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Research article

Few-weight quaternary codes via simplicial complexes

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Abstract: In this paper, we construct quaternary linear codes via simplicial complexes and we also determine the weight distributions of these codes. Moreover, we present an infinite family of minimal quaternary linear codes, which also meet the Griesmer bound.

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1. Introduction

Let m be a positive integer, q be a power prime, and $(V_m, \cdot)$ be an m-dimensional vector space over $\mathbb{F}_q$, where $\cdot$ denotes an inner product on V_m. For a linear code of length n over $\mathbb{F}_q$, there is a generic construction as follows:

$$C_D = \{(x \cdot d_1, x \cdot d_2, \cdots, x \cdot d_n) : x \in V_m\} \tag{1.1}$$

where, $D = \{d_1, \cdots, d_n\} \subseteq V_m$. The set D is called the defining set of the code C_D. Although different orderings of the elements of D result in different codes, these codes are permutation equivalent and have the same parameters. If the set D is properly chosen, the code C_D may have good parameters. The following two situations are common:

(1) When $V_m = \mathbb{F}_{q^m}$, $x \cdot y = \mathrm{Tr}_m(xy)$ for $x, y \in \mathbb{F}_{q^m}$ and Tr_m is the trace function from $\mathbb{F}_{q^m}$ to $\mathbb{F}_q$. In this case, the corresponding code C_D in (1.1) is called a trace code over $\mathbb{F}_q$. This generic construction was first introduced by Ding et al. [3]. Many known codes have been produced by selecting a proper defining set, see [6, 10] for examples. Note that defining sets here are almost all related with trace functions, and the computations of weight distributions of corresponding linear codes are heavily dependent on known results of exponential sums.

(2) When $V_m = \mathbb{F}_q^m$, $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^m x_i y_i$ for $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{F}_q^m$. This standard construction in (1.1) can be also found in [7]. Recently, Zhou et al. [13] investigated four infinite families of binary linear codes and obtained some binary linear complementary dual or self-orthogonal codes based on the above generic construction.

Based on the construction for linear codes from functions, Hyun et al. [9] constructed some infinite families of binary optimal linear codes by choosing the support set of a Boolean function as the complement of some simplicial complexes. After that, a more general situation was considered by Hyun et al. [8] by using posets, and they presented some optimal and minimal binary linear codes not satisfying the condition of Ashikhmin-Barg [1]. Notice that linear codes from the generic construction via posets are all over prime fields and the main difficulty is to calculate the frequencies of their codewords. It seems that new techniques are required to go beyond prime fields. In this paper, we will provide such a technique for linear codes over the finite field $\mathbb{F}_4$.

The rest of this paper is organized as follows. In Section 2, we will recall some concepts of simplicial complexes, generating functions and investigate the structure of the finite field $\mathbb{F}_4$. In Section 3, we determine the weight distributions of these quaternary codes and find a class of minimal quaternary linear codes.

2. Preliminaries

Let C be an $[n, k, d]$ linear code over $\mathbb{F}_q$. Assume that there are A_i codewords in C with Hamming weight i for $1 \leq i \leq n$. Then C has weight distribution $(1, A_1, \ldots, A_n)$ and weight enumerator $1 + A_1 z + \cdots + A_n z^n$. Moreover, if the number of nonzero A_i's in the sequence $(A_1, \ldots, A_n)$ is exactly equal to t, then the code is called *t-weight*. An $[n, k, d]$ code C is called *distance optimal* if there is no $[n, k, d+1]$ code (that is, this code has the largest minimum distance for given length n and dimension k), and it is called *almost optimal* if an $[n, k, d+1]$ code is distance optimal (refer to [7, Chapter 2]). On the other hand, the *Griesmer bound* [5] on an $[n, k, d]$ linear code over $\mathbb{F}_q$ is given by $\sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \leq n$, where $\lceil \cdot \rceil$ is the ceiling function. We say that a linear code is a Griesmer code if it meets the Griesmer bound with equality. One can verify that Griesmer codes are distance-optimal.

2.1. Simplicial complexes and generating functions

Let $\mathbb{F}_q$ be the finite field with order q. Assume that m is a positive integer. The support supp($\mathbf{v}$) of a vector $\mathbf{v} \in \mathbb{F}_q^m$ is defined by the set of nonzero coordinates. The Hamming weight $wt(\mathbf{v})$ of $\mathbf{v} \in \mathbb{F}_q^m$ is defined by the size of supp($\mathbf{v}$). For two subsets $A, B \subseteq [m]$, the set $\{x : x \in A \text{ and } x \notin B\}$ and the number of elements in the set A are denoted by $A \backslash B$ and $|A|$, respectively.

