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Research article

Predicting future order statistics with random sample size

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Abstract: We suggest a new method for constructing an efficient point predictor for the future order statistics when the sample size is a random variable. The suggested point predictor is based on some characterization properties of the distributions of order statistics. For several distributions, including the mixture distribution, the performance of the suggested predictor is evaluated by means of a comprehensive simulation study. Three examples of real lifetime data-sets are analyzed by using this method and compared with an efficient recent method given by Barakat et al. [1], that deals with non-random sample sizes. One of these examples predicts the accumulative new cases per million for infection of the new Coronavirus (COVID-19).

Keywords: order statistics; characterization of distributions; point D-predictor; random sample size; COVID-19
Mathematics Subject Classification: 62G30, 62E15

1. Introduction

Predictive inference of future, or censored observations, is a main interest to many applications of the reliability theory. The point prediction is a basic tool for predicting future observations, which is widely used in the reliability theory and lifetime problems. For several years in the reliability theory, especially in life-testing experiments, much of research has extensively focused on point prediction by using a limited number of methods such as the maximum likelihood predictor, best unbiased predictor, conditional median predictor and Bayesian predictor. Balakrishnan et al. [2], Barakat et al. [3], Dellaportas and Wright [4], Kaminsky and Rhodin [5], Kundu and Raqab [6], Raqab and Nagaraja [7], Saadati Nik et al. [8], and Volovskiy and Kamps [9] are some works about this subject. In addition, many authors have considered the prediction of future events, especially future order statistics and generalized order statistics, in the life-testing experiments. Among these authors are Aly et al. [10], Barakat et al. [1, 3, 11–15], El-Adll and Aly [16], Fan et al. [17], Hsieh [18], Kaminsky and Nelson [19], Lawless [20], Patel [21], Peng et al. [22], Raqab and Barakat [23], Raqab and Nagaraja [7], Shah et al. [24], and Valiollahi et al. [25].

The importance of the order statistics in the reliability theory is attributed to the fact that the rth order statistic $X_{r:n}$ represents the life length of a $(n - r + 1)$-out-of-n system made up of n identical components with independent life lengths. On the other hand, in dealing with censored samples, where the life-test is terminated after observing the rth failure (Type II censoring), or the termination of the test occurs after a given time lapse (Type I censoring), the complete survival times can not usually be observed (due to time or cost). In this case, we have to predict the future-failure times to choose a suitable censoring scheme. More specifically, in any lifetime experiment, a number of items, e.g. n (fixed or random), is put in a test. The times of failed items $X_{1:n} \leq X_{2:n} \leq ... \leq X_{r:n}$ (say $r < n$) are then observed and we have to predict, based on these observed times, the life span of the surviving items, $X_{r+1:n} \leq X_{r+2:n} \leq ... \leq X_{n:n}$, for choosing the suitable scheme.

In many biological and agriculture problems, we often come across a situation where the sample size is not deterministic because either some observations get lost for various reasons, or the size of the target population and its representative sample cannot be determined well. For example, assume that the inhabitants of a populous town are exposed to a dose of radiation resulting from an atomic accident, or exposed to an infection of an unknown epidemic. Furthermore, assume that our interest focuses on the time at which r persons would die among a big random sample of size n that is drawn from the residents of this town. Since the number of infected people in this town is unknown and changes randomly with time, the drawn sample contains a random number of infected and non-infected people. Accordingly, the sample size of the sub-sample of the infected people will be a nonnegative integer valued RV, e.g. N, and it will be described by a sequence of independent and identically distributed RVs $X_1, X_2, ..., X_N$. Therefore, the rth smallest order statistic will be denoted by $X_{r:N}$, which represents the time at which r persons will die.

Many authors have considered prediction problems based on samples of random sizes, including Al-Hussaini and Al-Awadhi [26], Barakat et al. [3], Louzada et al. [27], and Raqab and Barakat [23]. The present paper proposes a new efficient method to predict the future time-failure $X_{s:N}$, $1 < r < s < N$, by using some characterizations of the distribution functions (DFs) of the order statistics, where N is a positive-integer RV distributed as a left truncated DF at s. It is known that the characterization of a DF means finding a unique property enjoyed by that distribution. The literature abounds with many different results for these characterizations and their applications in terms of order statistics. Interested readers may refer to Khan el al. [28], Oncel et al. [29], Shah et al. [30–32] and Wesolowski and Ahsanullah [33].

Before going ahead, we have noticed an obstacle that seems, at first glance, to obstruct the implementation of our purpose. Namely, it is known that the DFs do not uniquely determine their RVs, especially for the discrete RVs with finite support, or even for some continuous RVs with DFs that do not strictly increase. However, for the continuous strictly increasing DFs with infinite supports, we anticipate that the coincidence of two DFs implies that the corresponding RVs will be significantly close together. For this reason, we only consider the RVs with infinite support and strictly increasing DFs. Since there is no a rigorous proof of this speculation or even a theoretical method to measure how the corresponding RVs are close together, the evaluation of the performance of the suggested method is achieved within simulation and practical studies. The new suggested method relies on the following definition given by Barakat et al. [1]:

Definition 1.1 (point D-predictor). Let Z be a continuous RV with strictly increasing DF, which has an infinite support. Then, a point predictor $\hat{Z}$ of Z is said to be a point D-predictor of Z if $\hat{Z} \overset{d}{=} Z$ where "$X \overset{d}{=} Y$"means that the RVs X and Y have the same DFs.

