Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 5088–5105.
DOI: 10.3934/math.2021301
Received: 03 December 2020
Accepted: 01 February 2021
Published: 09 March 2021

Research article

Ulam stabilities of nonlinear coupled system of fractional differential equations including generalized Caputo fractional derivative

Tamer Nabil[1,2,*]

[1] King Khalid University, College of Science, Department of Mathematics, Abha, Saudi Arabia
[2] Suez Canal University, Faculty of Computers and Informatics, Department of Basic Science, Ismailia, Egypt

* **Correspondence:** Email: t_3bdelsadek@yahoo.com.

Abstract: In this paper, we establish the existence and uniqueness of solution for a nonlinear coupled system of implicit fractional differential equations including ψ-Caputo fractional operator under nonlocal conditions. Schaefer's and Banach fixed–point theorems are applied to obtain the solvability results for the proposed system. Furthermore, we extend the results to investigate several types of Ulam stability for the proposed system by using classical tool of nonlinear analysis. Finally, an example is provided to illustrate the abstract results.

Keywords: fixed-point; ψ-Caputo operator; coupled implicit system; Ulam stability; existence of solution
Mathematics Subject Classification: 47H10, 26A33, 34B27, 39B82

1. Introduction

Fractional calculus is very significant and important part of mathematical analysis. As is well known, fractional calculus plays a very vital role for the study of fractional order systems. Fractional differential equations (FDEs, for short) are utilized in many applications of engineering and natural science topics, such as dynamics of biology, finance, optimal control, automation system, image analysis, fluid and elasticity [1–7].

Different natural problems can be modeled as mathematical problems. The solvability of these mathematical problems is an important aspect of modern mathematics. The fixed-point theory can be applied to investigate the solvability for those natural problems. A large number of works dealing with refinements, applications of fixed-point theory in different types of FDEs have appeared [8–11]. In particular, Schaefer's and Banach fixed-point theorems are effectively used to establish the solvability of a lot of nonlinear natural problems [12].

With the development of powerful mainframe computers and parallel computing methods, the investigation of stability of solutions of FDEs has become important tool to achieve and to confirm the study, which led that stability study to be one of the most important research subjects in functional analysis. Additionally, sometimes it is too difficult to find an exact solution for nonlinear FDEs, in such case stability analysis plays a very important role in the study. The theory of Ulam stability was introduced in 1940 by Ulam [13]. The following year, Hyers [14] established a type of stability in the Banach space which was more generalized than the concept of Ulam stability, and applied this type to achieve the stability conditions of functional equations. After that, Rassias [15] considered another definition of stability condition, that definition was more improved than Ulam and Hyers stabilities, and used this definition to study the stability of FDEs [16, 17].

Recently, more and more researchers are interested in applying the Hyers-Ulam stability [18–22]. In 2017, Ali et al. [23] established the existence of solutions for the coupled of FDEs involving Riemann-Liouville fractional operators and achieved Hyers-Ulam stability (HUS). In the same year, Khan et al. [24] considered the existence of solution, therefore HUS for the coupled system of FDEs in Caputo's sense. Recently, in 2019, Ali et al. [25] greatly given Ulam Stability conditions of solution for nonlinear coupled implicit FDEs.

There are several approaches to define the fractional derivative such as Riemann-Liouville, Hadamard, Erdely Kober and Caputo [26]. Recently, in 2017, Almeida [27] proposed new definition of the fractional derivative and called this operator ψ-Caputo derivative. This new definition is more generalized than Riemann-Liouville, Hadamard, Erdely Kober and Caputo operators types. He given a lot of properties of these new fractional derivative operator. After one year, Almeida et al. [28] studied the uniqueness of solution for the system of FDEs including ψ-Caputo derivative.

Nowadays, Abdo et al. [29] proposed an implicit nonlinear FDE including ψ-Caputo derivative under nonlocal condition. They obtained the existence and uniqueness of solution for this problem under some hypotheses.

In this paper, we investigate the existence and uniqueness of solution for the following couples system of FDEs

$$\begin{cases} {}^*D_{0^+}^{\alpha,\psi}u(t) = f(t, u(t), {}^*D_{0^+}^{\alpha,\psi}v(t)); \\ {}^*D_{0^+}^{\alpha,\psi}v(t) = g(t, v(t), {}^*D_{0^+}^{\alpha,\psi}u(t)); \\ \alpha \in (0,1), t \in I = [0,T], T \in \mathbb{R}^+; \\ \text{subject to the nonlocal conditions} \\ u(0) + h_1(u) = u_0; \\ v(0) + h_2(v) = v_0, \end{cases} \tag{1.1}$$

where $u, v \in C(I,\mathbb{R})$, $f, g : I \times \mathbb{R}^2 \to \mathbb{R}$, $h_1, h_2 : C(I,\mathbb{R}) \to \mathbb{R}$, $u_0, v_0 \in \mathbb{R}$ and ${}^*D^{\alpha,\psi}$ is ψ-Caputo derivative operator. We apply Schaefer's and Banach-fixed-point theorems to obtain the existence and the uniqueness of solution for the proposed system (1.1) under certain hypotheses. Furthermore, we study several kinds of Ulam stability for the proposed system (1.1). The system (1.1) is very general since it is involved fractional derivatives with unlike kernels. Further, it has been shown that these ψ-fractional derivatives unify a wide class of fractional derivatives such as the aforementioned one. On the other hand, the ψ-Caputo operators are found to be more effective and practical than the classical Caputo operators in the mathematical modeling of numerous phenomena and process in dynamics systems [30–35].

