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Research article

A pair of dual Hopf algebras on permutations

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Abstract: Hopf algebras are important objects in algebraic combinatorics since they have strong stability. In particular, its dual space is an important tool to study the properties of the original Hopf algebra. Based on the classical shuffle Hopf algebra structure, we have proved that the shuffle product and deconcatenation coproduct on the standard factorizations of permutations define a graded shuffle Hopf algebra on permutations. In this paper, we figure out a new product and a new coproduct on permutations to get the duality of this graded shuffle Hopf algebra.

Keywords: dual Hopf algebra; global descent; concatenation product; draw coproduct; antipode
Mathematics Subject Classification: 05A05, 08C20, 16T05

1. Introduction

In 1941, Hopf [1] put forward the concept of both algebraic structure and coalgebra structure in the study of cohomology algebra $H^*(G, K)$ of Lie group G. In 1965, Milnor and Moore [2] formally called them *Hopf algebras*, then Sweedler [3] and Artamonov [4] also studied it. Especially in recent two decades, the rise of quantum groups, the successful solution of Kaplansky's conjectures [5] and the development of Hopf algebra theory make it a new scientific system. Hopf algebras are widely used in various fields, for example, universal enveloping algebras in Lie theory and free Lie algebras [6, 7], nonlinear control theory [8], geometry algebras [9], degenerate versions of Drinfeld-Jimbo quantum groups [10], generalized Clifford algebras in aforementioned categories [11] and even the genetic inheritance in biology [12].

In 1984, Michel Van den Bergh [13], Blattner and Montgomery [14] introduced the duality of Hopf algebras. It is proved that the group ring is a classical special case of Hopf algebra. But the dual space of a general Hopf algebra is only an algebra, not a Hopf algebra. Only when a Hopf algebra is graded, its graded dual space has a corresponding graded Hopf algebra structure.

Because the rigidity of Hopf algebra structure often reveals the deep nature and connection in combinatorics, Hopf algebra has gradually become one of the important research contents in algebraic combinatorics. In 1979, Joni and Rota [15] found that the discrete structures in combinatorics have natural Hopf algebraic structures. In 2006, Aguiar, Bergeron and Sottle [16] proposed the concept of combinatorial Hopf algebras and studied the category of them. More and more combinatorial Hopf algebras have been discovered and studied. In 2009, Bergeron and Li [17] gave the axioms on a tower of algebras to guarantee that its Grothendieck groups are dual graded Hopf algebras. Later, Bergeron, Lam and Li [18] analyzed the relationship between combinatorial Hopf algebras and dual graded graphs. The Hopf algebra of symmetric functions is a self-dual Hopf algebra, which plays an important role in algebraic combinatorics. In 2016, using the duality Li, Morse and Shields [19] provided a dual approach to the structure constants for K-theory of Grassmannians. It is well-known that symmetric functions are closed related to symmetric groups. They can be applied to algebraic number theory [20] and substochastic matrices [21–24].

In combinatorics, a permutation of degree n is an arrangement of n elements. The symmetric group of degree n, denoted by S_n, contains all permutations of $[n] = \{1, 2, \ldots, n\}$. Let KS_n be the vector space spanned by S_n over field K. Define $KS := \bigoplus_{n\geq 0} KS_n$, the direct sum of KS_n, where $S_0 = \{\epsilon\}$ and ϵ is the empty permutation. Then KS is graded and its nth component is KS_n. In 1958, Ree [25] studied an algebra associated with shuffles, then the study of shuffles is further promoted [26, 27]. In 1995, Malvenuto and Reutenauer [28] constructed the product $*$ according to [29]. In fact, the product $*$ is the shuffle product $\sqcup\!\sqcup$ (Eq 2.2 in Section 2). However, this product $*$ is not commutative on permutations.

In 2005, Aguiar and Sottile [30] introduced *global descents* of permutations in symmetric group S_n. It plays a crucial role in this paper. In 2018, Bergeron, Ceballos and Pilaud [31] introduced the concept of *gaps* on permutations of S_n. Then Bergeron et al. linked the global descents and the shuffle product $\sqcup\!\sqcup$ together and defined a new shuffle product on permutations. The new shuffle product is based on the global descents of permutations, which requires that from a standard factorization to the original permutation we renumber the numbers in the atoms from right to left. This ensures the new shuffle product is commutative on permutations. We denote such a shuffle by $\sqcup\!\sqcup_G$ and define a coproduct Δ on KS such that $(KS, \sqcup\!\sqcup_G, \mu, \Delta, \nu)$ is a Hopf algebra [32]. There must be a graded Hopf algebra dual to the shuffle Hopf algebra $(KS, \sqcup\!\sqcup_G, \mu, \Delta, \nu)$, which could be seen from the definition of dual graded Hopf algebras.

The main result of this paper is to find the duality $(KS, \sqcup\!\sqcup_G^*, \mu, \Delta^*, \nu)$. In Section 2, we introduce the dual Hopf algebras of Malvenuto and Reutenauer in details. In Section 3, we first introduce the shuffle product $\sqcup\!\sqcup_G$ and deconcatenation coproduct Δ, then define a new product $\sqcup\!\sqcup_G^*$ and a new coproduct Δ^* on KS. In Section 4, we prove that the product $\sqcup\!\sqcup_G^*$ and the coproduct Δ^* are compatible so $(KS, \sqcup\!\sqcup_G^*, \mu, \Delta^*, \nu)$ becomes a bialgebra. Since the bialgebra is graded and connected, it is a graded Hopf algebra. Then, we give a closed-formula of its antipode and prove it. In Section 5, we prove that graded Hopf algebras $(KS, \sqcup\!\sqcup_G, \mu, \Delta, \nu)$ and $(KS, \sqcup\!\sqcup_G^*, \mu, \Delta^*, \nu)$ are dual to each. In Section 6, we apply products $\sqcup\!\sqcup_G$, $\sqcup\!\sqcup_G^*$ and coproduct Δ, Δ^* on different objects and get some other pairs of dual graded Hopf algebras and link some of them to dual graded graphs. Furthermore, we post some open questions on ungraded bialgebras.

2. A classic example of dual Hopf algebras on KS

One of the classic examples of dual graded Hopf algebras were introduced by Malvenuto and Reutenauer, who defined two products $*$ and $*'$, and two coproducts Δ and Δ' on KS. More details can be found in [28]. Here, we only briefly introduce these two graded Hopf algebras.

A bialgebra $(H, \psi, \mu, \Delta, \nu)$ over a field K, where ψ is a product, μ is a unit map, Δ is a coproduct and ν is a counit map, is a Hopf algebra if it admits a unique *antipode* θ satisfying the following identity

$$\psi \circ (\mathrm{id} \otimes \theta) \circ \Delta = \mu \circ \nu = \psi \circ (\theta \otimes \mathrm{id}) \circ \Delta. \tag{2.1}$$

A graded connected bialgebra $(H, \psi, \mu, \Delta, \nu)$ admits a unique antipode θ so is a graded Hopf algebra.

Let A be a set called an *alphabet* and the elements in it are called *letters*. A *word* f over alphabet A is composed of finite letters, *i.e.*, $f = f_1 f_2 \cdots f_n$ with $f_i \in A$ for all i. A word without any letters is called the *empty word*, denoted by ϵ.