In this paper, we use the result of Shah et al. [30] to suggest a new method for developing a prediction point for the future order statistics under the assumption that the sample size is random. The rest of the paper unfolds as follows. In Section 2, we present a new method to solve the aforesaid predicting problem, by deriving a point D-predictor for the future time-failure $X_{s:N}, 1 < r < s < N$, where N is a positive-integer RV distributed as a left truncated DF at the point s. In the sequel, this method will be denoted by CP. Section 3 is devoted to evaluating the performance of the method CP via a comprehensive simulation study, which is conducted on several DFs such as exponential, Weibull, normal, log-normal, generalized gamma DFs and a mixture of two general exponential DFs, and the RV N is assumed to have a Binomial or Poisson DF. In many practical situations, there is an uncertainty about the randomness of the sample sizes. Therefore, we have to compete between the prediction problems with random and non-random sample sizes. This problem is tackled in Section 4 by analyzing three real data sets via the CP method (the random size prediction problem) and the CP2 method, which has been recently suggested by Barakat et al. [1] for non-random sample size prediction problems.

2. The CP method

We begin this section by representing the result of Shah et al. [30], which is an essential pillar of our study.

Lemma 2.1 (Theorem 2.3 in Shah et al., [30])**.** *Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous DF F_X. Furthermore, let U_j, $j = 0, 1, ..., r-1$, be RVs, which are independent of X and satisfy the relation $X_{r+n-m-j:n} \overset{d}{=} X_{n-m:n} + U_j$, $1 \le r < m < n$, $j = 0, 1, ..., r-1$. Then $U_j \overset{d}{=} Y_{r-j:m}$, where $Y_{r-j:m}$ is the $(r-j)$th order statistic from a sample of size m drawn from the exponential DF $E_\alpha(y) = 1 - e^{-\alpha y}, y, \alpha > 0$, if and only if $X \sim E_\alpha(x)$.*

Theorem 2.1. *Let N be a positive-integer RV distributed as a left truncated DF at the point $R+k, R, k \ge 1$. Furthermore, let N_0 be any value in the support of the DF of the RV N and $X_{1:N_0} \le X_{2:N_0} \le ... \le X_{N_0:N_0}$ be order statistics drawn from a continuous strictly increasing DF F_X with an infinite support. Then, the point predictor $\hat{X}_{R+k:N_0}$ of $X_{R+k:N_0}$ is given by*

$$\hat{X}_{R+k:N_0} \overset{d}{=} F_X^{-1}(E_\alpha(E_\alpha^{-1}(F_X(X_{R:N_0})) + Y_{k:N_0-R})) \tag{2.1}$$

where $E_\alpha^{-1}(y) = \frac{1}{\alpha}\log(1-y)$, $0 \le y \le 1$, and $Y_{k:N_0-R}$ is the kth order statistic from a sample of size $N_0 - R$ drawn from the DF $E_\alpha(y)$.

Proof. We can easily show that the characterization property given in Lemma 2.1 can be written in the form

$$X_{R+k:N_0} \overset{d}{=} X_{R:N_0} + Y_{k:N_0-R}, R, k \ge 1. \tag{2.2}$$

The PC method can now be applied to the special DF $E_\alpha(x)$, by predicting $x_{R+k:N_0}, k, R \ge 1$, based on the observed value $x_{R:N_0}$. This can be achieved by generating an ordered random sample of size $N_0 - R$ from the exponential distribution, $E_\alpha(y)$, and determining $y_{k:N_0-R}$. In this way, a point D-prediction $\hat{x}_{R+k:N_0}$ of $x_{R+k:N_0}$ can be easily computed from (2.2) as $x_{R+k:N_0} = x_{R:N_0} + y_{k:N_0-R}$. Now, let $X \sim F_X$, where F_X is any continuous strictly increasing DF defined on an infinite support. Then, by using the probability integral transformation, (2.2) yields

$$E_\alpha^{-1}(F_X(X_{R+k:N_0})) \overset{d}{=} E_\alpha^{-1}(F_X(X_{R:N_0})) + Y_{k:N_0-R}$$

and

$$F_X(X_{R+k:N_0}) \overset{d}{=} E_\alpha(E_\alpha^{-1}(F_X(X_{R:N_0})) + Y_{k:N_0-R}).$$

Therefore, we get

$$X_{R+k:N_0} \overset{d}{=} F_X^{-1}(E_\alpha(E_\alpha^{-1}(F_X(X_{R:N_0})) + Y_{k:N_0-R})). \tag{2.3}$$

The theorem is proved. □

Remark 2.1. *Theorem 2.1 enables us to predict the value of $X_{R+k:N}$ based on the observed value of $X_{R:N}$, where $R, k \geq 1$. Namely, we first generate a discrete random sample $N_1, N_2, ..., N_M$ from the left truncated DF of the RV N at the point $R + k$. Then, we generate M ordered random samples of sizes $N_1 - R, N_2 - R, ..., N_M - R$, respectively, from the exponential distribution, $E_\alpha(y)$, and determine $y_{k:N_i-R}$, for each $i = 1, 2, ..., M$. In this way, a point D-prediction of the $(R + k)$th order statistic for each generated random sample can be computed from (2.1). Therefore, the average of the D-predictions resulting from the M random samples may be taken as a predictor of $X_{R+k:N}$, which is denoted by $\hat{X}_{R+k:N}$ and will be still called a D-predictor. In the next sections, via a comprehensive simulation study and real data examples, we will show that the performance of the predictor $\hat{X}_{R+k:N}$ is well above all expectations, regardless of the value of k.*