2. Preliminaries

In this section, we give some basic concepts and auxiliary results. The following definition was given in [25].

Definition 1. *Let $I = [0, T],\ T > 0$, and $C(I, \mathbb{R})$ be the space of all continuous functions $u : I \to \mathbb{R}$. It is easy to prove $C(I, \mathbb{R})$ is a Banach space with the norm, $\|u\|_C = \max_{t\in I} |u(t)|$. Therefore, $C^n(I, \mathbb{R}),\ n \in \mathbb{N}$, is the space of all $n-$times continuous and differentiable functions from I to $\mathbb{R}$.*

The following definition was given in [26, 27].

Definition 2. *Let $\psi \in C^n(I, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Consider $u : I \to \mathbb{R}$ be an integrable function. The $\psi-$Riemann-Liouville fractional integral of order $\alpha > 0,\ \alpha \in (n-1, n)$ for some $n \in \mathbb{N}$, of the function u is defined as*

$$J_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{\alpha-1}\ u(\zeta)\ d\zeta,$$

and the ψ-Riemann-Liouville fractional derivative of order $\alpha > 0,\ \alpha \in (n-1, n)$ for some $n \in \mathbb{N}$, of the function u is defined as

$$D_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{n-\alpha-1}\ u(\zeta)\ d\zeta.$$

The following definition was given in [27, 28].

Definition 3. *Let $\psi \in C^n(I, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Consider $u : I \to \mathbb{R}$ be an integrable function. The ψ-Caputo fractional derivative of order $\alpha > 0,\ \alpha \in (n-1, n)$ for some $n \in \mathbb{N}$, is defined by ψ-Riemann-Liouville fractional derivative as*

$$^*D_{0^+}^{\alpha,\psi} u(t) = D_{0^+}^{\alpha,\psi} [u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k]\ .$$

where, $u_\psi^{[k]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^k\ u(t)$. We recall the following Lemma which was given in [28].

Lemma 4. *Suppose that $u : I \to \mathbb{R}$, then*

(1) if $u \in C^n(I, \mathbb{R})$, then $^*D_{0^+}^{\alpha,\psi}\ J_{0^+}^{\alpha,\psi} u(t) = u(t)$,

(2) if $u \in C^n(I, \mathbb{R})$, then

$$J_{0^+}^{\alpha,\psi}\ {}^*D_{0^+}^{\alpha,\psi}\ u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k,$$

therefore, if $\alpha \in (0, 1)$, then

$$J_{0^+}^{\alpha,\psi}\ {}^*D_{0^+}^{\alpha,\psi}\ u(t) = u(t) - u(0).$$

Now we recall Schaefer's fixed-point theorem which was given in [36].

Theorem 5. ***(Schaefer's fixed–point theorem)*** *Let X be a Banach space and $T : X \to X$ be a continuous and compact map. Furthermore, let*

$$\Omega = \{x \in X : x = \lambda T x,\ \lambda \in (0, 1)\},$$

be a bounded set. Then T has at least one fixed-point in X.

Let $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_1,\ h_2 : C(I, \mathbb{R}) \to \mathbb{R}$. Then we study the Ulam stabilities of the following proposed problem.

$$\begin{cases} {}^*D^{\alpha,\psi}_{0^+}u(t) = f(t, u(t), {}^* D^{\alpha,\psi}_{0^+}v(t)); \\ {}^*D^{\alpha,\psi}_{0^+}v(t) = g(t, v(t), {}^* D^{\alpha,\psi}_{0^+}u(t)); \\ \alpha \in (0, 1),\ t \in I = [0, T],\ T \in \mathbb{R}^+; \\ \text{subject to the nonlocal conditions} \\ u(0) + h_1(u) = u_0; \\ v(0) + h_2(v) = v_0. \end{cases} \tag{2.1}$$

The following two Lemmas will be utilized throughout in the paper.

Lemma 6. *[29] Let $h \in C(I, \mathbb{R})$, then the FDE*

$$\begin{cases} {}^*D^{\alpha,\psi}_{0^+}u(t) = f(t, u(t), {}^* D^{\alpha,\psi}_{0^+}u(t)); \\ \alpha \in (0, 1),\ t \in I = [0, T],\ T > 0; \\ \text{subject to the nonlocal conditions} \\ u(0) + h(u) = u_0, \end{cases} \tag{2.2}$$

can be written as the following integral equation

$$u(t) = u_0 - h(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ f(t, u(t), {}^* D^{\alpha,\psi}_{0^+}u(t))\ ds.$$

From Lemma 6, we can show the following interesting Lemma.