The *shuffle product* ш on words is defined by

$$\begin{aligned} f ш g &= (f_1 f_2 \cdots f_n) ш (g_1 g_2 \cdots g_m) \\ &= f_1(f_2 \cdots f_n ш g) + g_1(f ш g_2 \cdots g_m), \end{aligned} \tag{2.2}$$

where $f = f_1 f_2 \cdots f_n$, $g = g_1 g_2 \cdots g_m$ and $f ш \epsilon = \epsilon ш f = f$.

For example, if $f = ab$ and $g = ac$, then

$$f ш g = ab ш ac = abac + 2aabc + 2aacb + acab.$$

For any permutation $\alpha \in S_n$, it can be written in the one-line notation as $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$, where $\alpha_i = \alpha(i) \in [n]$.

For $\alpha \in S_n$ and $\beta \in S_m$, define the *product* $*$ on permutations by

$$\alpha * \beta = \alpha_1 \alpha_2 \cdots \alpha_n ш (n + \beta_1)(n + \beta_2) \cdots (n + \beta_m), \tag{2.3}$$

where ш is the shuffle product above.

For example, $12 * 21 = 12 ш 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312$. The empty permutation ϵ serves as the unit, and $\alpha * \epsilon = \epsilon * \alpha = \alpha$.

The *unit* map is defined by μ: $K \to KS$, $\mu(1) = \epsilon$.

A *sequence* on $[n]$ is obtained by selecting finite different elements from the set $[n]$, denoted by $\beta = \beta_1 \beta_2 \cdots \beta_m$ $(1 \le m \le n)$ where $\beta_i \in [n]$. When no element is selected, we have the *empty sequence*, also denoted by ϵ. Let $\mathrm{alph}(\beta)$ be the set that consists of all elements in β.

The *standard form* [33, 34] of a sequence $\beta = \beta_1 \cdots \beta_m$ on $[n]$ $(1 \le m \le n)$ is a unique permutation in S_m, denoted by $\mathrm{st}(\beta)$, satisfying that

$$\begin{aligned} \mathrm{st}(\beta)_i < \mathrm{st}(\beta)_j &\Leftrightarrow \beta_i < \beta_j, \\ \mathrm{st}(\beta)_i > \mathrm{st}(\beta)_j &\Leftrightarrow \beta_i > \beta_j, \end{aligned} \tag{2.4}$$

where $1 \le i < j \le n$.

For instance, $\mathrm{st}(4769) = 1324$. And the standard form of the empty sequence is the empty permutation, *i.e.*, $\mathrm{st}(\epsilon) = \epsilon$.

The *coproduct* Δ on KS is defined by

$$\Delta(\alpha) = \sum_{i=0}^{n} \mathrm{st}(\alpha_1 \cdots \alpha_i) \otimes \mathrm{st}(\alpha_{i+1} \cdots \alpha_n), \tag{2.5}$$

for $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ in S_n.

For example,

$$\begin{aligned}\Delta(4123) &= \epsilon\otimes 4123 + \mathrm{st}(4)\otimes\mathrm{st}(123) + \mathrm{st}(41)\otimes\mathrm{st}(23) + \mathrm{st}(412)\otimes\mathrm{st}(3) + 4123\otimes\epsilon\\ &= \epsilon\otimes 4123 + 1\otimes 123 + 21\otimes 12 + 312\otimes 1 + 4123\otimes\epsilon.\end{aligned}$$

The *counit* is defined by ν: $KS \to K$, $\nu(\epsilon) = 1$ and $\nu(\alpha) = 0$ for any $\alpha \in S_n$, $n \geq 1$.

Malvenuto and Reutenauer also defined product $*'$ and coproduct Δ' as follows.

For $\alpha \in S_n$ and $\beta \in S_m$, define the *product* $*'$ by

$$\alpha *' \beta = \sum_{\substack{\mathrm{st}(u)=\alpha,\\ \mathrm{st}(v)=\beta}} uv, \tag{2.6}$$

where the sum is over all sequences u, v satisfying the following conditions:

1) uv is the concatenation of u and v;

2) $\mathrm{alph}(u)\bigcup\mathrm{alph}(v) = [n+m]$;

3) $uv \in S_{n+m}$.

For example, $12 *' 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$.

Here neither product $*$ nor product $*'$ is commutative.

For a permutation $\alpha \in S_n$ and $0 \leq i \leq n$, let $\alpha_{[i]}$ consist of all elements of $[i]$ in the one-line notation of α. Then $\alpha_{[i]}$ is a permutation over $[i]$, which is obtained by removing all digits greater than i in permutation α. For the permutation $\alpha = 3642751$ we have $\alpha_{[5]} = 34251$ and $\alpha_{[0]} = \epsilon$ because $[0] = \emptyset$.

For any permutation $\alpha \in S_n$, define the *coproduct* Δ' by

$$\Delta'(\alpha) = \sum_{i=0}^{n} \alpha_{[i]} \otimes \mathrm{st}(\alpha_{[n]\setminus[i]}). \tag{2.7}$$

For example, $\Delta'(2413) = \epsilon\otimes 2413 + 1\otimes\mathrm{st}(243) + 21\otimes\mathrm{st}(43) + 213\otimes\mathrm{st}(4) + 2413\otimes\epsilon = \epsilon\otimes 2413 + 1\otimes 132 + 21\otimes 21 + 213\otimes 1 + 2413\otimes\epsilon$.

Then $(KS, *, \mu, \Delta, \nu)$ and $(KS, *', \mu, \Delta', \nu)$ are both graded connected bialgebras, and so they are both Hopf algebras. In 2005, Aguiar and Sottile gave a detailed description of their antipodes [30, Theorem 5.4].

If there is a bilinear pairing $<,>$: $H\otimes H^{*gr} \to K$ satisfying the following identities:

$$\begin{gathered} < P\otimes Q, M\otimes N > = < P, M >< Q, N >,\\ < \psi(P\otimes Q), M > = < P\otimes Q, \Delta^*(M) >,\\ < \Delta(P), M\otimes N > = < P, \psi^*(M\otimes N) >,\\ < \mu(1), M > = \nu(M),\\ < P, \mu(1) > = \nu(P),\end{gathered}$$

for any P, $Q \in H$, M, $N \in H^{*gr}$, then H and H^{*gr} are graded dual to each other.

Define a bilinear pairing $<,>$ on $KS\otimes KS$ by

$$< \alpha, \beta > = \begin{cases} 1, \text{ if } \alpha = \beta,\\ 0, \text{ if } \alpha \neq \beta,\end{cases} \tag{2.8}$$

for any permutations α and β.

For $x, y, z \in KS$,

$$< x * y, z > =< x \otimes y, \Delta'(z) >,$$
$$< \Delta(z), x \otimes y > =< z, x *' y > .$$

Thus the graded Hopf algebras $(KS, *, \mu, \Delta, \nu)$ and $(KS, *', \mu, \Delta', \nu)$ are dual to each.

3. Concatenation product and draw coproduct

For a permutation α in S_n in one-line notation, from left to right the positions in front of the first number, between two adjacent numbers and behind the last number indexed by $\{0, \ldots, n\}$ are called the *gaps* of α.

For example, consider permutation 78645312

permutation	$_\bigcirc 7_\bigcirc 8_\bigcirc 6_\bigcirc 4_\bigcirc 5_\bigcirc 3_\bigcirc 1_\bigcirc 2_\bigcirc$
	$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$
gaps	0 1 2 3 4 5 6 7 8

where each $_\bigcirc$ represents a position of 78645312 and corresponds to a gap. Thus, the gaps of 78645312 are 0, 1, 2, 3, 4, 5, 6, 7 and 8.