For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^m$, we say $\mathbf{v} \subseteq \mathbf{u}$ if supp($\mathbf{v}$) $\subseteq$ supp($\mathbf{u}$). We say that a family $\Delta \subseteq \mathbb{F}_2^m$ is a *simplicial complex* if $\mathbf{u} \in \Delta$ and $\mathbf{v} \subseteq \mathbf{u}$ imply $\mathbf{v} \in \Delta$. For a simplicial complex Δ, a maximal element of Δ is one that is not properly contained in any other element of Δ. Let $\mathcal{F} = \{F_1, \ldots, F_l\}$ be the family of maximal elements of Δ. For each $F \subseteq [m]$, the simplicial complex Δ_F generated by F is defined to be the family of all subsets of F.

Let X be a subset of $\mathbb{F}_2^m$. Hyun et al. [2] introduced the following m-variable generating function associated with the set X:

$$\mathcal{H}_X(x_1, x_2, \ldots, x_m) = \sum_{u \in X} \prod_{i=1}^{m} x_i^{u_i} \in \mathbb{Z}[x_1, x_2, \ldots, x_m],$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{F}_2^m$ and $\mathbb{Z}$ is the ring of integers.

The following lemma plays an important role in determining the weight distributions of the quaternary codes defined in (1.1).

Lemma 2.1. *[2, Theorem 1]* Let Δ be a simplicial complex of $\mathbb{F}_2^m$ with the set of maximal elements $\mathcal{F}$. Then

$$\mathcal{H}_\Delta(x_1, x_2, \ldots, x_m) = \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (1 + x_i),$$

where $\cap S$ denotes the intersection of all elements in S. In particular, we also have $|\Delta| = \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} 2^{|\cap S|}$.

Example 2.2. Let Δ be a simplicial complex of $\mathbb{F}_2^4$ with the set of maximal elements $\mathcal{F} = \{(1, 1, 0, 0), (1, 0, 1, 1)\}$. Then

$$\begin{aligned}\mathcal{H}_\Delta(x_1, x_2, x_3, x_4) &= \prod_{i \in \{1,2\}} (1 + x_i) + \prod_{i \in \{1,3,4\}} (1 + x_i) - \prod_{i \in \{1\}} (1 + x_i) \\ &= (1 + x_1)(1 + x_2 + x_3 + x_4 + x_3x_4).\end{aligned}$$

Example 2.3. Let Δ be a simplicial complex of $\mathbb{F}_2^4$ with the set of maximal elements $\mathcal{F} = \{(1, 1, 0, 0), (0, 0, 1, 1)\}$. Then

$$\begin{aligned}\mathcal{H}_\Delta(x_1, x_2, x_3, x_4) &= \prod_{i \in \{1,2\}} (1 + x_i) + \prod_{i \in \{3,4\}} (1 + x_i) \\ &= (1 + x_1)(1 + x_2) + (1 + x_3)(1 + x_4) - 1.\end{aligned}$$

2.2. *The structure of* $\mathbb{F}_4$

In the paper [12], the authors first constructed linear codes over the finite ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$ and obtained many optimal binary linear codes by Gray map. After that, Wu et al. [11] also considered the case of $\mathbb{F}_p + u\mathbb{F}_p$ with $u^2 = 0$ and p is an odd prime number. Let $\mathbb{Z}_4$ be the ring of integers modulo 4. For each $u \in \mathbb{Z}_4$ there is a unique representation $u = a + 2b$, where $a, b \in \mathbb{F}_2$. Here the element 2 in $\mathbb{Z}_4$ plays a similar role, which like u for the ring $\mathbb{F}_2 + u\mathbb{F}_2$, and the only difference is the characteristics of the two rings.