Lemma 7. *The solution of the system (1.1) is equivalent to the following coupled system of nonlinear integral equations*

$$\begin{array}{l} u(t) = u_0 - h_1(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ f(t, u(t), {}^* D^{\alpha,\psi}_{0^+}v(t))\ ds, \\ \textit{and} \\ v(t) = v_0 - h_2(v) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ g(t, v(t), {}^* D^{\alpha,\psi}_{0^+}u(t))\ ds. \end{array} \tag{2.3}$$

3. Existence and uniqueness

In this partition we will investigate the solvability of the proposed system (1.1). The system (1.1) can be written as

$$\begin{cases} {}^*D^{\alpha,\psi}_{0^+}u(t) = x(t) = f(t, u(t), y(t)); \\ {}^*D^{\alpha,\psi}_{0^+}v(t) = y(t) = g(t, v(t), x(t)); \\ \alpha \in (0, 1),\ t \in I = [0, T],\ T \in \mathbb{R}^+; \\ \text{subject to the nonlocal conditions} \\ u(0) + h_1(u) = u_0; \\ v(0) + h_2(v) = v_0. \end{cases} \tag{3.1}$$

Now, define the following two operators,

$$T_1 : C(I, \mathbb{R}) \to C(I, \mathbb{R}) \quad , \qquad T_2 : C(I, \mathbb{R}) \to C(I, \mathbb{R}),$$

such that

$$\begin{cases} T_1u(t) = u_0 - h_1(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \ f(t, u(t),^* D_{0^+}^{\alpha,\psi} v(t)) \ ds \quad , \\ T_2v(t) = v_0 - h_2(v) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \ g(t, v(t),^* D_{0^+}^{\alpha,\psi} u(t)) \ ds \quad , \end{cases} \tag{3.2}$$

where $t \in I$ and $u, v \in C(I, \mathbb{R})$.

We will study the proposed system under the following conditions:

(C1) $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;

(C2) for all $t \in I$, there exists $L_i \in (0, 1)$, $i = 1, 2, 3, 4$, such that

$$|f(t, w_1, z_1) - f(t, w_2, z_2)| \le L_1|w_1 - w_2| + L_2|z_1 - z_2|,$$

$$|g(t, w_1, z_1) - g(t, w_2, z_2)| \le L_3|w_1 - w_2| + L_4|z_1 - z_2|,$$

$\forall w_1, w_2, z_1, z_2 \in \mathbb{R}$;

(C3) for all $t \in I$, there exists $A_f, B_f, C_f, A_g, B_g, C_g \in C(I, \mathbb{R})$ such that

$$|f(t, u(t), w(t))| \le A_f(t) + B_f(t)|u(t)| + C_f(t)|w(t)|,$$

$$|g(t, u(t), w(t))| \le A_g(t) + B_g(t)|u(t)| + C_g(t)|w(t)|,$$

$\forall u, w \in C(I, \mathbb{R})$, with

$$A_f^* = \sup_{t\in I} A_f(t), \quad B_f^* = \sup_{t\in I} B_f(t), \quad C_f^* = \sup_{t\in I} C_f(t), \quad A_g^* = \sup_{t\in I} A_g(t), \quad B_g^* = \sup_{t\in I} B_g(t), \quad C_g^* = \sup_{t\in I} C_g(t),$$

(C4) $h_1, h_2 : C(I, \mathbb{R}) \to \mathbb{R}$, are continuous and there exists $L_5, L_6 \in (0, 1)$ such that

$$|h_1(u) - h_2(v)| \le L_5|u - v|,$$

$$|h_2(u) - h_3(v)| \le L_6|u - v|,$$

for all $u, v \in C(I, \mathbb{R})$.

Let X be the product space $X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ with the norm

$$\|(u, v)\|_X = \|u(t)\|_C + \|v(t)\|_C,$$

we can prove that X is Banach space. Define $F : X \to X$ such that

$$F(u, v) = (T_1u, T_2v)(t).$$

The existence of solution for the proposed system (1.1) is equivalent the existence of the fixed-point for the operator equation

$$(u, v) = F(u, v) = (T_1u, T_2v)(t).$$

Theorem 8. *Let the conditions (C1)–(C4) hold. Then F is compact.*

Proof. Arzela-Ascoli's theorem will be applied by doing the following steps.

Step 1. F is continuous.