If a gap γ of α satisfies $\alpha([\gamma]) = [n]\backslash[n-\gamma]$, then γ is a *global descent*, where gaps 0 and n are always are globe descents. An *atom* is a permutation that has no other global descent except 0 and n. For example, permutations 1, 12, 132 and 1423 are all atoms.

Consequently, for α in S_n putting a bullet $\bullet$ at each global descent except 0 and n we get the *factorization* of $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$. Denote the *length* of α by $|\alpha| = s$, which is always less than or equal to its degree n. It is obvious that all $\mathrm{st}(\alpha_i)$'s are atoms. Replacing all α_i's in the factorization of α by their standard form $\mathrm{st}(\alpha_i)$'s, we get the *standard factorization* of α, *i.e.*, $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s = \mathrm{st}(\alpha_1) \bullet \mathrm{st}(\alpha_2) \bullet \cdots \bullet \mathrm{st}(\alpha_s)$. From the standard factorization to the original permutation, we renumber all the numbers in the standard factorization from right to left then eliminate all bullets.

For example, $321 = 3 \bullet 2 \bullet 1 = \mathrm{st}(3) \bullet \mathrm{st}(2) \bullet \mathrm{st}(1) = 1 \bullet 1 \bullet 1$ and $312 = 3 \bullet 12 = \mathrm{st}(3) \bullet \mathrm{st}(12) = 1 \bullet 12$.

From now on, all factorizations are standard.

Define the *shuffle product* $ш_G$ on KS recursively by

$$\alpha\, ш_G\, \epsilon = \alpha = \epsilon\, ш_G\, \alpha$$

and

$$\begin{aligned} \alpha\, ш_G\, \beta &= (\alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s)\, ш_G\, (\beta_1 \bullet \beta_2 \bullet \cdots \bullet \beta_t) \\ &:= \alpha_1 \bullet (\alpha_2 \bullet \cdots \bullet \alpha_s\, ш_G\, \beta) + \beta_1 \bullet (\alpha\, ш_G\, \beta_2 \bullet \cdots \bullet \beta_t), \end{aligned} \tag{3.1}$$

where $\alpha = \alpha_1 \bullet \cdots \bullet \alpha_i \bullet \cdots \bullet \alpha_s \in S_n$ and $\beta = \beta_1 \bullet \cdots \bullet \beta_j \bullet \cdots \bullet \beta_t \in S_m$, α_i $(0 \le i \le s \le n)$ and β_j $(0 \le j \le t \le n)$ are atoms.

Obviously, the shuffle product $ш_G$ is commutative and $\alpha\, ш_G\, \beta \in KS_{n+m}$. Thus, $(KS, ш_G, \mu)$ is a graded algebra.

Example 1.

$$321\, ш_G\, 312$$

$=(1 \bullet 1 \bullet 1)ш_G(1 \bullet 12)$

$=1 \bullet 1 \bullet 1 \bullet 1 \bullet 12 + 1 \bullet 1 \bullet 1 \bullet 1 \bullet 12 + 1 \bullet 1 \bullet 1 \bullet 12 \bullet 1 + 1 \bullet 1 \bullet 1 \bullet 1 \bullet 12 + 1 \bullet 1 \bullet 1 \bullet 12 \bullet 1$

$+ 1 \bullet 1 \bullet 12 \bullet 1 \bullet 1 + 1 \bullet 1 \bullet 1 \bullet 1 \bullet 12 + 1 \bullet 1 \bullet 1 \bullet 12 \bullet 1 + 1 \bullet 1 \bullet 12 \bullet 1 \bullet 1 + 1 \bullet 12 \bullet 1 \bullet 1 \bullet 1$

$= \underbrace{654312 + 654312 + 654312 + 654312}_{4\times 654312} + \underbrace{654231 + 654231 + 654231}_{3\times 654231} + \underbrace{653421 + 653421}_{2\times 653421} + \underbrace{645321}_{1\times 645321}.$

From the example above, if we color the atoms of α blue and the atoms of β red then atoms 1 and 1 are different and we can consider all permutations in the shuffle product distinct even them are same, for example, 654312, 654312, 654312 and 654312 are distinct. In this paper, we color the atoms of different permutations in different colors.

For any permutaion $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$, let I be a subset of $[s]$. Define α_I be a permutation consisting of atoms index by I and keeping their relative positions in the factorization of α. Therefore, $\alpha_\emptyset = \epsilon$ and $\alpha_{[s]} = \alpha$.

Define the *deconcatenation coproduct* Δ on KS by

$$\begin{aligned}\Delta(\alpha) := & \sum_{i=0}^{s}(\alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_i) \otimes (\alpha_{i+1} \bullet \cdots \bullet \alpha_s) \\ = & \sum_{i=0}^{s} \alpha_{[i]} \otimes \alpha_{[s]\setminus[i]},\end{aligned} \tag{3.2}$$

where $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$ is its factorization with length $|\alpha| = s$ and $\Delta(\epsilon) = \epsilon \otimes \epsilon$.

From (3.2), each term in the deconcatenation coproduct of a permutation is putting a tensor sign at one of its global descents.

Example 2.

$$\Delta(321) = \Delta(1 \bullet 1 \bullet 1) = \epsilon \otimes 321 + 1 \otimes 21 + 21 \otimes 1 + 321 \otimes \epsilon,$$

$$\Delta(312) = \Delta(1 \bullet 12) = \epsilon \otimes 312 + 1 \otimes 12 + 312 \otimes \epsilon.$$

The following lemma was first mentioned in [31] without a proof. Zhao and Li proved it and the result is published in a journal in Chinese [32]. Here we give a sketch of the proof.

Lemma 1. *The shuffle product* $ш_G$ *and the deconcatenation coproduct* Δ *are compatible, i.e.,*

$$\Delta(\alpha ш_G \beta) = \Delta(\alpha) ш_G \Delta(\beta), \tag{3.3}$$

where α *and* β *are permutations and* $\Delta(\alpha ш_G \beta) = \sum(\alpha_{[i]} ш_G \beta_{[j]}) \otimes (\alpha_{[s]\setminus[i]} ш_G \beta_{[t]\setminus[j]})$*, if* $\Delta(\alpha) = \sum \alpha_{[i]} \otimes \alpha_{[s]\setminus[i]}$ *and* $\Delta(\beta) = \sum \beta_{[j]} \otimes \alpha_{[t]\setminus[j]}$.

Proof. Let $\alpha = \alpha_1 \bullet \cdots \bullet \alpha_s$ and $\beta = \beta_1 \bullet \cdots \bullet \beta_t$. Denote $\Delta(\alpha ш_G \beta)|_{j=k}$ to be the sum of terms in $\Delta(\alpha ш_G \beta)$ with the number of atoms of β on the left side of the tensor sign is k, $0 \leq k \leq t$.