For the finite field $\mathbb{F}_4$, as we known $\mathbb{F}_4 \cong \mathbb{F}_2[x]/\langle x^2 + x + 1\rangle$, where $x^2 + x + 1$ is the only irreducible polynomial of degree two in $\mathbb{F}_2[x]$. Let w be an element in some extend field of $\mathbb{F}_2$ such that $w^2 + w + 1 = 0$. Then we have $\mathbb{F}_4 = \mathbb{F}_2(w)$ and for each $u \in \mathbb{F}_4$ there is a unique representation $u = a + wb$, where $a, b \in \mathbb{F}_2$. Let m be a positive integer, and $\mathbb{F}_4^m$ be the set of m-tuples over $\mathbb{F}_4$. Any vector $\mathbf{x} \in \mathbb{F}_4^m$ can be written as $\mathbf{x} = \mathbf{a} + w\mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{F}_2^m$.

3. Weight distributions of quaternary codes

In this section, we will construct some quaternary codes via simplicial complexes and determine the weight distributions of these codes.

There is a bijection between $\mathbb{F}_2^m$ and $2^{[m]}$ being the power set of $[m] = \{1, \ldots, m\}$, defined by $\mathbf{v} \mapsto \mathrm{supp}(\mathbf{v})$. Throughout this paper, we will identify a vector in $\mathbb{F}_2^m$ with its support.

Let A, B be two subsets of $[m]$ and $D = \Delta_A^c + w\Delta_B = \mathbb{F}_2^m \backslash \Delta_A + w\Delta_B \subset \mathbb{F}_4^m$, where w is an element in some extension field of $\mathbb{F}_2$ such that $w^2 + w + 1 = 0$. We define a quaternary code as follows:

$$C_D = \{c_{\mathbf{a}} = (\mathbf{a} \cdot \mathbf{d})_{\mathbf{d} \in D} : \mathbf{a} \in \mathbb{F}_4^m\}. \tag{3.1}$$

First of all, from (3.1), it is easy to check that the code C_D is a linear quaternary code. The length of the code C_D is $|D|$. If $\mathbf{a} = \mathbf{0}$, then the Hamming weight of the codeword $c_{\mathbf{a}}$ is equal to $\mathrm{wt}(c_{\mathbf{a}}) = 0$. Next we assume that $\mathbf{a} \neq \mathbf{0}$. Suppose that $\mathbf{a} = \boldsymbol{\alpha} + w\boldsymbol{\beta}$, $\mathbf{d} = \mathbf{d_1} + w\mathbf{d_2}$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_m), \boldsymbol{\beta} = (\beta_1, \cdots, \beta_m) \in \mathbb{F}_2^m$, $\mathbf{d_1} \in \Delta_A^c$, and $\mathbf{d_2} \in \Delta_B$. Then

$$\begin{aligned} \mathrm{wt}(c_{\mathbf{a}}) &= \mathrm{wt}(((\boldsymbol{\alpha} + w\boldsymbol{\beta}) \cdot (\mathbf{d_1} + w\mathbf{d_2}))_{\mathbf{d_1} \in \Delta_A^c, \mathbf{d_2} \in \Delta_B}) \\ &= \mathrm{wt}((\boldsymbol{\alpha}\mathbf{d_1} + w(\boldsymbol{\alpha}\mathbf{d_2} + \boldsymbol{\beta}\mathbf{d_1}) + w^2\boldsymbol{\beta}\mathbf{d_2})_{\mathbf{d_1} \in \Delta_A^c, \mathbf{d_2} \in \Delta_B}) \\ &= \mathrm{wt}((\boldsymbol{\alpha}\mathbf{d_1} + \boldsymbol{\beta}\mathbf{d_2} + w(\boldsymbol{\beta}\mathbf{d_2} + \boldsymbol{\alpha}\mathbf{d_2} + \boldsymbol{\beta}\mathbf{d_1}))_{\mathbf{d_1} \in \Delta_A^c, \mathbf{d_2} \in \Delta_B}). \end{aligned} \tag{3.2}$$

By the definition of Hamming weight of vector $\mathbf{x} = \mathbf{y} + w\mathbf{z} \in \mathbb{F}_4^m$ with $\mathbf{y}, \mathbf{z} \in \mathbb{F}_2^m$, $\mathrm{wt}(\mathbf{x}) = 0$ if and only if $\mathbf{y} = \mathbf{z} = \mathbf{0}$. Hence