Remark 2.2 (stability of the point D-predictors with respect to α)**.** *Clearly, the RV $X_{R+k:N}$ and its DF do not depend on the parameter α. Thus, the relation (2.3) reveals an interesting and useful property of the point D-predictor (2.1), that is, it is stable with respect to α. More specifically, changing α does not affect the value of the corresponding point D-predictor. We carried out a simulation study for several values of α to check this stability property. For the sake of brevity, we did not include this study in the paper. Therefore, in the next sections we always take $\alpha = 1$.*

3. Simulation study

In this section, a comprehensive simulation study is conducted for some important lifetime distributions, such as exponential and Weibull DFs (denoted by $W(a, b; x) = 1 - \exp(-(\frac{x}{b})^a), a, b > 0, x \geq 0$), and normal DF (denoted by $\mathcal{N}(x; \mu, \sigma)$ with mean μ and variance σ^2) to evaluate the efficiency of the method CP. Besides the preceding lifetime DFs, we consider two more complex lifetime DFs, namely the log-normal DF $L\mathcal{N}(x; \mu, \sigma)$ (its probability density function (PDF) is given by $f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp(-(\frac{(\log x-\mu)^2}{2\sigma^2})), \ -\infty < \mu < \infty, \sigma > 0, x > 0$) and generalized gamma DF $GG(x; b, a, k)$(its PDF is given by $f(x) = \frac{b}{\Gamma(k)} \frac{x^{bk-1}}{a^{bk}} \exp(-\left(\frac{x}{a}\right)^b), \ a, b, k, x > 0$, where $\Gamma(.)$ denotes the gamma function). Namely, we consider the DFs $E_2(x)$, $W(3, 5; x)$, and $\mathcal{N}(x; 2, 100)$, $L\mathcal{N}(x; 1, 0.6)$, and $GGamma(x; 2, 3, 5)$. For each of these distributions, we generate 10,000 ordered random samples, each of which has an integer size, which is generated from the left truncated DF of the RV N at $R + k$. The average value $\bar{N}$ of N (i.e., the sample mean of N) for these observed samples is determined as well as its integer part $\tilde{N}$, when the random sample size N is assumed to be distributed as:

1. Binomial distribution $Bin(50, p)$, for $p = 0.1, 0.4, 0.7$, or
2. Poisson distribution $Pois(\lambda)$, for $\lambda = 24, 28, 32$.

Algorithm for implementing the CP method
Step 1: Select a distribution, from which the data come, i.e., F_X, and its parameters,
Step 2: Determine the values of R and k (say $R = 20$ and $k = 1, 2, ..., 5$),
Step 3: Generate a random integer, N_0, from the truncated binomial distribution (denoted by $Bin(50, p|R + k), p = 0.1, 0.4, 0.7$), or the truncated Poisson distribution (denoted by $Pois(\lambda|R + k), \lambda = 24, 28, 32$), at $R + k$.
Step 4: Generate an ordered random sample of size N_0 from F_X and determine $x_{R:N_0}$ and $x_{R+k:N_0}$,
Step 5: Generate an ordered random sample of size $N_0 - R$ from $E_1(y)$ and determine $y_{k:N_0-R}$,
Step 6: Calculate $\hat{x}_{R+k:N_0}$ by using (2.1),
Step 7: Repeat the steps 3–6, 10,000 times,
Step 8: Compute the average of $x_{R+k:N_0}$ and $\hat{x}_{R+k:N_0}$ over these random samples (10,000) (the point D-prediction of $x_{R+k:N_0}$ is $\hat{x}_{R+k:N_0}$). For simplicity, we denote these averages by $x_{R+k:N}$ and $\hat{x}_{R+k:N}$, respectively,
Step 9: Compute the mean square error (MSE) of the point D-predictor $\hat{X}_{R+k:N}$.

The computations are carried out by R 4.0.2 and the results of the simulation are presented in Tables 1 and 2. Tables 1 and 2 show that the CP method yields very close point estimates for future order statistics for all DFs under consideration. Moreover, the accuracy of these estimates are stable with increasing k, i.e., the distance between the last observed order statistic and the future one (the predicted order statistic).

Remark 3.1. *Although the main aim of the paper is studying point predictions, which is highly important, we can use the average perdition over the* $M = 10,000$ *samples, as an average, e.g.* $\overline{x}$, *of the sample of large size and construct a* 95% *confidence interval (CI) to the future failure. Namely,* $\overline{x} \pm \frac{S}{\sqrt{M}} z^\star$, *where* $\overline{x} = \hat{x}_{R+k:N_0}$, $z^\star$ *represents the appropriate* $z^\star$*−value from the standard normal distribution for the desired confidence level and* S *is the sample standard deviation. For example,*

1. (1.3022; 1.3170) *is the* 95% *CI for* $x_{R+k:N}$ (= 1.3091) *in Table 1 with* $E_2(x)$, $N \sim Bin(50, 0.1|R+k)$, $k = 1$, *where* $\overline{x} = \hat{x}_{R+k:N} = 1.3096, S = 0.3769$;
2. (6.8250; 6.8508) *is the* 95% *CI for* $x_{R+k:N}$ (= 6.8350) *in Table 1 with* $W(3; 5; x)$, $N \sim Bin(50, 0.1|R + k)$, $k = 1$, *where* $\overline{x} = \hat{x}_{R+k:N} = 6.8379, S = 0.6566$;
3. (143.6749; 145.2233) *is the* 95% *CI for* $x_{R+k:N}$ (= 144.6539) *in Table 1 with* $\mathcal{N}(x; 2, 100)$, $N \sim Bin(50, 0.1|R + k)$, $k = 1$, *where* $\overline{x} = \hat{x}_{R+k:N} = 144.4491, S = 39.4994$.