Then

$$
\begin{aligned}
\Delta(\alpha ш_G \beta)|_{j=k} &= \beta_1 \bullet \cdots \bullet \beta_k \otimes (\alpha ш_G \beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&\quad + (\alpha_1 ш_G \beta_1 \bullet \cdots \bullet \beta_k) \otimes (\alpha_2 \bullet \cdots \bullet \alpha_s ш_G \beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&\quad + (\alpha_1 \bullet \alpha_2 ш_G \beta_1 \bullet \cdots \bullet \beta_k) \otimes (\alpha_3 \bullet \cdots \bullet \alpha_s ш_G \beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&\quad + \cdots + \\
&\quad + (\alpha ш_G \beta_1 \bullet \cdots \bullet \beta_k) \otimes (\beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&= \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i ш_G \beta_1 \bullet \cdots \bullet \beta_k) \otimes (\alpha_{i+1} \bullet \cdots \bullet \alpha_s ш_G \beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&= \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G (\beta_1 \bullet \cdots \bullet \beta_k \otimes \beta_{k+1} \bullet \cdots \bullet \beta_t).
\end{aligned}
\tag{3.4}
$$

Hence

$$
\begin{aligned}
\Delta(\alpha ш_G \beta) &= \Delta(\alpha ш_G \beta)|_{j=0} + \Delta(\alpha ш_G \beta)|_{j=1} + \cdots + \Delta(\alpha ш_G \beta)|_{j=k} + \cdots + \Delta(\alpha ш_G \beta)|_{j=t} \\
&= \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G (\epsilon \otimes \beta) \\
&\quad + \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G (\beta_1 \otimes \beta_2 \bullet \cdots \bullet \beta_t) \\
&\quad + \cdots + \\
&\quad + \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G (\beta_1 \bullet \cdots \bullet \beta_k \otimes \beta_{k+1} \bullet \cdots \bullet \beta_t) \\
&\quad + \cdots + \\
&\quad + \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G (\beta \otimes \epsilon) \\
&= \sum_{i=0}^{s} (\alpha_1 \bullet \cdots \bullet \alpha_i \otimes \alpha_{i+1} \bullet \cdots \bullet \alpha_s) ш_G \sum_{j=0}^{t} (\beta_1 \bullet \cdots \bullet \beta_j \otimes \beta_{j+1} \bullet \cdots \bullet \beta_t) \\
&= \Delta(\alpha) ш_G \Delta(\beta).
\end{aligned}
\tag{3.5}
$$

□

From Lemma 1 the deconcatenation coproduct Δ is an algebra homomorphism, thus $(KS, ш_G, \mu, \Delta, \nu)$ is a bialgebra, where the unit μ and counit ν are the same in Section 2. A graded connected bialgebra is a Hopf algebra so the bialgebra $(KS, ш_G, \mu, \Delta, \nu)$ is a Hopf algebra.

Next, we would introduce another product and coproduct on KS.

Definition 1. *Defined the* concatenation product $\diamond$ *on KS by*

$$\alpha \diamond \beta = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s \bullet \beta_1 \bullet \beta_2 \bullet \cdots \bullet \beta_t, \tag{3.6}$$

where $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$, $\beta = \beta_1 \bullet \beta_2 \bullet \cdots \bullet \beta_t$ are their standard factorizations, and $\alpha \diamond \epsilon = \alpha = \epsilon \diamond \alpha$.

The renumbering of the product is still from right to left and we notice that the concatenation product $\diamond$ does not satisfy commutativity.

Example 3. *Take* $\alpha = 312 = 1 \bullet 12 \in S_3, \beta = 4213 = 1 \bullet 213 \in S_4$, *then*

$$312 \diamond 4213 = (1 \bullet 12) \diamond (1 \bullet 213) = 1 \bullet 12 \bullet 1 \bullet 213 = 7564213.$$

It is easy to check that μ is also a unit for the concatenation product $\diamond$.

In $(KS, \diamond, \mu)$, the product of any two permutations of degrees p and q is a permutation of degree $p+q$. So $(KS, \diamond, \mu)$ is a graded algebra.

Definition 2. *The* draw coproduct Δ^* *on* KS *is defined by*

$$\Delta^*(\alpha) = \sum_{I \subseteq [s]} \alpha_I \otimes \alpha_{[s] \setminus I}, \tag{3.7}$$

for $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$, *where the sum is over all subsets* I *in* $[s]$.

Example 4. *Take* $\alpha = 564312 \in S_6$. *It has global descents at positions 0, 2, 3, 4 and 6. Then its factorization is* $\alpha = 56 \bullet 4 \bullet 3 \bullet 12 = 12 \bullet 1 \bullet 1 \bullet 12$. *Thus,*

$$\begin{aligned}
\Delta^*(\alpha) &= \Delta^*(12 \bullet 1 \bullet 1 \bullet 12) \\
&= \epsilon \otimes 12 \bullet 1 \bullet 1 \bullet 12 + 12 \otimes 1 \bullet 1 \bullet 12 + 1 \otimes 12 \bullet 1 \bullet 12 + 1 \otimes 12 \bullet 1 \bullet 12 + 12 \otimes 12 \bullet 1 \bullet 1 \\
&\quad + 12 \bullet 1 \otimes 1 \bullet 12 + 12 \bullet 1 \otimes 1 \bullet 12 + 12 \bullet 12 \otimes 1 \bullet 1 + 1 \bullet 1 \otimes 12 \bullet 12 + 1 \bullet 12 \otimes 12 \bullet 1 \\
&\quad + 1 \bullet 12 \otimes 12 \bullet 1 + 12 \bullet 1 \bullet 1 \otimes 12 + 12 \bullet 1 \bullet 12 \otimes 1 + 12 \bullet 1 \bullet 12 \otimes 1 + 1 \bullet 1 \bullet 12 \otimes 12 \\
&\quad + 12 \bullet 1 \bullet 1 \bullet 12 \otimes \epsilon \\
&= \epsilon \otimes 564312 + 12 \otimes 4312 + 1 \otimes 45312 + 1 \otimes 45312 + 12 \otimes 3421 + 231 \otimes 312 + 231 \otimes 312 \\
&\quad + 3412 \otimes 21 + 21 \otimes 3412 + 312 \otimes 231 + 312 \otimes 231 + 3421 \otimes 12 + 45312 \otimes 1 + 45312 \otimes 1 \\
&\quad + 4312 \otimes 12 + 564312 \otimes \epsilon \\
&= \epsilon \otimes 564312 + 12 \otimes 4312 + 2 \times 1 \otimes 45312 + 12 \otimes 3421 + 2 \times 231 \otimes 312 \\
&\quad + 3412 \otimes 21 + 21 \otimes 3412 + 2 \times 312 \otimes 231 + 3421 \otimes 12 + 2 \times 45312 \otimes 1 \\
&\quad + 4312 \otimes 12 + 564312 \otimes \epsilon.
\end{aligned}$$

Since $\Delta^*(KS_n) \subseteq \bigoplus_{k=0}^{n}(KS_k \otimes KS_{n-k})$ and ν is also a counit for the draw coproduct Δ^*, (KS, Δ^*, ν) is a graded coalgebra. We will prove that $(KS, \diamond, \mu, \Delta^*, \nu)$ is a graded bialgebra in Section 4.

4. A Hopf algebra on permutations

From Section 3, we have a graded algebra $(KS, \diamond, \mu)$ and a graded coalgebra (KS, Δ^*, ν). In this section, we would prove $(KS, \diamond, \mu, \Delta^*, \nu)$ is a Hopf algebra and give a closed-formula of its antipode.