Define $D_r = \{(u,v) \in X : \|(u,v)\|_X \leq r\}$. Consider $\{(u_n, v_n)\}$ be a sequence in X such that $\{(u_n, v_n)\}$ converges to (u,v) in D_r. Hence, $u_n \to u$ and $v_n \to v$. Thus, we have that

$$\begin{aligned} |T_1(u_n)(t) - T_1(u)(t)| &\leq |h_1(u_n) - h_1(u)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |x_n(s) - x(s)| \, ds \quad , \\ |T_2(v_n)(t) - T_2(v)(t)| &\leq |h_2(v_n) - h_2(v)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |y_n(s) - y(s)| \, ds \quad , \end{aligned} \tag{3.3}$$

where $y_n, y, x_n, x \in C(I, \mathbb{R})$ and

$$x_n = f(t, u_n(t), y_n(t)) \quad , \quad y_n = g(t, v_n(t), x_n(t)).$$

From condition (C2), we get that

$$|x_n(t) - x(t)| \leq L_1 |u_n(t) - u(t)| + L_2 |y_n(t) - y(t)| \ , \tag{3.4}$$

$$|y_n(t) - y(t)| \leq L_3 |v_n(t) - v(t)| + L_4 |x_n(t) - x(t)| \ , \tag{3.5}$$

which implies that

$$|x_n(t) - x(t)| \leq L_1 |u_n(t) - u(t)| + L_2 L_3 |v_n(t) - v(t)| + L_2 L_4 |x_n(t) - x(t)|, \tag{3.6}$$

therefore,

$$|x_n(t) - x(t)| \leq \frac{L_1}{1 - L_2 L_4} |u_n(t) - u(t)| + \frac{L_2 L_3}{1 - L_2 L_4} |v_n(t) - v(t)|, \tag{3.7}$$

and

$$|y_n(t) - y(t)| \leq \frac{L_1 L_4}{1 - L_2 L_4} |u_n(t) - u(t)| + (L_3 + \frac{L_2 L_3 L_4}{1 - L_2 L_4}) |v_n(t) - v(t)|. \tag{3.8}$$

Thus, we have that

$$\begin{aligned} |T_1(u_n)(t) - T_1(u)(t)| \leq L_5 |u_n(t) - u(t)| &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &[\frac{L_1}{1 - L_2 L_4} |u_n(s) - u(s)| + \frac{L_2 L_3}{1 - L_2 L_4} |v_n(s) - v(s)|] \, ds \quad , \\ |T_2(v_n)(t) - T_2(v)(t)| \leq L_6 |v_n(t) - v(t)| &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &[\frac{L_1 L_4}{1 - L_2 L_4} |u_n(s) - u(s)| + (L_3 + \frac{L_2 L_3 L_4}{1 - L_2 L_4}) |v_n(s) - v(s)|] \, ds \quad . \end{aligned} \tag{3.9}$$

Since $(u_n, v_n) \to (u, v)$ as $n \to \infty$, for all $t \in I$ and from Lebesgue Dominated Converges theorem [36], we have

$$\|T_1(u_n)(t) - T_1(u)(t)\|_C \to 0 \quad , \quad \|T_2(v_n)(t) - T_2(v)(t)\|_C \to 0,$$

as $n \to \infty$. Also,

$$\|F(u_n, v_n)(t) - F(u, v)(t)\|_X = \|T_1(u_n)(t) - T_1(u)(t)\|_C + \|T_2(v_n)(t) - T_2(v)(t)\|_C.$$

Thus,

$$\|F(u_n, v_n)(t) - F(u, v)(t)\|_X \to 0,$$

as $n \to \infty$. Thus F is continuous operator.

Step 2. ***F*** **is bounded operator in** ***X*****.**

By using (C3), we get

$$|x(t)| \leq A_f(t) + B_f(t)|u(t)| + C_f(t)|y(t)| \leq A_f^* + B_f^*\, r + C_f^*\, |y(t)|,$$

similarly, we find that

$$|y(t)| \leq A_g^* + B_g^*\, r + C_g^*\, |x(t)|,$$

therefore, we have

$$|x(t)| \leq A_f^* + B_f^*\, r + C_f^*\, (A_g^* + B_g^*\, r + C_g^*|x(t)|).$$

It follows that

$$\|x\|_C \leq \frac{A_f^* + B_f^*\, r + C_f^*\, A_g^* + C_f^* B_g^*\, r}{1 - C_f^*\, C_g^*} = M_1.$$

Similarly, we have

$$\|y\|_C \leq A_g^* + B_g^*\, r + C_g^*\, M_1 = M_2.$$

Thus, we have

$$\begin{aligned} |T_1(u)(t)| &\leq |u_0| + |h_1(u)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\, |x(s)|\, ds \\ &\leq |u_0| + |h_1(0)| + L_5 r + \tfrac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha} M_1, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |T_2(v)(t)| &\leq |v_0| + |h_2(v)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\, |y(s)|\, ds \\ &\leq |v_0| + |h_2(0)| + L_6 r + \tfrac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha} M_2\ . \end{aligned} \tag{3.11}$$

Therefore, we have

$$\|T_1 u\|_C \leq |u_0| + |h_1(0)| + L_5 r + \frac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha} M_1 = A_1,$$

$$\|T_2 v\|_C \leq |v_0| + |h_2(0)| + L_6 r + \frac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha} M_2 = A_2.$$

Hence, we have

$$\|F(u, v)(t)\|_X \leq A_1 + A_2 = A.$$

Thus, F is bounded in X .