Table 1. CP method for $E_2(x)$, $W(3,5;x)$ and $\mathcal{N}(x;2,100)$, when $R = 20,\ k = 1,2,...,5$.

$E_2(x)$

$N \sim Bin(50,0.1\mid R+k)$				$N \sim Bin(50,0.4\mid R+k)$				$N \sim Bin(50,0.7\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	1.3091	1.3096	1	24	1.0563	1.0592	1	35	0.4609	0.4612
2	23	1.3347	1.3342	2	25	1.1033	1.1038	2	35	0.4994	0.4987
3	24	1.3577	1.3643	3	25	1.1538	1.1528	3	35	0.5404	0.5405
4	25	1.3889	1.3875	4	26	1.1994	1.1988	4	35	0.5862	0.5866
5	26	1.4884	1.4915	5	27	1.2362	1.2457	5	35	0.6377	0.6365
MSE=1.1058E-05				MSE=1.9889E-05				MSE=4.5115E-07			

$N \sim Pois(24\mid R+k)$				$N \sim Pois(28\mid R+k)$				$N \sim Pois(32\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	26	0.8412	0.8413	1	29	0.6987	0.6977	1	32	0.5676	0.5674
2	27	0.8891	0.8937	2	29	0.7492	0.7510	2	33	0.6096	0.6113
3	28	0.9400	0.9417	3	30	0.8007	0.8045	3	33	0.6608	0.6601
4	28	0.9786	0.9843	4	30	0.8517	0.8551	4	33	0.7105	0.7113
5	29	1.0291	1.0302	5	31	0.8986	0.9049	5	33	0.7599	0.7597
MSE=1.1535E-05				MSE=1.4111E-05				MSE=8.6515E-07			

$W(3,5;x)$

$N \sim Bin(50,0.1\mid R+k)$				$N \sim Bin(50,0.4\mid R+k)$				$N \sim Bin(50,0.7\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	6.8350	6.8379	1	24	6.3380	6.3380	1	35	4.8188	4.8190
2	23	6.8746	6.8802	2	25	6.4422	6.4455	2	35	4.9482	4.9504
3	24	6.9211	6.9213	3	25	6.5460	6.5452	3	35	5.0830	5.0863
4	25	6.9742	6.9821	4	26	6.6148	6.6202	4	35	5.2197	5.2244
5	26	7.1175	7.1263	5	27	6.6886	6.6915	5	35	5.3612	5.3662
MSE=3.5633E-05				MSE=9.8913E-06				MSE=1.2601E-05			

$N \sim Pois(24\mid R+k)$				$N \sim Pois(28\mid R+k)$				$N \sim Pois(32\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	27	5.8413	5.8376	1	29	5.4782	5.4781	1	32	5.1180	5.1190
2	27	5.9643	5.9603	2	29	5.6235	5.6205	2	32	5.2519	5.2555
3	28	6.0721	6.0706	3	30	5.7461	5.7400	3	33	5.3893	5.3888
4	28	6.1678	6.1640	4	30	5.8534	5.8516	4	33	5.5261	5.5353
5	29	6.2639	6.2598	5	31	5.9647	5.9709	5	33	5.6378	5.6398
MSE=1.2797E-05				MSE=1.7777E-05				MSE=2.0344E-05			

$\mathcal{N}(x;2,100)$

$N \sim Bin(50,0.1\mid R+k)$				$N \sim Bin(50,0.4\mid R+k)$				$N \sim Bin(50,0.7\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	144.6539	144.4491	1	24	115.1733	115.4217	1	35	24.9693	24.9320
2	23	147.0919	147.3464	2	25	121.3190	120.9976	2	35	32.7856	32.7156
3	24	149.2766	149.2806	3	25	125.8034	125.3318	3	35	40.5828	40.6840
4	25	153.6074	152.9934	4	26	132.1379	131.7823	4	35	48.5879	48.5560
5	26	161.6238	161.7251	5	27	136.6317	136.5249	5	35	57.1199	57.2327
MSE=0.0988				MSE=0.1051				MSE=0.0061			

$N \sim Pois(24\mid R+k)$				$N \sim Pois(28\mid R+k)$				$N \sim Pois(32\mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	27	85.4340	85.2440	1	29	64.5565	64.4336	1	32	42.8026	42.7798
2	27	92.0580	91.8196	2	29	72.4807	72.5209	2	33	50.7889	50.8142
3	28	98.7348	98.4784	3	30	79.6402	79.4052	3	33	58.6090	58.8497
4	28	104.2157	104.4301	4	30	86.8052	86.8753	4	33	66.2443	66.2244
5	29	109.7479	110.1024	5	31	93.5239	93.9755	5	33	74.1083	74.4022
MSE=0.0661				MSE=0.0562				MSE=0.0292			

Table 2. CP method for $\mathcal{LN}(x; 1, 0.6)$ and $GGamma(x; 2, 3, 5)$, when $R = 20,\ k = 1, 2, ..., 5$.