Theorem 1. $(KS, \diamond, \mu, \Delta^*, \nu)$ *is a bialgebra.*

To prove this we need to prove the following properties:

(1) $\nu(\mu(1)) = 1$,

(2) $\nu(\alpha\beta) = \nu(\alpha)\nu(\beta)$, for any α, β in KS,

(3) $\Delta^*(\mu(1)) = \mu(1) \otimes \mu(1)$,

(4) $\Delta^*(\alpha \diamond \beta) = \sum(\alpha_i \diamond \beta_j) \otimes (\alpha'_i \diamond \beta'_j)$, if $\Delta^*(\alpha) = \sum \alpha_i \otimes \alpha'_i$ and $\Delta^*(\beta) = \sum \beta_j \otimes \beta'_j$, for any α, β in KS.

Conditions (1)–(3) are obvious, so here we only prove Condition (4) which is equivalent to

$$\Delta^*(\alpha \diamond \beta) = \Delta^*(\alpha) \diamond \Delta^*(\beta), \tag{4.1}$$

for any two permutations α and β, *i.e.,* the coproduct Δ^* is an algebra homomorphism.

Proof. Let $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s$ and $\beta = \beta_1 \bullet \beta_2 \bullet \cdots \bullet \beta_t$ in standard factorization form. Then $\alpha \diamond \beta = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_s \bullet \beta_1 \bullet \beta_2 \bullet \cdots \bullet \beta_t$.

Denote $\Delta^*(\alpha \diamond \beta)|_{y=k}$ to be the sum of terms in $\Delta^*(\alpha \diamond \beta)$ with the number of atoms of β on the left side of the tensor sign is k, $0 \le k \le t$.

Therefore,

$$\begin{aligned}
&\Delta^*(\alpha \diamond \beta)|_{y=0} \\
=&\sum_{X \subseteq [s]} \alpha_X \otimes \alpha_{[s]\setminus X} \bullet \beta \\
=&(\sum_{X \subseteq [s]} \alpha_X \otimes \alpha_{[s]\setminus X}) \diamond (\epsilon \otimes \beta) \\
=&\Delta^*(\alpha) \diamond (\epsilon \otimes \beta),
\end{aligned} \tag{4.2}$$

where X is a subset of $[s]$ and the sum is over all subsets X of $[s]$.

And

$$\begin{aligned}
&\Delta^*(\alpha \diamond \beta)|_{y=1} \\
=&\sum_{X \subseteq [s]} \sum_{j=1}^{t} \alpha_X \bullet \beta_j \otimes \alpha_{[s]\setminus X} \bullet \beta_{[t]\setminus\{j\}} \\
=&(\sum_{X \subseteq [s]} \alpha_X \otimes \alpha_{[s]\setminus X}) \diamond \sum_{j=1}^{t} (\beta_j \otimes \beta_{[t]\setminus\{j\}}).
\end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned}
&\Delta^*(\alpha \diamond \beta)|_{y=k} \\
=&\sum_{X \subseteq [s]} \sum_{Y \subseteq [t], |Y|=k} \alpha_X \bullet \beta_Y \otimes \alpha_{[s]\setminus X} \bullet \beta_{[t]\setminus Y} \\
=&(\sum_{X \subseteq [s]} \alpha_X \otimes \alpha_{[s]\setminus X}) \diamond \sum_{Y \subseteq [t], |Y|=k} (\beta_Y \otimes \beta_{[t]\setminus Y}).
\end{aligned} \tag{4.4}$$

Then

$$\begin{aligned}
\Delta^*(\alpha \diamond \beta) &= \Delta^*(\alpha \diamond \beta)|_{y=0} + \Delta^*(\alpha \diamond \beta)|_{y=1} + \cdots + \Delta^*(\alpha \diamond \beta)|_{y=k} + \cdots + \Delta^*(\alpha \diamond \beta)|_{y=t} \\
&= (\sum_{X \subseteq [s]} \alpha_X \otimes \alpha_{[s]\setminus X}) \diamond (\sum_{Y \subseteq [t]} \beta_Y \otimes \beta_{[t]\setminus Y}) \\
&= \Delta^*(\alpha) \diamond \Delta^*(\beta),
\end{aligned} \tag{4.5}$$

where the sums are over all subsets X of $[s]$ and Y of $[t]$ respectively. □

Corollary 1. *The bialgebra $(KS, \diamond, \mu, \Delta^*, \nu)$ is a Hopf algebra.*

Proof. The bialgebra $(KS, \diamond, \mu, \Delta^*, \nu)$ is graded and connected, so it is a Hopf algebra. □

Because the bialgebra $(KS, \diamond, \mu, \Delta^*, \nu)$ is a Hopf algebra, it possesses an antipode θ. The antipode of $(KS, \diamond, \mu, \Delta^*, \nu)$ is given by the following proposition.

Proposition 1. *The antipode θ of Hopf algebra $(KS, \diamond, \mu, \Delta^*, \nu)$ is given by*

$$\theta(\alpha) = (-1)^t(\alpha_t \bullet \alpha_{t-1} \bullet \cdots \bullet \alpha_2 \bullet \alpha_1), \tag{4.6}$$

for any permutation $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_{t-1} \bullet \alpha_t$ and $\theta(\epsilon) = \epsilon$.

Proof. According to the definition of the antipode, we are supposed to verify that all elements of the Hopf algebra $(KS, \diamond, \mu, \Delta^*, \nu)$ are all satisfy Eq (4.7).

Since the both sides of the equal signs in Eq (4.7) are symmetric and the coproduct Δ^* is cocommutative, we just need to prove one side of the formula

$$\diamond \circ (\mathrm{id} \otimes \theta) \circ \Delta^* = \mu \circ \nu. \tag{4.7}$$

First, for a permutation α and $|\alpha| = t$, then

$$\Delta^*(\alpha) = \sum_{I \subseteq [t]} \alpha_I \otimes \alpha_{[t]\backslash I}. \tag{4.8}$$

where the sum is over all subsets I of $[t]$.

It is easy to check that $\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\epsilon) = \epsilon = \nu \circ \mu(\epsilon)$.

Assume $\alpha \neq \epsilon$. From the definition of coproduct Δ^*, we have

$$\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha) = \sum_{I \subseteq [t]} \theta(\alpha_I) \bullet \alpha_{[t]\backslash I}. \tag{4.9}$$

Denote

$$\begin{aligned} \Delta^*(\alpha)|_{x=k} &= \sum_{I_k \subseteq [t]} (\alpha_{I_k}) \otimes \alpha_{[t]\backslash I_k} \\ &= \sum_{I_{k-1} \subseteq [t]\backslash\{1\}} \alpha_1 \bullet (\alpha_{I_{k-1}}) \otimes \alpha_{[t]\backslash (I_{k-1} \cup \{1\})} \\ &\quad + \sum_{I_k \subseteq [t]\backslash\{1\}} (\alpha_{I_k}) \otimes \alpha_1 \bullet \alpha_{[t]\backslash (I_k \cup \{1\})}, \end{aligned} \tag{4.10}$$

where I_k is a subset of $[t]$ consist of k elements of $[t]$ and α_{I_k} are permutations consisting of all atoms indexed by I_k.