Step 3. ***F*** **is equicontinuous operator in** ***X*****.**

Let $t_1, t_2 \in I$ such that $t_1 > t_2$. Then, we get

$$\begin{aligned} |T_1(u)(t_1) - T_1(u)(t_2)| &= |\tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} x(s)\, ds - \tfrac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} x(s)\, ds| \\ &\leq \tfrac{1}{\Gamma(\alpha)} | \int_0^{t_1} \psi'(s)[(\psi(t_1) - \psi(s))^{\alpha-1} - \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}] x(s)\, ds| \\ &\quad + \tfrac{1}{\Gamma(\alpha)} | \int_{t_1}^{t_2} \psi'(s)[(\psi(t_1) - \psi(s))^{\alpha-1} x(s) ds| \\ &\leq \tfrac{A_1}{\Gamma(\alpha+1)} [(\psi(t_2) - \psi(0))^{\alpha} - (\psi(t_1) - \psi(0))^{\alpha} - 2(\psi(t_2) - \psi(t_1))^{\alpha}]. \end{aligned} \tag{3.12}$$

Therefore, we obtain that

$$|T_2(v)(t_1) - T_2(v)(t_2)| \leq \tfrac{A_2}{\Gamma(\alpha+1)}[(\psi(t_2) - \psi(0))^{\alpha} - (\psi(t_1) - \psi(0))^{\alpha} - 2(\psi(t_2) - \psi(t_1))^{\alpha}]. \tag{3.13}$$

When $t_1 \to t_2$, we get

$$|T_1(u)(t_1) - T_1(u)(t_2)| \to 0, \tag{3.14}$$

and

$$|T_2(v)(t_1) - T_2(v)(t_2)| \to 0, \tag{3.15}$$

thus, we find that

$$|F(u,v)(t_1) - F(u,v)(t_2)| \to 0. \tag{3.16}$$

Hence, F is equicontinuous. Thus, F is completely continuous. From Arzela-Ascoli's theorem [37], F is compact operator. □

Theorem 9. *Let the conditions (C3) and (C4) hold. The proposed system (1.1) has at least one solution if* $\mathbb{L} < 1$, *where*

$$\mathbb{L} = \frac{2\ell}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha} + \max\{L_5, L_6\},$$

and

$$\ell = \max\{A_f^* + C_f^* A_g^*, \frac{B_f^*}{1 - C_f^* C_g^*}, \frac{C_f^* B_f^*}{1 - C_f^* C_g^*}, (A_g^* + A_f^* C_g^* + C_f^* C_g^* A_g^*), \frac{C_g^*\, B_f^*}{1 - C_f^*\, C_g^*}, \frac{C_f^* B_f^*\, C_g^*}{1 - C_f^*\, C_g^*}\}.$$

Proof. Let $\mathbb{B}$ defined as

$$\mathbb{B} = \{(u,v) \in X : (u,v) = \lambda F(u,v),\ 0 < \lambda < 1\}.$$

We will show that $\mathbb{B}$ is bounded. Let $(u,v) \in \mathbb{B}$ such that

$$u(t) = \lambda T_1 u(t),\ v(t) = \lambda T_2 v(t),\ \lambda \in (0,1).$$

Then for each $t \in I$, we have that

$$\begin{aligned} |u(t)| &= |\lambda(u_0 - h_1(u) + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ f(t, u(t), {}^* D_{0^+}^{\alpha,\psi} v(t))\ ds)| \\ &\leq |u_0| + L_5|u(t)| + |h_1(0)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|x(s)|ds, \\ |v(t)| &= |\lambda(v_0 - h_2(v) + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ g(t, v(t), {}^* D_{0^+}^{\alpha,\psi} u(t))\ ds)| \\ &\leq |v_0| + L_6|v(t)| + |h_2(0)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|y(s)|ds. \end{aligned} \tag{3.17}$$

From (C3), we get

$$|x(t)| \leq A_f^* + B_f^*\ |u(t)| + C_f^*\ |y(t)|\ \ , \tag{3.18}$$

$$|y(t)| \leq A_g^* + B_g^*\ |v(t)| + C_g^*\ |x(t)|\ \ . \tag{3.19}$$

Thus, we have

$$|x(t)| \leq A_f^* + C_f^* A_g^* + \frac{B_f^*}{1 - C_f^* C_g^*}|u(t)| + \frac{C_f^* B_f^*}{1 - C_f^* C_g^*}|v(t)|\ \ , \tag{3.20}$$

$$|y(t)| \leq (A_g^* + A_f^* C_g^* + C_f^* C_g^* A_g^*) + \frac{C_g^* \, B_f^*}{1 - C_f^* \, C_g^*} \, |u(t)| + \frac{C_f^* B_f^* \, C_g^*}{1 - C_f^* \, C_g^*} \, |v(t)| \quad . \tag{3.21}$$