$\mathcal{LN}(x; 1, 0.6)$											
$N \sim Bin(50, 0.1\|R + k)$				$N \sim Bin(50, 0.4\|R + k)$				$N \sim Bin(50, 0.7\|R + k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	6.5873	6.5763	1	24	5.5508	5.5557	1	35	3.1603	3.1600
2	23	6.6769	6.6931	2	25	5.7548	5.7434	2	35	3.3156	3.3146
3	24	6.7675	6.7688	3	25	5.9019	5.8809	3	35	3.4777	3.4800
4	25	6.9460	6.9153	4	26	6.1252	6.1068	4	35	3.6531	3.6533
5	26	7.3399	7.3381	5	27	6.2895	6.2851	5	35	3.8534	3.8564
MSE=0.0003				MSE=0.0002				MSE=3.1667E-06			
$N \sim Pois(24\|R + k)$				$N \sim Pois(28\|R + k)$				$N \sim Pois(32\|R + k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	27	4.6707	4.6623	1	29	4.1019	4.1027	1	32	3.5777	3.5763
2	27	4.8502	4.8434	2	29	4.3152	4.3147	2	33	3.7616	3.7610
3	28	5.0480	5.0376	3	30	4.5018	4.4906	3	33	3.9520	3.9596
4	28	5.2108	5.2164	4	30	4.7010	4.7042	4	33	4.1396	4.1375
5	29	5.3838	29.1105	5	31	4.8904	4.9033	5	33	4.3420	4.3551
MSE=7.0578E-05				MSE=6.0784E-05				MSE=4.7215E-05			
$GGamma(x; 2, 3, 5)$											
$N \sim Bin(50, 0.1\|R + k)$				$N \sim Bin(50, 0.4\|R + k)$				$N \sim Bin(50, 0.7\|R + k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	8.7163	8.7195	1	24	8.2472	8.2370	1	35	6.8318	6.8313
2	23	8.7629	8.7672	2	25	8.3438	8.3545	2	35	6.9484	6.9478
3	24	8.8173	8.8188	3	25	8.4462	8.4339	3	35	7.0667	7.0643
4	25	8.8695	8.8662	4	26	8.5174	8.5116	4	35	7.1957	7.1927
5	26	9.0048	9.0019	5	27	8.5825	8.5884	5	35	7.3260	7.3208
MSE=1.0300E-05				MSE=8.7148E-05				MSE=8.3946E-06			
$N \sim Pois(24\|R + k)$				$N \sim Pois(28\|R + k)$				$N \sim Pois(32\|R + k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	26	7.7938	7.7855	1	29	7.4508	7.4448	1	32	7.1149	7.1156
2	27	7.8884	7.8898	2	29	7.5653	7.5671	2	32	7.2342	7.2317
3	28	7.9916	7.9903	3	30	7.6914	7.6880	3	33	7.3528	7.3559
4	28	8.0797	8.0766	4	30	7.7926	7.7905	4	33	7.4734	7.4687
5	29	8.1755	8.1671	5	31	7.9055	7.9001	5	33	7.5740	7.5796
MSE=3.0767E-05				MSE=1.7015E-05				MSE=1.4171E-05			

Further simulation study (mixture of DFs)

One of the most important DFs in the reliability theory is the finite mixture of DFs. A mixture of exponential distributions $E_{\lambda_i}(x - \mu_i)$, $x \geq \mu_i$, $i = 1, 2$, is a possible lifetime distribution given by

$$\begin{aligned} F(x\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2) &= aE_{\lambda_1}(x - \mu_1) + \overline{a}E_{\lambda_1}(x - \mu_1) \\ &= 1 - a\exp(-\lambda_1(x - \mu_1)) - \overline{a}\exp(-\lambda_2(x - \mu_2)), \end{aligned}$$

where $0 \leq a \leq 1$ and $\overline{a} = 1 - a$. The parameter a is known as the mixing proportion. The quantile function (QF) $Q(y\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2)$, $0 < y < 1$, i.e., the inverse function of $F(x\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2)$, is the only obstacle that encounters us to get the point D-predictor for the mixture DF $F(x\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2)$ by the CP method. The following theorem explicitly gives this QF.