For convenience, we denote $(\alpha)^{-1} = \alpha_t \bullet \alpha_{t-1} \bullet \cdots \bullet \alpha_2 \bullet \alpha_1$ for $\alpha = \alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_{t-1} \bullet \alpha_t$. Obviously, $(\epsilon)^{-1} = \epsilon$. Then

$$\begin{aligned} (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k} &= (-1)^k \sum_{I_k \subseteq [t]} (\alpha_{I_k})^{-1} \otimes \alpha_{[t]\backslash I_k} \\ &= (-1)^k \sum_{I_{k-1} \subseteq [t]\backslash\{1\}} (\alpha_{I_{k-1}})^{-1} \bullet \alpha_1 \otimes \alpha_{[t]\backslash (I_{k-1} \cup \{1\})} \\ &\quad + (-1)^k \sum_{I_k \subseteq [t]\backslash\{1\}} (\alpha_{I_k})^{-1} \otimes \alpha_1 \bullet \alpha_{[t]\backslash (I_k \cup \{1\})}. \end{aligned} \tag{4.11}$$

Hence

$$\begin{aligned}\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k} &= (-1)^k \sum_{I_k \subseteq [t]} (\alpha_{I_k})^{-1} \bullet \alpha_{[t]\setminus I_k} \\ &= (-1)^k \sum_{I_{k-1} \subseteq [t]\setminus\{1\}} (\alpha_{I_{k-1}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_{k-1}\cup\{1\})} \\ &\quad + (-1)^k \sum_{I_k \subseteq [t]\setminus\{1\}} (\alpha_{I_k})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_k\cup\{1\})}.\end{aligned} \tag{4.12}$$

From Eq (4.12), when $x = k - 1$, we have

$$\begin{aligned}\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k-1} &= (-1)^{k-1} \sum_{I_{k-1} \subseteq [t]} (\alpha_{I_{k-1}})^{-1} \bullet \alpha_{[t]\setminus I_{k-1}} \\ &= (-1)^{k-1} \sum_{I_{k-2} \subseteq [t]\setminus\{1\}} (\alpha_{I_{k-2}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_{k-2}\cup\{1\})} \\ &\quad + (-1)^{k-1} \sum_{I_{k-1} \subseteq [t]\setminus\{1\}} (\alpha_{I_{k-1}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_{k-1}\cup\{1\})}.\end{aligned} \tag{4.13}$$

When $x = k + 1$, we have

$$\begin{aligned}\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k+1} &= (-1)^{k+1} \sum_{I_{k+1} \subseteq [t]} (\alpha_{I_{k+1}})^{-1} \bullet \alpha_{[t]\setminus I_{k+1}} \\ &= (-1)^{k+1} \sum_{I_k \subseteq [t]\setminus\{1\}} (\alpha_{I_k})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_k\cup\{1\})} \\ &\quad + (-1)^{k+1} \sum_{I_{k+1} \subseteq [t]\setminus\{1\}} (\alpha_{I_{k+1}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_{k+1}\cup\{1\})}.\end{aligned} \tag{4.14}$$

It is easy to see that $\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k}$ would be cancelled by part of $\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k-1}$ and part of $\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k+1}$.

When $x = 0$, we have

$$\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=0} = \alpha_1 \bullet \cdots \bullet \alpha_t. \tag{4.15}$$

When $x = 1$, we have

$$\begin{aligned}\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=1} &= (-1) \sum_{i=1}^{t} \alpha_i \bullet \alpha_1 \bullet \cdots \bullet \alpha_{i-1} \bullet \alpha_{i+1} \cdots \bullet \alpha_t \\ &= (-1)\alpha_1 \bullet \cdots \bullet \alpha_t \\ &\quad + (-1) \sum_{I_1 \subseteq [t]\setminus\{1\}} (\alpha_{I_1})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_1\cup\{1\})}.\end{aligned} \tag{4.16}$$

When $x = t - 1$, we have

$$\begin{aligned}\diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=t-1} &= (-1)^{t-1} \sum_{I_{t-1} \subseteq [t]} (\alpha_{I_{t-1}})^{-1} \bullet \alpha_{[t]\setminus I_{t-1}} \\ &= (-1)^{t-1} \sum_{I_{t-2} \subseteq [t]\setminus\{1\}} (\alpha_{I_{t-2}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\setminus(I_{t-2}\cup\{1\})} \\ &\quad + (-1)^{t-1} (\alpha_t \bullet \cdots \bullet \alpha_2) \bullet \alpha_1.\end{aligned} \tag{4.17}$$

When $x = t$, we have

$$\diamond \circ(\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=t} = (-1)^t \alpha_t \bullet \cdots \bullet \alpha_1. \tag{4.18}$$

From the results above (4.12–4.18)

$$\begin{aligned}
&\diamond \circ(\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha) \\
=& \diamond \circ(\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=0} + \cdots + \diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=k} + \cdots + \diamond \circ (\theta \otimes \mathrm{id}) \circ \Delta^*(\alpha)|_{x=t} \\
=& \alpha_1 \bullet \cdots \bullet \alpha_t + (-1)\alpha_1 \bullet \cdots \bullet \alpha_t + (-1) \sum_{I_1 \subseteq [t]\backslash\{1\}} (\alpha_{I_1})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\backslash(I_1 \cup \{1\})} \\
&+ \cdots + \\
&+ (-1)^{k-1} \sum_{I_{k-2} \subseteq [t]\backslash\{1\}} (\alpha_{I_{k-2}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\backslash(I_{k-2} \cup \{1\})} \\
&+ (-1)^{k-1} \sum_{I_{k-1} \subseteq [t]\backslash\{1\}} (\alpha_{I_{k-1}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\backslash(I_{k-1} \cup \{1\})} \\
&+ \cdots + \\
&+ (-1)^{t-1} \sum_{I_{t-2} \subseteq [t]\backslash\{1\}} (\alpha_{I_{t-2}})^{-1} \bullet \alpha_1 \bullet \alpha_{[t]\backslash(I_{t-2} \cup \{1\})} \\
&+ (-1)^{t-1}(\alpha_t \bullet \cdots \bullet \alpha_2) \bullet \alpha_1 + (-1)^t \alpha_t \bullet \cdots \bullet \alpha_1 \\
=& 0.
\end{aligned} \tag{4.19}$$

□

We notice that the closed-formula of the antipode is similar to the antipode in [7, Proposition 1.10], because for a graded connected Hopf algebra we always can compute its antipode recursively.

5. Duality of the shuffle Hopf algebra

A *sub-permutation* of a permutation α is a subsequence composed of some atoms of α with the same relative positions.

We can see the sub-permutations of any permutation from its *contact map*. Let $\alpha_1 = 1, \alpha_2 = 12, \alpha_3 = 123$ and $\alpha = \alpha_1 \bullet \alpha_2 \bullet \alpha_3 \bullet \alpha_1 \bullet \alpha_1 \bullet \alpha_3 \bullet \alpha_2 \bullet \alpha_1 \in S_{14}$. See Figure 1 for an illustration. In order to see the sub-permutations of a permutation clearly, we omit bullets $\bullet$ of the standardization factorization in the figures.

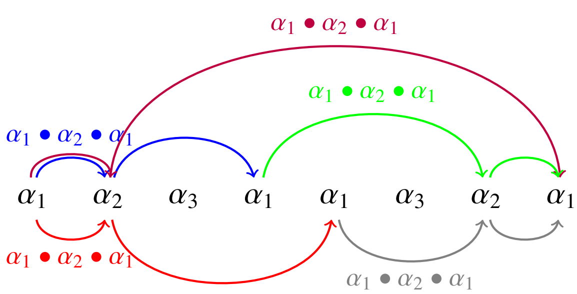
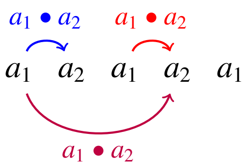


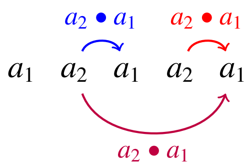
Figure 1. The contact map of sub-permutation 4•23•1 (or $\alpha_1 \bullet \alpha_2 \bullet \alpha_1$) of α.