$$\begin{aligned} \mathrm{wt}(c_{\mathbf{a}}) &= |D| - \sum_{\mathbf{d_1} \in \Delta_A^c} \sum_{\mathbf{d_2} \in \Delta_B} \Big(\frac{1}{2} \sum_{y \in \mathbb{F}_2} (-1)^{(\alpha\mathbf{d_1} + \beta\mathbf{d_2})y}\Big)\Big(\frac{1}{2} \sum_{z \in \mathbb{F}_2} (-1)^{(\alpha\mathbf{d_2} + \beta(\mathbf{d_1} + \mathbf{d_2}))z}\Big) \\ &= |D| - \frac{1}{4} \sum_{\mathbf{d_1} \in \Delta_A^c} \sum_{\mathbf{d_2} \in \Delta_B} (1 + (-1)^{\alpha\mathbf{d_1} + \beta\mathbf{d_2}})(1 + (-1)^{\alpha\mathbf{d_2} + \beta(\mathbf{d_1} + \mathbf{d_2})}) \\ &= \frac{3}{4}|D| - \frac{1}{4}\Big(\sum_{\mathbf{d_1} \in \Delta_A^c} (-1)^{\alpha\mathbf{d_1}}\Big)\Big(\sum_{\mathbf{d_2} \in \Delta_B} (-1)^{\beta\mathbf{d_2}}\Big) \\ &- \frac{1}{4}\Big(\sum_{\mathbf{d_1} \in \Delta_A^c} (-1)^{\beta\mathbf{d_1}}\Big)\Big(\sum_{\mathbf{d_2} \in \Delta_B} (-1)^{(\alpha+\beta)\mathbf{d_2}}\Big) \\ &- \frac{1}{4}\Big(\sum_{\mathbf{d_1} \in \Delta_A^c} (-1)^{(\alpha+\beta)\mathbf{d_1}}\Big)\Big(\sum_{\mathbf{d_2} \in \Delta_B} (-1)^{\alpha\mathbf{d_2}}\Big). \end{aligned} \tag{3.3}$$

Theorem 3.1. Let A, B be two subsets of $[m]$ and $D = \Delta_A^c + w\Delta_B \subset \mathbb{F}_4^m$. Then C_D in (3.1) is a $[(2^m - 2^{|A|})2^{|B|}, m]$ quaternary code and its weight distribution is presented in Table 1.

Table 1. Weight distribution of the code in Theorem 3.1.

Weight	Frequency
0	1
$3\times 2^{m+\lvert B\rvert-2}-3\times 2^{\lvert A\rvert+\lvert B\rvert-2}$	$3(2^m-2^{m-\lvert A\cap B\rvert})+(2^m-1)(2^m-2)-(2^{m-\lvert A\rvert}-1)(2^{m-\lvert B\rvert}-2)$
$3\times 2^{m+\lvert B\rvert-2}-2^{\lvert A\rvert+\lvert B\rvert-1}$	$3(2^{m-\lvert A\cap B\rvert}-2^{m-\lvert B\rvert})+(2^{m-\lvert A\rvert}-1)(2^{m-\lvert B\rvert}-2)-(2^{m-\lvert A\cup B\rvert}-1)(2^{m-\lvert A\cup B\rvert}-2)$
$2^{m+\lvert B\rvert-1}-2^{\lvert A\rvert+\lvert B\rvert-1}$	$3(2^{m-\lvert B\rvert}-2^{m-\lvert A\cup B\rvert})$
$2^{m+\lvert B\rvert-1}$	$3(2^{m-\lvert A\cup B\rvert}-1)$
$3\times 2^{m+\lvert B\rvert-2}$	$(2^{m-\lvert A\cup B\rvert}-1)(2^{m-\lvert A\cup B\rvert}-2)$

Proof. It is easy to check that the length of the code C_D is $|D|=(2^m-2^{|A|})2^{|B|}$. To compute the weight and frequency of a codeword, we need to introduce the following notation.

For X a subset of $\mathbb{F}_2^m$, we use $\chi(\mathbf{u}|X)$ to denote a Boolean function in m-variable, and $\chi(\mathbf{u}|X)=1$ if and only if $\mathbf{u}\bigcap X=\emptyset$. For a vector $\mathbf{u}=(u_1,\ldots,u_m)\in\mathbb{F}_2^m$ and a nonempty simplicial complex Δ_A, by Lemma 2.1 we have

$$\begin{aligned}\sum_{\mathbf{x}\in\boldsymbol{\Delta}_\mathbf{A}}(-1)^{\mathbf{u}\cdot\mathbf{x}}=\mathcal{H}_{\Delta_A}((-1)^{u_1},(-1)^{u_2},\ldots,(-1)^{u_m})&=\prod_{i\in A}(1+(-1)^{u_i})\\&=\prod_{i\in A}(2-2u_i)=2^{|A|}\prod_{i\in A}(1-u_i)=2^{|A|}\chi(\mathbf{u}|A).\end{aligned}\qquad(3.4)$$

