^+}(v;p)\right](v-u)^2\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta+m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).
\end{aligned}
$$

Proof. For $x \in [u, v]$, we have

$$(x-u)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x-u)^{\mu}; p) \leq (v-u)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(v-u)^{\mu}; p), \eta > 0. \tag{2.31}$$

As the function ϕ is exponentially $(\alpha, h-m)$-convex, for $\tau \in \mathbb{R}$, we have:

$$\phi(x) \leq h\left(\frac{x-u}{v-u}\right)^{\alpha} \frac{\phi(v)}{e^{\tau v}} + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}} h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right). \tag{2.32}$$

Multiplying (2.31) with (2.32) and then integrating over $[u, v]$, we get

$$\begin{aligned}
&\int_u^v (x-u)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x-u)^{\mu}; p)\phi(x)dx \\
&\leq (v-u)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(v-u)^{\mu}; p)\left(\frac{\phi(v)}{e^{\tau v}} \int_u^v h\left(\frac{x-u}{v-u}\right)^{\alpha} dx\right. \\
&\left.+m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}} \int_u^v h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right)dx\right).
\end{aligned}$$

Further it takes the following forms

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,v^-}\phi\right)(u; p) \leq (v-u)^{\beta+1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(v-u)^{\mu}; p) \\
&\left(\frac{\phi(v)}{e^{\tau v}} \int_0^1 h(\theta^{\alpha})d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}} \int_0^1 h(1-\theta^{\alpha})d\theta\right),
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,v^-}\phi\right)(u; p) \leq (v-u)^{2} J_{\beta-1,v^-}(u; p) \\
&\left(\frac{\phi(v)}{e^{\tau v}} \int_0^1 h(\theta^{\alpha})d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}} \int_0^1 h(1-\theta^{\alpha})d\theta\right).
\end{aligned} \tag{2.34}$$

Now on the other hand for $x \in [u, v]$, we have

$$(v-x)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-x)^{\mu}; p) \leq (v-u)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-u)^{\mu}; p), \xi > 0. \tag{2.35}$$

Multiplying (2.32) with (2.35) and then integrating over $[u, v]$, we get

$$\begin{aligned}
&\int_u^v (v-x)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-x)^{\mu}; p)\phi(x)dx \\
&\leq (v-u)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-u)^{\mu}; p)\left(\frac{\phi(v)}{e^{\tau v}} \int_u^v h\left(\frac{x-u}{v-u}\right)^{\alpha} dx\right. \\
&\left.+m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}} \int_u^v h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right)dx\right).
\end{aligned}$$

Further it takes the following forms

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v; p) \leq (v-u)^{\alpha+1} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-u)^{\mu}; p) \tag{2.36}$$

$$\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right),$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p) \leq (v-u)^2 J_{\xi-1,u^+}(v;p) \tag{2.37}$$
$$\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).$$

Adding (2.34) and (2.37), we get;

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(u;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p)\\ &\leq \left[J_{\eta-1,v^-}(u;p) + J_{\xi-1,u^+}(v;p)\right]\\ &(v-u)^2\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).\end{aligned} \tag{2.38}$$

Multiplying (2.27) with $(x-u)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x-u)^\mu;p)$ and integrating over $[u,v]$, we get

$$\begin{aligned}&\phi\left(\frac{u+mv}{2}\right)\int_u^v (x-u)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x-u)^\mu;p)dx\\ &\leq \frac{1}{e^{\sigma x}}\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\int_u^v (x-u)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x-u)^\mu;p)\phi(x)dx.\end{aligned} \tag{2.39}$$

By using (1.8) and (1.11), we get

$$\phi\left(\frac{u+mv}{2}\right)J_{\eta+1,v^-}(u;p) \leq \frac{1}{e^{\sigma x}}\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(u;p). \tag{2.40}$$

Multiplying (2.27) with $(v-x)^\xi E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(v-x)^\mu;p)$ and integrating over $[u,v]$, also using (1.7) and(1.11), we get

$$\phi\left(\frac{u+mv}{2}\right)J_{\xi+1,u^+}(v;p) \leq \frac{1}{e^{\sigma x}}\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p). \tag{2.41}$$

Adding (2.40) and (2.41), we get;

$$\begin{aligned}&\frac{e^{\sigma x}}{h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)}\phi\left(\frac{u+mv}{2}\right)\left[J_{\eta+1,v^-}(u;p)+J_{\xi+1,u^+}(v;p)\right]\\ &\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,v^-}\phi\right)(u;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p).\end{aligned} \tag{2.42}$$

Now combining (2.38) and (2.42), inequality (2.30) can be established. □

Corollary 5. *If we put $\xi = \eta$ in (2.30), then the following inequality is obtained:*

$$\frac{e^{\sigma x}}{h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)}\phi\left(\frac{u+mv}{2}\right)\left[J_{\xi+1,v^-}(u;p)+J_{\xi+1,u^+}(v;p)\right] \tag{2.43}$$

$$\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,v^-}\phi\right)(u;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,u^+}\phi\right)(v;p)$$
$$\leq \left[J_{\xi-1,v^-}(u;p) + J_{\xi-1,u^+}(v;p)\right](v-u)^2$$
$$\left(\frac{\phi(v)}{e^{\tau v}}\int_0^1 h(\theta^\alpha)d\theta + m\frac{\phi\left(\frac{u}{m}\right)}{e^{\sigma\frac{u}{m}}}\int_0^1 h(1-\theta^\alpha)d\theta\right).$$

Remark 5. *(i) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.30), then we get result for exponentially convex function.*
(ii) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.43), then we get result for exponentially convex function.
(iii) If we take $h(t) = t$ and $\sigma = \tau = 0$ in (2.30), then we get [20, Theorem 2.3].
(v) If we take $h(t) = t$ and $\sigma = \tau = 0$ in (2.43), then we get [20, Corollary 2.3].
(vi) If we take $\alpha = 1$ and $\sigma = \tau = 0$ in (2.30), then we get [21, Theorem 3].
(vi) If we take $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$ in (2.30), then we get [21, Corollary 3].
(vii) If we take $\alpha = 1$ and $\sigma = \tau = 0$ in (2.30), then we get [22, Theorem 3].
(viii) If we take $\alpha = m = 1$, $h(t) = t$ and $\sigma = \tau = 0$ in (2.30), then we get [23, Theorem 3].

3. Conclusions

In this research, we present the bounds of fractional integral operators containing Mittag-Leffler (ML) functions by using exponential $(\alpha, h-m)$-convexity. Also we provide the generalization of various results already determined in [20–26]. The boundedness and continuity of several known integral operators defined in [11–13, 15, 16] are also mentioned. Also we have established upper and lower bounds in the form of the Hadamard like inequality. The reader can derive a plenty of fractional integral inequalities for various kinds of convex functions.

Conflict of interest

It is declared that the author have no competing interests.

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