Given two permutations α, β, the number of sub-permutations equal to β in α is called the *binomial coefficient* [15, 27] of β in α, denoted by $\binom{\alpha}{\beta}$. Notice that $\binom{\alpha}{\epsilon} = 1$ and $\binom{\alpha}{\alpha} = 1$.

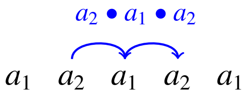
Example 5. *(1) Permutation* $7564231 = \mathrm{st}(7) \bullet \mathrm{st}(56) \bullet \mathrm{st}(4) \bullet \mathrm{st}(23) \bullet \mathrm{st}(1) = 1 \bullet 12 \bullet 1 \bullet 12 \bullet 1$. *Let* $a_1 = 1$, $a_2 = 12$, *then* $7564231 = a_1 \bullet a_2 \bullet a_1 \bullet a_2 \bullet a_1$.



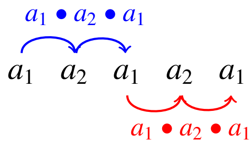
$\binom{7564231}{312}$=*3*



$\binom{7564231}{231}$=*3*



$\binom{7564231}{45312}$=*1*



$\binom{7564231}{4231}$=*2.*

(2) The binomial coefficients also work on subwords of a word. For example,

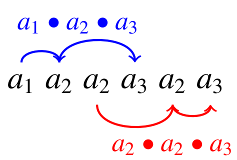
$$\binom{ababa}{aba} = 4, \binom{abbab}{bab} = 2.$$

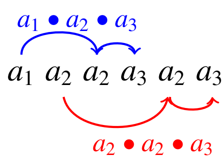
From the above examples, for any permutations α and β, $\binom{\alpha}{\beta}$ is also a useful tool for determining whether β is a sub-permutation of α. If β is not sub-permutation of α, then $\binom{\alpha}{\beta} = 0$.

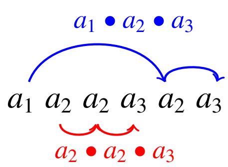
For a sub-permutation α of ω, the remaining atoms of ω compose another sub-permutation β (keeping the relative positions), denoted by $\beta = \omega \backslash \alpha$. Obviously, $|\alpha| + |\beta| = |\omega|$.

Thus, we define that $\left[\begin{matrix}\omega \\ \alpha, \beta\end{matrix}\right]$ is the number of ways to chose α as a sub-permutation in ω with $\beta = \omega \backslash \alpha$. Then $\left[\begin{matrix}\omega \\ \epsilon, \omega\end{matrix}\right] = \left[\begin{matrix}\omega \\ \omega, \epsilon\end{matrix}\right] = 1$. Therefore, if $\omega \backslash \alpha \neq \beta$ or $|\alpha| + |\beta| \neq |\omega|$ then $\left[\begin{matrix}\omega \\ \alpha, \beta\end{matrix}\right] = 0$.

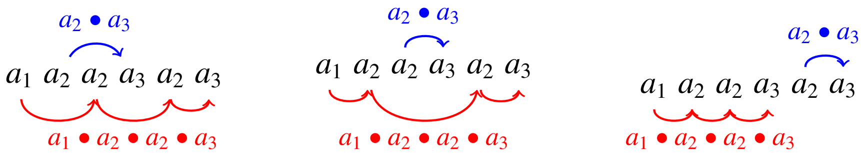
Example 6. *Let a permutation* $\omega = a_1 \bullet a_2 \bullet a_2 \bullet a_3 \bullet a_2 \bullet a_3$.







$$\left[\begin{matrix} a_1 \bullet a_2 \bullet a_2 \bullet a_3 \bullet a_2 \bullet a_3 \\ a_1 \bullet a_2 \bullet a_3, \quad a_2 \bullet a_2 \bullet a_3 \end{matrix}\right] = 3.$$



$$\left[\begin{array}{c} a_1 \bullet a_2 \bullet a_2 \bullet a_3 \bullet a_2 \bullet a_3 \\ a_2 \bullet a_3, \quad a_1 \bullet a_2 \bullet a_2 \bullet a_3 \end{array}\right] = 3.$$

Hence, the draw coproduct Δ^* can be written as follows

$$\Delta^*(\omega) = \sum_{\substack{|\alpha|+|\beta|=|\omega| \\ \omega\backslash\alpha=\beta}} \left[\begin{array}{c} \omega \\ \alpha,\beta \end{array}\right] \alpha \otimes \beta. \tag{5.1}$$

Here $\left[\begin{array}{c} \omega \\ \alpha,\beta \end{array}\right]$ is the coefficient of $\alpha \otimes \beta$ in $\Delta^*(\omega)$.

Example 7. *We also can define draw coproduct Δ^* on words by binomial coefficients. For example,*

$$\begin{aligned}
\Delta^*(abba) = & \left[\begin{array}{c} abba \\ \epsilon, abba \end{array}\right] \epsilon \otimes abba + \left[\begin{array}{c} abba \\ a, bba \end{array}\right] a \otimes bba + \left[\begin{array}{c} abba \\ b, aba \end{array}\right] b \otimes aba + \left[\begin{array}{c} abba \\ a, abb \end{array}\right] a \otimes abb \\
& + \left[\begin{array}{c} abba \\ ab, ba \end{array}\right] ab \otimes ba + \left[\begin{array}{c} abba \\ aa, bb \end{array}\right] aa \otimes bb + \left[\begin{array}{c} abba \\ abb, a \end{array}\right] abb \otimes a + \left[\begin{array}{c} abba \\ aba, b \end{array}\right] aba \otimes b \\
& + \left[\begin{array}{c} abba \\ bba, a \end{array}\right] bba \otimes a + \left[\begin{array}{c} abba \\ abba, \epsilon \end{array}\right] abba \otimes \epsilon \\
= & \ \epsilon \otimes abba + a \otimes bba + 2b \otimes aba + a \otimes abb + 2ab \otimes ba + aa \otimes bb + abb \otimes a \\
& + 2aba \otimes b + bba \otimes a + abba \otimes \epsilon.
\end{aligned}$$

From the definition of $\left[\begin{array}{c} \omega \\ \alpha,\beta \end{array}\right]$, we can think it as a top-down calculation, that is, it counts the number of ways that the upper permutation ω can be divided into the lower two sub-permutations α and β with $\beta = \omega\backslash\alpha$. We can also think it as a bottom-up calculation, that is, it counts the number of ways that the lower permutations α and β compose the upper permutation ω with with $\beta = \omega\backslash\alpha$.

Therefore, the shuffle product $ш_G$ can be written as

$$\alpha\, ш_G\, \beta = \sum_{\substack{|\alpha|+|\beta|=|\omega| \\ \omega\backslash\alpha=\beta}} \left[\begin{array}{c} \omega \\ \alpha,\beta \end{array}\right] \omega, \tag{5.2}$$

for any permutations α and β in S.