By (3.3) and (3.4)

$$\begin{aligned}\mathrm{wt}(c_\mathbf{a})&=\frac{3}{4}|D|-\frac{1}{4}(2^m\delta_{\mathbf{0},\boldsymbol{\alpha}}-2^{|A|}\chi(\boldsymbol{\alpha}|A))2^{|B|}\chi(\boldsymbol{\beta}|B)\\&-\frac{1}{4}(2^m\delta_{\mathbf{0},\boldsymbol{\beta}}-2^{|A|}\chi(\boldsymbol{\beta}|A))2^{|B|}\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B)\\&-\frac{1}{4}(2^m\delta_{\mathbf{0},\boldsymbol{\alpha}+\boldsymbol{\beta}}-2^{|A|}\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A))2^{|B|}\chi(\boldsymbol{\alpha}|B),\end{aligned}\qquad(3.5)$$

where δ is the Kronecker delta function.

Next we need to consider the following cases:

Case 1. $\boldsymbol{\alpha}=\mathbf{0}$ and $\boldsymbol{\beta}\neq\mathbf{0}$. Then

$$\begin{aligned}\mathrm{wt}(c_\mathbf{a})&=\frac{3}{4}|D|-\frac{1}{4}(2^m-2^{|A|})2^{|B|}\chi(\boldsymbol{\beta}|B)+\frac{1}{4}2^{|A|+|B|}\chi(\boldsymbol{\beta}|A)(\chi(\boldsymbol{\beta}|B)+1)\\&=\begin{cases}\frac{3}{4}(2^m-2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\beta}|B)=0 \text{ and } \chi(\boldsymbol{\beta}|A)=0,\\ \frac{3}{4}(2^m-2^{|A|})2^{|B|}+\frac{1}{4}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\beta}|B)=0 \text{ and } \chi(\boldsymbol{\beta}|A)=1,\\ \frac{1}{2}(2^m-2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\beta}|B)=1 \text{ and } \chi(\boldsymbol{\beta}|A)=0,\\ \frac{1}{2}(2^m-2^{|A|})2^{|B|}+\frac{1}{2}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\beta}|B)=1 \text{ and } \chi(\boldsymbol{\beta}|A)=1.\end{cases}\end{aligned}$$

(1) Note that $\chi(\boldsymbol{\beta}|B)=0$ and $\chi(\boldsymbol{\beta}|A)=0$ if and only if $\boldsymbol{\beta}\cap(A\cap B)\neq\emptyset$. The number of such $\boldsymbol{\beta}$ is $2^{m-|A\cap B|}(2^{|A\cap B|}-1)=2^m-2^{m-|A\cap B|}$.

(2) Note that $\chi(\boldsymbol{\beta}|B)=1$ and $\chi(\boldsymbol{\beta}|A)=1$ if and only if $\boldsymbol{\beta}\cap(A\cup B)=\emptyset$. The number of such $\boldsymbol{\beta}$ is $2^{m-|A\cup B|}-1$.

(3) Note that $\chi(\boldsymbol{\beta}|B) = 1$ and $\chi(\boldsymbol{\beta}|A) = 0$ if and only if $\boldsymbol{\beta} \cap B = \emptyset$ and $\boldsymbol{\beta} \cap A \neq \emptyset$. The number of such $\boldsymbol{\beta}$ is $(2^{m-|B|} - 1) - (2^{m-|A\cup B|} - 1) = 2^{m-|B|} - 2^{m-|A\cup B|}$.

(4) The number of $\boldsymbol{\beta}$ such that $\chi(\boldsymbol{\beta}|B) = 0$ and $\chi(\boldsymbol{\beta}|A) = 1$ is $2^m - 1 - (2^m - 2^{m-|A\cap B|}) - (2^{m-|A\cup B|} - 1) - (2^{m-|B|} - 2^{m-|A\cup B|}) = 2^{m-|A\cap B|} - 2^{m-|B|}$.