Theorem 3.1. *For $y \in (0, 1)$, the QF for the DF $F(x\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2)$ is*

$$Q(y\,{:}\,a; \mu_1, \lambda_1; \mu_2, \lambda_2) = \mu_2 - \frac{1}{\lambda_2}\log(1 - \beta_0), \quad 0 \leq a \leq 1,\ \lambda_1,\ \lambda_2 > 0,$$

where $\beta_0 \in (0,1)$ *is the minimum root of the non-linear equation (of* β*)*

$$a[1 - e^{\lambda_1(\mu_1-\mu_2)}(1-\beta)^{\frac{\lambda_1}{\lambda_2}}] + \overline{a}\beta = y, \ 0 \le \beta \le 1. \tag{3.1}$$

Proof. By adopting the method of Bernard and Vanduffel [34], the QF $Q(y : a; \mu_1, \lambda_1; \mu_2, \lambda_2)$, for $y \in (0,1)$, is given by

$$Q(y : a; \mu_1, \lambda_1; \mu_2, \lambda_2) = \max\{E_{\lambda_1}^{-1}(\theta_* - \mu_1), E_{\lambda_2}^{-1}(\beta_* - \mu_2)\}, \tag{3.2}$$

where θ_* and β_* are defined by

$$\theta_* = \inf\{\theta \in (0,1) \mid \exists \beta \in (0,1);\ E_{\lambda_1}^{-1}(\theta_* - \mu_1) \ge E_{\lambda_2}^{-1}(\beta_* - \mu_2);\ a\theta + \overline{a}\beta = y\} \tag{3.3}$$

and $\beta_* = \frac{p - a\theta_*}{\overline{a}} \in [0,1]$, respectively. Now, the relation $E_{\lambda_1}^{-1}(\theta_* - \mu_1) \ge E_{\lambda_2}^{-1}(\beta_* - \mu_2)$ clearly leads to $\theta \ge 1 - e^{\lambda_1(\mu_1-\mu_2)}(1-\beta)^{\frac{\lambda_1}{\lambda_2}}$. Therefore, by (3.3), we get $\theta_* = 1 - e^{\lambda_1(\mu_1-\mu_2)}(1-\beta_0)^{\frac{\lambda_1}{\lambda_2}}$ and $\beta_* = \frac{p - a[1 - e^{\lambda_1(\mu_1-\mu_2)}(1-\beta_0)^{\frac{\lambda_1}{\lambda_2}}]}{\overline{a}}$, where β_0 is the minimum root of the non-linear equation (3.1). Finally, since both the mixture components $E_{\lambda_i}(x - \mu_i),\ i = 1, 2$, are continuous and monotone increasing, then by using the result of Bernard and Vanduffel (2014), we get $E_{\lambda_1}^{-1}(\theta_* - \mu_1) = E_{\lambda_2}^{-1}(\beta_* - \mu_2)$. Thus, (3.2) implies $Q(y : a; \mu_1, \lambda_1; \mu_2, \lambda_2) = E_{\lambda_1}^{-1}(\theta_* - \mu_1) = E_{\lambda_2}^{-1}(\beta_* - \mu_2) = \mu_2 - \frac{1}{\lambda_2}\log(1 - \beta_0)$. The theorem is proved. □

By using Theorem 3.1 we can now apply the aforesaid Algorithms for the implementation of the CP method to carry out a simulation study for the mixture model $F(x : a; \mu_1, \lambda_1; \mu_2, \lambda_2)$. Table 3 displays the result of this simulation study for $F(x : 0.2 : 5, 0.6; 7, 0.1)$. Again, Table 3 shows that the CP method yields very close point estimates for future order statistics for the DF under consideration (mixture DF). Moreover, the accuracy of these estimates is stable with increasing k.

Table 3. CP method for $F(x : 0.2; 5, 0.6; 7, 0.1)$ when $R = 20,\ k = 1, 2, ..., 5$.

$N \sim Bin(50, 0.1 \mid R+k)$				$N \sim Bin(50, 0.4 \mid R+k)$				$N \sim Bin(50, 0.7 \mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	22	31.1213	31.0776	1	24	25.9358	25.8828	1	35	14.0011	13.9997
2	23	31.4706	31.3951	2	25	26.9543	26.9086	2	35	14.7553	14.7586
3	24	32.0642	31.9205	3	25	27.9817	27.9114	3	35	15.5903	15.6042
4	25	32.7166	32.6573	4	26	28.7172	28.6755	4	35	16.5025	16.5061
5	26	34.6037	34.5996	5	27	29.4947	29.3302	5	35	17.4883	17.5074
MSE=0.0064				MSE=0.0077				MSE=0.0001			
$N \sim Pois(24 \mid R+k)$				$N \sim Pois(28 \mid R+k)$				$N \sim Pois(32 \mid R+k)$			
k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$	k	$\tilde{N}$	$x_{R+k:N}$	$\hat{x}_{R+k:N}$
1	27	21.5964	21.5423	1	29	18.7141	18.7048	1	32	16.0340	16.0471
2	27	22.6279	22.5567	2	29	19.8979	19.8337	2	32	17.0549	17.0459
3	28	23.5521	23.4860	3	30	20.8200	20.7374	3	33	18.0709	18.0387
4	28	24.4502	24.3322	4	30	21.6331	21.6293	4	33	19.0787	19.0957
5	29	25.3155	25.2471	5	31	22.7450	22.7166	5	33	19.9040	19.8981
MSE=0.0062				MSE=0.0024				MSE=0.0003			

4. Numerical examples for real lifetime data sets

In this section, three examples of real lifetime data are presented to demonstrate the importance of the suggested CP method. Moreover, we compete between the CP method (for the random sample size prediction problems) and the CP2 method (for the non-random sample size prediction problems).