Then, we have that

$$|u(t)| \leq |u_0| + L_5|u(t)| + |h_1(0)| + \tfrac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha}[A_f^* + C_f^* A_g^* + \tfrac{B_f^*}{1-C_f^* C_g^*}|u(t)| + \tfrac{C_f^* B_f^*}{1-C_f^* C_g^*}|v(t)|], \tag{3.22}$$

and

$$|v(t)| \leq |v_0| + L_6|v(t)| + |h_2(0)| + \tfrac{1}{\Gamma(\alpha+1)}(\psi(T) - \psi(0))^{\alpha}[(A_g^* + A_f^* C_g^* + C_f^* C_g^* A_g^*) + \tfrac{C_g^* \, B_f^*}{1-C_f^* \, C_g^*} \, |u(t)| + \tfrac{C_f^* B_f^* \, C_g^*}{1-C_f^* \, C_g^*} \, |v(t)|]. \tag{3.23}$$

Using the last two inequalities, we obtain that

$$\begin{aligned} \|(u, v)(t)\|_X &\leq |u(t)| + |v(t)| \\ &\leq \mathbb{L} \, \|(u, v)\|_X + \mathbb{K}. \end{aligned} \tag{3.24}$$

Hence, we have

$$\|(u, v)(t)\|_X \leq \tfrac{\mathbb{K}}{1-\mathbb{L}}, \tag{3.25}$$

where

$$\mathbb{L} = \frac{2\ell}{\Gamma(\alpha + 1)}(\psi(T) - \psi(0))^{\alpha} + \max\{L_5, L_6\},$$

$$\ell = \max\{A_f^* + C_f^* A_g^*, \frac{B_f^*}{1 - C_f^* C_g^*}, \frac{C_f^* B_f^*}{1 - C_f^* C_g^*}, (A_g^* + A_f^* C_g^* + C_f^* C_g^* A_g^*), \frac{C_g^* \, B_f^*}{1 - C_f^* \, C_g^*}, \frac{C_f^* B_f^* \, C_g^*}{1 - C_f^* \, C_g^*}\},$$

$$\mathbb{K} = |u_0| + |h_1(0)| + |v_0| + |h_2(0)|) + \frac{\hat{\ell}}{\Gamma(\alpha + 1)}(\psi(T) - \psi(0))^{\alpha},$$

and

$$\hat{\ell} = A_f^* + C_f^* \, A_g^* + A_g^* + A_f^* C_g^* + C_f^* C_g^* A_g^*.$$

Since $\mathbb{L} < 1$, than $\mathbb{B}$ is bounded. Thus, F possess at least a fixed-point. □

Theorem 10. *Suppose that the conditions (C1) and (C2) hold. If $\mathbb{L}^* < \frac{1}{2}$, where*

$$\begin{aligned} \mathbb{L}^* = \max\{&[L_5 + \tfrac{L_1}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})], (\tfrac{L_2L_3}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})), \\ &\tfrac{L_1L_4}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})], [L_6 + (L_3 + \tfrac{L_2L_3L_4}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})]\}, \end{aligned} \tag{3.26}$$

then the system (1.1) has a unique solution in X.

Proof. We apply Banach contraction approach to prove F has a unique fixed-point. Let $(u, v), (\overline{u}, \overline{v}) \in X$. Then, we have

$$\begin{aligned} |T_1(u)(t) - T_1(\overline{u})(t)| &\leq L_5|u(t) - \overline{u}(t)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \, |x(s) - \overline{x}(s)| \, ds \quad , \\ |T_2(v)(t) - T_1(\overline{v})(t)| &\leq L_6|v(t) - \overline{v}(t)| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \, |y(s) - \overline{y}(s)| \, ds \quad , \end{aligned} \tag{3.27}$$

where $x, \overline{x}, y, \overline{y} \in C(I, \mathbb{R})$, such that

$$\overline{x} = f(t, \overline{u}, \overline{y}) \quad , \quad \overline{y} = g(t, \overline{v}, \overline{x}).$$

Therefore, we get

$$|x-\overline{x}| \leq \frac{L_1}{1-L_2L_4}|u-\overline{u}| + \frac{L_2L_3}{1-L_2L_4}|v-\overline{v}|,$$

$$|y-\overline{y}| \leq \frac{L_1L_4}{1-L_2L_4}|u-\overline{u}| + (L_3+\frac{L_2L_3L_4}{1-L_2L_4})|v-\overline{v}|.$$