Case 2. $\boldsymbol{\alpha} \neq \mathbf{0}$ and $\boldsymbol{\beta} = \mathbf{0}$. Then

$$\begin{aligned} \mathrm{wt}(c_{\mathbf{a}}) &= \frac{3}{4}|D| - \frac{1}{4}(2^m - 2^{|A|})2^{|B|}\chi(\boldsymbol{\alpha}|B) + \frac{1}{4}2^{|A|+|B|}\chi(\boldsymbol{\alpha}|A)(\chi(\boldsymbol{\alpha}|B) + 1) \\ &= \begin{cases} \frac{3}{4}(2^m - 2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 0 \text{ and } \chi(\boldsymbol{\alpha}|A) = 0, \\ \frac{3}{4}(2^m - 2^{|A|})2^{|B|} + \frac{1}{4}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 0 \text{ and } \chi(\boldsymbol{\alpha}|A) = 1, \\ \frac{1}{2}(2^m - 2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 1 \text{ and } \chi(\boldsymbol{\alpha}|A) = 0, \\ \frac{1}{2}(2^m - 2^{|A|})2^{|B|} + \frac{1}{2}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 1 \text{ and } \chi(\boldsymbol{\alpha}|A) = 1. \end{cases} \end{aligned}$$

Similar to Case 1, the numbers of such $\boldsymbol{\alpha}$ can be determined.

Case 3. $\boldsymbol{\alpha} = \boldsymbol{\beta} \neq \mathbf{0}$. Then

$$\begin{aligned} \mathrm{wt}(c_{\mathbf{a}}) &= \frac{3}{4}|D| - \frac{1}{4}(2^m - 2^{|A|})2^{|B|}\chi(\boldsymbol{\alpha}|B) + \frac{1}{4}2^{|A|+|B|}\chi(\boldsymbol{\alpha}|A)(\chi(\boldsymbol{\alpha}|B) + 1) \\ &= \begin{cases} \frac{3}{4}(2^m - 2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 0 \text{ and } \chi(\boldsymbol{\alpha}|A) = 0, \\ \frac{3}{4}(2^m - 2^{|A|})2^{|B|} + \frac{1}{4}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 0 \text{ and } \chi(\boldsymbol{\alpha}|A) = 1, \\ \frac{1}{2}(2^m - 2^{|A|})2^{|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 1 \text{ and } \chi(\boldsymbol{\alpha}|A) = 0, \\ \frac{1}{2}(2^m - 2^{|A|})2^{|B|} + \frac{1}{2}2^{|A|+|B|}, & \text{if } \chi(\boldsymbol{\alpha}|B) = 1 \text{ and } \chi(\boldsymbol{\alpha}|A) = 1. \end{cases} \end{aligned}$$

Similar to Case 1, the numbers of such $\boldsymbol{\alpha}$ can be determined.

Case 4. $\boldsymbol{\alpha} \neq 0, \boldsymbol{\beta} \neq 0$, and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Then

$$\begin{aligned} \mathrm{wt}(c_{\mathbf{a}}) &= \frac{3}{4}|D| \\ &+ \frac{1}{4}2^{|A|+|B|}[\chi(\boldsymbol{\alpha}|A)\chi(\boldsymbol{\alpha}|B) + \chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B) + \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}|B)]. \end{aligned} \tag{3.6}$$

Let $T = \chi(\boldsymbol{\alpha}|A)\chi(\boldsymbol{\alpha}|B) + \chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B) + \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}|B)$. We divide the proof into the following subcases:

(1) $T = 3$. In this case we have $\mathrm{wt}(c_{\mathbf{a}}) = \frac{3}{4}(2^m - 2^{|A|})2^{|B|} + \frac{3}{4}2^{|A|+|B|}$ and

$$\chi(\boldsymbol{\alpha}|A) = \chi(\boldsymbol{\alpha}|B) = \chi(\boldsymbol{\beta}|A) = \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B) = \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A) = 1,$$

which is equivalent to

$$\boldsymbol{\alpha} \cap (A \cup B) = \emptyset \text{ and } \boldsymbol{\beta} \cap (A \cup B) = \emptyset.$$

The number of such $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is $(2^{m-|A\cup B|} - 1)(2^{m-|A\cup B|} - 2)$.