Example 4.1. *The following set of real lifetime data was reported by Badar and Priest [35], and was also analyzed by Raqab and Kundu [36]. This data set represents strength measured in GPA for single carbon fibers as well as impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm. The 63 ordered data are given in Table 4. The earlier studies on this data set (Badar and Priest, [35] and Raqab and Kundu, [36]) showed that the data was fitted by the Weibull distribution* $W(x; a, b)$ *with MLE with parameters* $a = 5.049422$ *and* $b = 3.314562$. *We checked the validity of the Weibull model by using the Kolmogorov-Smirnov (K-S) test. It is observed that the K-S distance and the corresponding p-value are K-S=0.087616 and p-value=0.7188336, respectively. We assume that we only observed the first 10 observations and we want to predict the next observations for different values of k (1, 3, 5, 10, 15, 20, 30, 40, 50). We apply the CP method given in Theorem 2.1 and Remark 2.1 with* $M = 10,000$ *and compute the averages to obtain suggested point D-predictions. The DF of N is assumed to be* $Bin(100, 0.6|R + k)$, $R + k = 11, 13, 15, 20, 25, 30, 40, 50, 60,$ *or to be* $Pois(60|R + k)$, $R + k = 11, 13, 15, 20, 25, 30, 40, 50, 60$. *Also, the CP2 method is applied as suggested in Barakat et al. [1]. The result is presented in Table 5. Generally speaking, Table 5 shows that the CP2 method (for non-random sample size) is more favorable than the method CP (for random sample size) for describing this model. On the other hand, the CP method's performance under the assumption* $N \sim Bin(100, 0.6|R + k)$ *is better than the assumption* $N \sim Pois(60|R + k)$.

Table 4. Data set (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Table 5. Point D-prediction of future observations by the CP and CP2 methods.

k	Exact Value	$Bin(100, 0.6\|10 + k)$	$Pois(60\|10 + k)$	n=63 (CP2)
1	2.454	2.488	2.489	2.485
3	2.518	2.567	2.569	2.560
5	2.525	2.641	2.644	2.630
10	2.618	2.809	2.814	2.789
15	2.740	2.961	2.967	2.933
20	2.937	3.105	3.114	3.069
30	3.243	3.392	3.411	3.335
40	3.501	3.731	3.716	3.630
50	4.027	4.052	3.953	4.093
	MSE	0.023	0.024	0.014

Remark 4.1. *When we take N as either binomial or Poisson DFs, we found (via large number of trials) that to get the best result in a reasonable run time, we should take the average of these DFs as equal to or greater than the value at which the distribution of N will be truncated, i.e., 60. There is a theoretical*

explanation for the inevitability of this choice. Namely, for the binomial DF $Bin(n, p)$*, the right end-point is* n*; so if one decided to truncate it at 60 on the left, we have to choose* $n \geq 60$ *(because we cannot choose the value of left-truncation greater than the right end-point of the distribution). In this case, if we choose the average to be 60, we get* $60 = np < n$*, for all values* p*,* $0 < p < 1$*. On the other hand, bearing in mind that p should be chosen as greater than* 0.5 *to get a reasonable time run, we took the average 60, p=0.6 and then we got* $n = 100$*. These choices give an excellent result in a reasonable time run. The same argument can be applied to the Poisson DF, since the average =* $\lambda \approx np$*.*

Example 4.2. *The data set given in Table 6 (20 ordered observations) was handled by Ahmad and Ali [37] and also analyzed by Ateya [38]. This data set includes the lifetimes of electronic components. The data was fitted by the 2-component mixture of the exponential distributions,* $F(x : a; 0, \lambda_1; 0, \lambda_2)$*, with EM algorithm with parameters* $a = 0.02084921$*,* $\lambda_1 = 0.22009927$*, and* $\lambda_2 = 0.40215270$*. We checked the validity of this mixture model by using the K-S test. It is observed that the K-S distance and the corresponding p-value are K-S=0.14188 and p-value=0.7645, respectively . We assume that we observed the first 10 observations and we want to predict the next observations. We apply the CP method given in Theorem 2.1 and Remark 2.1 with* $M = 10,000$ *and compute the averages to obtain suggested point D-predictions. The DF of N is assumed to be* $Bin(50, 0.5|R + k)$*,* $R + k = 11, 12, ..., 20$*, or to be* $Pois(25|R + k)$*,* $R + k = 11, 12, ..., 20$*. On the other hand, we apply the CP2 method. The result is presented in Table 7. Generally speaking, Table 7 shows the results of CP method are excellent if compared with the CP2 method. Moreover, the CP method's performance under the assumption* $N \sim Bin(50, 0.5|R + k)$ *is better than the assumption* $N \sim Pois(25|R + k)$*.*

Table 6. Lifetimes of 20 electronic components.

0.03	012	0.22	0.35	0.73	0.79	1.25	1.41	1.52	1.79
1.8	1.94	2.38	2.4	2.87	2.99	3.14	3.17	4.72	5.09

Table 7. Point D-prediction of future observations by the CP and CP2 methods.

k	Exact Value	$Bin(50, 0.5\|10 + k)$	$Pois(25\|10 + k)$	n=20 (CP2)
1	1.80	1.97	1.98	2.04
2	1.94	2.16	2.19	2.32
3	2.38	2.38	2.42	2.64
4	2.40	2.61	2.67	3.00
5	2.87	2.88	2.93	3.42
6	2.99	3.16	3.22	3.93
7	3.14	3.48	3.51	4.57
8	3.17	3.80	3.82	5.43
9	4.72	4.15	4.11	6.67
10	5.09	4.54	4.42	9.29
	MSE	0.13	0.16	3.05

Example 4.3. *At the start of 2020, the new Coronavirus (COVID-19) has spread widely in Egypt, and a large number of people have become infected. On March 11, 2020, the World Health Organization declared a new pneumonia outbreak a "global pandemic". Therefore, it has become necessary to find*

new ways to anticipate the number of infected people with the epidemic. Based on the official data modeling, Li et al. [39] have studied the transmission process of the Coronavirus disease in China in 2019. Read et al. [40] have studied early estimation of epidemiological parameters and epidemic prediction.