From substitution the last two equations in (3.27), we get

$$|T_1(u)(t)-T_1(\overline{u})(t)| \leq [L_5+\tfrac{L_1}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]|u-\overline{u}| + (\tfrac{L_2L_3}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})|v-\overline{v}|, \tag{3.28}$$

$$|T_2(v)(t)-T_2(\overline{v})(t)| \leq \tfrac{L_1L_4}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]|u-\overline{u}| + [L_6+(L_3+\tfrac{L_2L_3L_4}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]|v-\overline{v}|, \tag{3.29}$$

where

$$\begin{array}{c}\mathbb{L}^* = \max\{[L_5+\tfrac{L_1}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})], (\tfrac{L_2L_3}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})), \\ \tfrac{L_1L_4}{1-L_2L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})], [L_6+(L_3+\tfrac{L_2L_3L_4}{1-L_2L_4})(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]\}.\end{array} \tag{3.30}$$

By adding the inequalities (3.28) and (3.29) and taking the maximum over I , we get

$$\|F(u,v)(t)-F(\overline{u},\overline{v})(t)\|_X \leq 2\mathbb{L}^* \parallel (u,v)-(\overline{u},\overline{v}) \parallel_X .$$

If $\mathbb{L}^* < \frac{1}{2}$, then F has a unique fixed-point in X. □

4. Stability analysis

In this section, some types of Ulam stability will be considered. First we recall the definitions of those types of Ulam stability. For more details, see [38].

Definition 11. *The system (1.1) is called Ulam-Hyers stable(UHS, for short) if there exists $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that for every ε_1 , $\varepsilon_2 > 0$ and each $(u,v) \in X$ of the inequalities*

$$\begin{array}{l}|{}^*D_{0^+}^{\alpha,\psi}u(t) - f(t,u(t),{}^*D_{0^+}^{\alpha,\psi}v(t))| \leq \varepsilon_1, \\ |{}^*D_{0^+}^{\alpha,\psi}v(t) - g(t,v(t),{}^*D_{0^+}^{\alpha,\psi}u(t))| \leq \varepsilon_2,\end{array} \tag{4.1}$$

where $t \in I$, there exists a unique solution $(\overline{u},\overline{v}) \in X$ of the system (1.1) such that

$$\begin{array}{l}|u(t)-\overline{u}(t)| \leq \lambda_1\ \varepsilon_1,\ \forall\ t \in I, \\ |v(t)-\overline{v}(t)| \leq \lambda_2\ \varepsilon_2,\ \forall\ t \in I.\end{array} \tag{4.2}$$

Definition 12. *The system (1.1) is called generalized Ulam-Hyers stable (GUHS, for short) if there exists $\varphi_1,\ \varphi_2\ \in C(\mathbb{R}^+,\mathbb{R}^+)$, $\varphi_1(0)=\varphi_2(0)=0$, such that for every ε_1 , $\varepsilon_2 > 0$ and each $(u,v) \in X$ solution of the inequalities*

$$\begin{array}{l}|{}^*D_{0^+}^{\alpha,\psi}u(t) - f(t,u(t),{}^*D_{0^+}^{\alpha,\psi}v(t))| \leq \varepsilon_1 \quad , \\ |{}^*D_{0^+}^{\alpha,\psi}v(t) - g(t,v(t),{}^*D_{0^+}^{\alpha,\psi}u(t))| \leq \varepsilon_2 \quad ,\end{array} \tag{4.3}$$

where $t \in I$, there exists a unique solution $(\overline{u}, \overline{v}) \in X$ of the system (1.1) such that

$$\begin{aligned} |u(t) - \overline{u}(t)| &\leq \varphi_1(\varepsilon_1), \ \forall \ t \in I, \\ |v(t) - \overline{v}(t)| &\leq \varphi_2(\varepsilon_2), \ \forall \ t \in I. \end{aligned} \tag{4.4}$$

Definition 13. *The system (1.1) is called Ulam-Hyers-Rassias stable (UHRS, for short) with respect to $\varphi_1, \varphi_2 \in C(I, \mathbb{R}^+)$, if there exists $\kappa_{\varphi_1}, \ \kappa_{\phi_2} \in \mathbb{R}^+$ such that for every ε_1 , $\varepsilon_2 > 0$ and each $(u, v) \in X$ solution of the inequalities*

$$\begin{aligned} |^*D_{0^+}^{\alpha,\psi} u(t) - f(t, u(t), ^*D_{0^+}^{\alpha,\psi} v(t))| &\leq \varepsilon_1 \ \varphi_1(t), \\ |^*D_{0^+}^{\alpha,\psi} v(t) - g(t, v(t), ^*D_{0^+}^{\alpha,\psi} u(t))| &\leq \varepsilon_2 \ \varphi_2(t), \end{aligned} \tag{4.5}$$

where $t \in I$, there exists a unique solution $(\overline{u}, \overline{v}) \in X$ of the system (1.1) such that

$$\begin{aligned} |u(t) - \overline{u}(t)| &\leq \kappa_{\varphi_1} \ \varepsilon_1 \ \varphi_1(t), \ \forall \ t \in I, \\ |v(t) - \overline{v}(t)| &\leq \kappa_{\varphi_2} \ \varepsilon_2 \ \varphi_2(t), \ \forall \ t \in I. \end{aligned} \tag{4.6}$$