(2) $T = 2$. Suppose that $\chi(\boldsymbol{\alpha}|A)\chi(\boldsymbol{\alpha}|B) = 0$. We have

$$\begin{cases} \chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B) = 1 \\ \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}|B) = 1 \end{cases} \Longleftrightarrow \begin{cases} \boldsymbol{\beta} \cap A = (\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap B = \emptyset \\ \boldsymbol{\alpha} \cap B = (\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A = \emptyset. \end{cases}$$

Hence $\boldsymbol{\alpha} \cap A \neq \emptyset$. Note that the support of the vector $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is equal to $(\mathrm{supp}(\boldsymbol{\alpha}) \cup \mathrm{supp}(\boldsymbol{\beta})) \backslash (\mathrm{supp}(\boldsymbol{\alpha}) \cap \mathrm{supp}(\boldsymbol{\beta}))$. From $\boldsymbol{\alpha} \cap A \neq \emptyset$ and $\boldsymbol{\beta} \cap A = \emptyset$, we have $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A \neq \emptyset$, which is a contradiction with $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A = \emptyset$. Similarly, we have that there is no $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that $T = 2$.

(3) $T = 1$. In this case we have $\mathrm{wt}(c_{\mathbf{a}}) = \frac{3}{4}(2^m - 2^{|A|})2^{|B|} + \frac{1}{4}2^{|A|+|B|}$. Because $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha}+\boldsymbol{\beta}$ have the same status, without loss of generality, we suppose that $\chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\beta}|B) = 1$. Then we have

$$\begin{cases} \chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\beta}|B) = 1 \\ \chi(\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|B) = 0 \\ \chi(\boldsymbol{\alpha}+\boldsymbol{\beta}|A)\chi(\boldsymbol{\alpha}|B) = 0 \end{cases} \Longleftrightarrow \begin{cases} \boldsymbol{\alpha} \cap A = \emptyset, \boldsymbol{\beta} \cap B = \emptyset \\ \boldsymbol{\beta} \cap A \neq \emptyset \text{ or } (\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap B \neq \emptyset \\ (\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A \neq \emptyset \text{ or } \boldsymbol{\alpha} \cap B \neq \emptyset \end{cases}$$

$$\Longleftrightarrow \begin{cases} \boldsymbol{\alpha} \cap A = \emptyset, \boldsymbol{\beta} \cap B = \emptyset \\ (\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap (A \cup B) \neq \emptyset. \end{cases}$$

The number of such $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is $(2^{m-|A|} - 1)(2^{m-|B|} - 2) - (2^{m-|A\cup B|} - 1)(2^{m-|A\cup B|} - 2)$.

(4) $T = 0$. In this case we have $\mathrm{wt}(c_{\mathbf{a}}) = \frac{3}{4}(2^m - 2^{|A|})2^{|B|}$. The number of such $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is $(2^m - 1)(2^m - 2) - (2^{m-|A\cup B|} - 1)(2^{m-|A\cup B|} - 2) - (2^{m-|A|} - 1)(2^{m-|B|} - 2) + (2^{m-|A\cup B|} - 1)(2^{m-|A\cup B|} - 2) = (2^m - 1)(2^m - 2) - (2^{m-|A|} - 1)(2^{m-|B|} - 2)$.

This completes the proof. □

Corollary 3.2. Let B be a subset of $[m]$ and $D = \mathbb{F}_2^m + w\Delta_B \subset \mathbb{F}_4^m$. Then C_D is a $[2^{m+|B|}, m, 2^{m+|B|-1}]$ two-weight quaternary code and its weight distribution is presented in Table 2.

Table 2. Weight distribution of the code in Corollary 3.2.

Weight	Frequency
0	1
$2^{m+\lvert B\rvert-1}$	$3 \times (2^{m-\lvert B\rvert} - 1)$
$3 \times 2^{m+\lvert B\rvert-2}$	$2^{2m} - 1 - 3 \times (2^{m-\lvert B\rvert} - 1)$

Corollary 3.3. Let A be a subset of $[m]$ and $D = \Delta_A^c + w\mathbb{F}_2^m \subset \mathbb{F}_4^m$. Then C_D is a $[2^m(2^m - 2^{|A|}), m, 3 \times 2^{2m-2} - 3 \times 2^{|A|+m-2}]$ two-weight quaternary code and its weight distribution is presented in Table 3.

Table 3. Weight distribution of the code in Corollary 3.3.