In this example, we consider the prediction problem concerning the total cases per million in Egypt from 1-5-2020 to 23-8-2020. This time interval yields a set of 115 ordered observations. This study is based on the official data that is available to all countries of the world at the link "https://ourworldindata.org/coronavirus-source-data", and includes the new cases, cumulative cases, new cases per million, deaths numbers per million and population density of all countries of the world. Moreover, this data is updated daily. The difference between every two consecutive observations in this data has important indications. If the difference is large, this indicates an increase in the number of virus infections; while if the difference is small, this indicates a small number of infected people, and finally if it is zero, this indicates the absence of new infections.

Naturally, the continuity issue of this data appears at the beginning, but the crucial factor in our case is whether the DF that could successfully fit (approximately) this data is of a continuous type. Here, by using the moment method, the data were fitted by a continuous DF, which is Johnson SB distribution with parameters $\gamma = -0.19512$, $\delta = 0.29836$, $\xi = 80$, *and* $\lambda = 881.49$. *We checked the validity of the model by using the K-S test. It is observed that the K-S distance and the corresponding p-value are K-S=0.04532 and p-value=0.9582, respectively. We assume that we observed the first 100 observations and we want to predict the next observations. We apply the CP method given in Theorem 2.1 and Remark 2.1 with* $M = 10,000$, *and compute the averages to obtain suggested point D-predictions. The DF of N is assumed to be* $Bin(200, 0.7|R + k)$, $R + k = 101, 102, ..., 115$, *or to be* $Pois(140|R + k)$, $R + k = 101, 102, ..., 115$. *On the other hand, we apply the CP2 method. The result is presented in Table 8. Table 8 shows that the results of the CP method are excellent when compared with the result of the CP2 method. Moreover, the CP method's performance under the assumption* $N \sim Bin(200, 0.7|100 + k)$ *is better than the assumption* $N \sim Pois(140|100 + k)$.

Table 8. Point D-prediction of future observations by the CP and CP2 methods.

k	Exact Value	$Bin(200, 0.7\|100 + k)$	$Pois(140\|100 + k)$	n=115(CP2)
1	931.397	931.584	931.743	934.090
2	933.137	933.331	933.592	937.919
3	934.837	935.006	935.345	941.416
4	936.479	936.604	936.978	944.527
5	937.739	938.115	938.565	947.272
6	939.156	939.574	940.081	949.703
7	940.251	940.958	941.446	951.868
8	941.384	942.291	942.796	953.813
9	942.743	943.552	944.097	955.479
10	943.866	944.759	945.234	956.904
11	943.866	945.895	946.305	958.118
12	947.032	946.992	947.362	959.150
13	948.117	948.029	948.381	959.971
14	949.319	949.017	949.324	960.639
15	950.189	949.950	950.270	961.139
	MSE	0.4987	1.042	11.3361

5. Conclusions

We proposed a new method, CP, based on some characterizations of order statistics via their DFs for constructing point predictions for the observed values of the future order statistics (especially in the type I and II censoring). An implemented comprehensive simulation study revealed that the proposed method works very well if both the parent and the distribution for N are known.

Undoubtedly, there are some situations where the information needed about the data's distributions (the parent distribution F and the distribution for N) is available due to the earlier experiences, or the nature of the problem itself. For example, the earlier experiences about the data given in Example 4.1 (where in Example 4.1, we checked this claim) show that the value of the strength in GPA for single carbon fibers as well as impregnated 1000-carbon fiber tows follows the Weibull distribution. On the other hand, under some practical conditions, the extreme value distributions would appear as distributions for some material strength in the life-time tests, where this fact was theoretically proved (cf. Leadbetter et al. [41] Chap. 14). For several other examples of situations where the parametric assumption (i.e., the information about the data's distributions are available) is realistic see Nelson [42].

Three real data examples treated the problem of not knowing the parent distribution F and the DF of N (especially Examples 4.2 and 4.3). Moreover, these examples showed how we can trade-off between the random and non-random sample size of the given model. According to this study, in any real data prediction problem, where we have a lack of knowledge about the data's distributions. we suggest the following methodology:

1. Comparatively divide the available data into two parts, one is big and the other is small;
2. Choose a DF that best fits the data (i.e., a DF that best fits the data among all the DFs that fit this data via the statistical package that you use) belonging to the big part;
3. Use this selected DF to apply the two methods, CP2 for non-random sample size (Barakat et al., [1]) and the suggested method CP (by considering a potential number of DFs for the discrete random size) to the big part in order to predict the items in the small part.
4. Check which method is more favorable to the given data.

Examples 4.2 and 4.3 show that the proposed methodology is reliable and that it gives very good results. On the other hand, Example 4.1 shows that the suggested method is efficient even if the predicted future items are far from the last observed value.

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Conflict of interest

The authors have no conflict of interest.

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