Similarly, for a subword f of η, the remaining letters of η compose another subword g (keeping the relative positions), denoted by $g = \eta\backslash f$. Thus, we define that $\left[\begin{array}{c} \eta \\ f,g \end{array}\right]$ is the number of ways to chose f as a subword in η with $g = \eta\backslash f$.

Then, Eq (2.2) can be rewritten as

$$f\, ш\, g = \sum_{\substack{|f|+|g|=|\eta| \\ \eta\backslash f=g}} \left[\begin{array}{c} \eta \\ f,g \end{array}\right] \eta. \tag{5.3}$$

Theorem 2. *Hopf algebras* $(KS, ш_G, \mu, \Delta, \nu)$ *and* $(KS, \diamond, \mu, \Delta^*, \nu)$ *are dual to each.*

Proof. According to the definition of dual Hopf algebras, we first prove the three conditions:

1) $< P \otimes Q, M \otimes N >=< P, M >< Q, N >$,

2) $< \mu(1), M >= \nu(M)$,

3) $< P, \mu(1) >= \nu(P)$.

Let P, $Q \in (KS, ш_G, \mu, \Delta, \nu)$, M, $N \in (KS, \diamond, \mu, \Delta^*, \nu)$. If $P = M$, $Q = N$, then $< P \otimes Q, M \otimes N >=< P, M >< Q, N >= 1$; if $P \neq Q$ or $Q \neq N$, then $< P \otimes Q, M \otimes N >=< P, M >< Q, N >= 0$.

If $M = \epsilon$, then $< \mu(1), \epsilon >=< \epsilon, \epsilon >= 1 = \nu(\epsilon)$; if $M \neq \epsilon$, then $< \epsilon, M >= 0 = \nu(M)$. Similarly, $< P, \mu(1) >= \nu(P)$.

Then we prove Conditions 4) and 5):

4) $< ш_G(P \otimes Q), M >=< P \otimes Q, \Delta^*(M) >$,

5) $< \Delta(P), M \otimes N >=< P, \diamond(M \otimes N) >$.

By Eqs (5.1) and (5.2), we have

$$< ш_G(P \otimes Q), M >=< \sum_{\substack{|P|+|Q|=|W| \\ W \backslash P=Q}} \begin{bmatrix} W \\ P, Q \end{bmatrix} W, M >, \tag{5.4}$$

$$< P \otimes Q, \Delta^*(M) >=< P \otimes Q, \sum_{\substack{|\alpha|+|\beta|=|M| \\ M \backslash \alpha=\beta}} \begin{bmatrix} M \\ \alpha, \beta \end{bmatrix} \alpha \otimes \beta > . \tag{5.5}$$

When $M = W, \alpha = P, \beta = Q$, then

$$< ш_G(P \otimes Q), M >= \begin{bmatrix} M \\ P, Q \end{bmatrix} < M, M >, \tag{5.6}$$

$$< P \otimes Q, \Delta^*(M) >= \begin{bmatrix} M \\ P, Q \end{bmatrix} < P \otimes Q, P \otimes Q > . \tag{5.7}$$

Thus Condition 4) is proved.

From Eq (3.2) and Example 2 we can notice that in the result of $\Delta(P)$ each term is different. That is to say the coefficient of each term of $\Delta(P)$ all is 1. Thus, when $P = M \bullet N$, we have $< \Delta(P), M \otimes N >= 1$; meanwhile, since $M \diamond N = P$, so $< P, \diamond(M \otimes N) >= 1$. That proves Condition 5). □

Here, we verified Conditions 4) and 5) by the example following.

Example 8. *For permutations* $P = 231$*,* $Q = 231$*,* $M = 564231$ *, then*

$$\begin{aligned} < ш_G(231 \otimes 231), 564231 > &=< 2 \times 564231 + 4 \times 563421, 564231 > \\ &=< 2 \times 564231, 564231 > + < 4 \times 563421, 564231 > \\ &= 2. \end{aligned}$$

Because

$$\begin{aligned} \Delta^*(564231) = {} & \epsilon \otimes 564231 + 12 \otimes 4231 + 1 \otimes 45231 + 12 \otimes 3421 + 1 \otimes 45312 \\ & + 2 \times (231 \otimes 231) + 3412 \otimes 21 + 231 \otimes 312 + 312 \otimes 231 + 21 \otimes 3412 \\ & + 45312 \otimes 1 + 3421 \otimes 12 + 4231 \otimes 12 + 564231 \otimes \epsilon, \end{aligned}$$

and

$$< 231 \otimes 231, \Delta^*(564231) >=< 231 \otimes 231, 2 \times (231 \otimes 231) >= 2,$$

hence

$$< ш_G(231 \otimes 231), 564231 >=< 231 \otimes 231, \Delta^*(564231) > .$$

And

$$\Delta(564231) = \epsilon \otimes 564231 + 12 \otimes 4231 + 231 \otimes 231 + 45312 \otimes 1 + 564231 \otimes \epsilon,$$

then

$$< \Delta(564231), 231 \otimes 231 > =< 231 \otimes 231, 231 \otimes 231 >,$$
$$< \Delta(564231), 231 \otimes 231 > =< 564231, \diamond(231 \otimes 231) > .$$

6. Conclusions

Let A be a set called an *alphabet* and the elements in it are called *letters*. A finite sequence consisting of several letters is called a *word* over A. Denoted A^* as the set of all such words with the *empty word* ϵ. Thus, if we replace all atoms by letters in A or all standard factorizations by words in A^*, then $(KA^*, ш_G, \mu, \Delta, \nu)$ and $(KA^*, \diamond, \mu, \Delta^*, \nu)$ are also dual graded Hopf algebras.

Furthermore, let $X = \{x_1, x_2, \ldots\}$. The set of polynomials on noncommutative variables X equipped shuffle product $ш_G$ and deconcatenation coprodcut Δ, concatenation product $\diamond$ and draw coprodcut Δ^*, respectively, is a pair of dual graded Hopf algebras. In particular, when $X = \{x\}$, the corresponding dual graded graphs [18] are the chain and weighed chain [35]. When $X = \{x_1, x_2\}$, the corresponding dual graded graphs are the binary tree and the weighed binary tree [35]. Similarly, when $X = \{x_1, x_2, \ldots, x_n\}$, the corresponding dual graded graphs are the n-ary tree and weighted n-ary tree.

Similarly, if we use global ascents [36] instead of global descents of permutations and renumber atoms from left to right, then we obtain dual graded Hopf algebras $(KS, ш_G, \mu, \Delta, \nu)$ and $(KS, \diamond, \mu, \Delta^*, \nu)$.

In this paper we consider a pair of dual graded Hopf algebras. A *graded* Hopf algebra H has a direct sum decomposition $H = \bigoplus_{i=0} H_i$, where the degree of elements in H_i is i, satisfying that the product of two elements of degree p and degree q is an element of degree $p + q$. If a bialgebra has a direct sum decomposition $H = \bigoplus_{i=0} H_i$ but the product of two elements of degree p and degree q is not an element of degree $p + q$, then this biagebra is not graded. If we have a such bialgebra, we can study the following problems:

1. How to verify that a such bialgebra is a Hopf algebra or not?
2. If it is a Hopf algebra, how to figure out its antipode?
3. What is the duality of the bialgebra or Hopf algebra?

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Conflict of interest

The authors declare there is no conflict of interests.

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