Weight	Frequency
0	1
$3 \times 2^{2m-2} - 2^{\lvert A\rvert+m-1}$	$3 \times (2^{m-\lvert A\rvert} - 1)$
$3 \times 2^{2m-2} - 3 \times 2^{\lvert A\rvert+m-2}$	$2^{2m} - 1 - 3 \times (2^{m-\lvert A\rvert} - 1)$

We give the following examples to illustrate our main results.

Example 3.4. Let $m = 4$, $|B| = 2$, and $D = \mathbb{F}_2^m + w\Delta_B \subset \mathbb{F}_4^m$. By Corollary 3.3, C_D is a $[64, 4, 32]$ quaternary code and its weight distribution is given by $1 + 9z^{32} + 246z^{48}$.

Example 3.5. Let $m = 4$, $|A| = 3$, and $D = \Delta_A^c + w\mathbb{F}_2^m \subset \mathbb{F}_4^m$. By Corollary 3.3 and database in [4], C_D is a $[128, 4, 96]$ quaternary optimal code and its weight distribution is given by $1 + 252z^{96} + 3z^{128}$.

Proposition 3.6. The code in Corollary 3.3 is a Griesmer code.

Proof. By Corollary 3.3, the parameters of the code are

$$[2^m(2^m - 2^{|A|}), m, 3 \times 2^{2m-2} - 3 \times 2^{|A|+m-2}].$$

By the Griesmer bound, we have

$$\begin{aligned}
&\sum_{i=0}^{m-1} \left\lceil \frac{3(2^{2m-2} - 2^{m+|A|-2})}{4^i} \right\rceil \\
&= \sum_{i=0}^{m-1} \frac{3 \times 2^{2m-2}}{4^i} - \sum_{i=0}^{m-1} \left\lfloor \frac{3 \times 2^{m+|A|-2}}{4^i} \right\rfloor \\
&= 3 \times 2^{2m-2} + 3 \times 2^{2m-4} + \cdots + 3 \\
&-(3 \times 2^{m+|A|-2} + 3 \times 2^{m+|A|-4} + \cdots + X + Y),
\end{aligned}$$

where $X = 3$ and $Y = 0$ if $m + |A| - 2$ is even; and $X = 6$ and $Y = 1$ if $m + |A| - 2$ is odd. Then

$$\begin{aligned}
&\sum_{i=0}^{m-1} \left\lceil \frac{3(2^{2m-2} - 2^{m+|A|-2})}{4^i} \right\rceil \\
&= \frac{3 \times 2^{2m-2} - 3 \times \frac{1}{4}}{1 - \frac{1}{4}} - \frac{3 \times 2^{m+|A|-2} - X \times \frac{1}{4}}{1 - \frac{1}{4}} - Y \\
&= 2^{2m} - 1 - (2^{m+|A|} - 1) = 2^m(2^m - 2^{|A|}).
\end{aligned}$$

This completes the proof. □

For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^m$, we say that $\mathbf{u}$ covers $\mathbf{v}$ if $\mathrm{supp}(\mathbf{v}) \subseteq \mathrm{supp}(\mathbf{u})$. A nonzero codeword $\mathbf{u}$ in a linear code C is said to be *minimal* if $\mathbf{u}$ covers the zero vector and the $\mathbf{u}$ itself but no other codewords in the code C. A linear code C is said to be *minimal* if every nonzero codeword in the code C is minimal.

The following lemma developed by Aschikhmin and Barg [1] is a useful criterion for a linear code to be minimal.

Lemma 3.7. A linear code C over $\mathbb{F}_q$ with minimum distance $w_{\min}$ is minimal provided that $w_{\min}/w_{\max} > (q-1)/q$, where $w_{\max}$ denotes the maximum nonzero Hamming weight in the code C.

Corollary 3.8. Let A be a proper subset of $[m]$ and $D = \Delta_A^c + w\mathbb{F}_2^m \subset \mathbb{F}_4^m$ in Corollary 3.3. Then the code C_D is minimal.

Proof. The result follows from Lemma 3.7 and

$$\frac{w_{\min}}{w_{\max}} = \frac{3 \times 2^{2m-2} - 3 \times 2^{|A|+m-2}}{3 \times 2^{2m-2} - 2^{|A|+m-1}} = 1 - \frac{1}{2(3 \times 2^{m-|A|} - 1)} > \frac{3}{4}.$$

□

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Conflict of interest

Authors declare no conflict of interest in this paper.

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