Definition 14. *The system (1.1) is called generalized Ulam-Hyers-Rassias stable (GUHRS, for short) with respect to $\varphi_1, \varphi_2 \in C(I, \mathbb{R})$, if $\exists \ \kappa_{\varphi_1}, \ \kappa_{\phi_2} \in \mathbb{R}^+$ such that for each $(u, v) \in X$ solution of the inequalities*

$$\begin{aligned} |^*D_{0^+}^{\alpha,\psi} u(t) - f(t, u(t), ^*D_{0^+}^{\alpha,\psi} v(t))| &\leq \varphi_1(t), \\ |^*D_{0^+}^{\alpha,\psi} v(t) - g(t, v(t), ^*D_{0^+}^{\alpha,\psi} u(t))| &\leq \varphi_2(t), \end{aligned} \tag{4.7}$$

where $t \in I$, there exists a unique solution $(\overline{u}, \overline{v}) \in X$ of the system (1.1) such that

$$\begin{aligned} |u(t) - \overline{u}(t)| &\leq \kappa_{\varphi_1} \ \varphi_1(t), \ \forall \ t \in I, \\ |v(t) - \overline{v}(t)| &\leq \kappa_{\varphi_2} \ \varphi_2(t), \ \forall \ t \in I. \end{aligned} \tag{4.8}$$

Lemma 15. *Let $\alpha \in (0, 1)$, if $(z_1, z_2) \in X$ is the solution of the inequalities (4.1), then (z_1, z_2) is the solution of the following system*

$$\begin{aligned} |z_1(t) - N_1(t)| &\leq \left(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right)\varepsilon_1 \quad , \\ |z_2(t) - N_2(t)| &\leq \left(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right)\varepsilon_2 \quad . \end{aligned} \tag{4.9}$$

Proof. Let $(z_1, z_2) \in X$ be any solution of the inequalities (4.1), then there exists $(\Theta_1, \Theta_2) \in X$ (Θ_1 dependent of z_1 and Θ_2 dependent of z_2) such that

$$\begin{cases} ^*D_{0^+}^{\alpha,\psi} u(t) = x(t) = f(t, u(t), y(t)) + \Theta_1(t); \\ ^*D_{0^+}^{\alpha,\psi} v(t) = y(t) = g(t, v(t), x(t)) + \Theta_2(t); \\ \alpha \in (0, 1), \ t \in I = [0, T], \ T \in \mathbb{R}^+; \\ \text{subject to the nonlocal conditions} \\ u(0) + h_1(u) = u_0; \\ v(0) + h_2(v) = v_0, \end{cases} \tag{4.10}$$

and

$$|\Theta_1(t)| \leq \varepsilon_1 \quad , \quad |\Theta_2(t)| \leq \varepsilon_2 \quad , \forall t \in I.$$

Thus, the system (4.10) is equivalent to the following system

$$z_1(t) = u_0 - h_1(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ x(s)\ ds + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ \Theta_1(s)\ ds,$$
$$z_2(t) = v_0 - h_2(v) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ y(s)\ ds + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ \Theta_2(s)\ ds\ . \tag{4.11}$$

Let

$$\begin{aligned} N_1(t) &= u_0 - h_1(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ x(s)\ ds \quad , \\ N_2(t) &= v_0 - h_2(v) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ y(s)\ ds \quad . \end{aligned} \tag{4.12}$$

Thus, we have

$$\begin{aligned} |z_1(t) - N_1(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ |\Theta_1(s)|\ ds \le (\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 \quad , \\ |z_2(t) - N_2(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ |\Theta_2(s)|\ ds \le (\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_2 \quad . \end{aligned} \tag{4.13}$$

□

Theorem 16. *Suppose that (C1) and (C2) hold. Then the system (1.1) will be UHS and consequently GUHS under the following condition*

$$\max\{(\frac{L_1\ L_4}{1-L_2\ L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}), \frac{L_2\ L_3}{1-L_2\ L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}), (\frac{L_1\ L_4}{1-L_2\ L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}), [L_6 + (L_3 + \frac{L_2\ L_3\ L_4}{1-L_2\ L_4})(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})]\} < \frac{1}{2}. \tag{4.14}$$

Proof. Let $(z_1, z_2) \in X$ be solution of the system (4.1) and (u, v) be the unique solution of the proposed system (1.1). Hence, we get $|N_1(t)| \le \varepsilon_1$ and $|N_2(t)| \le \varepsilon_2$, $\forall t \in I$. Therefore

$$\begin{aligned} |z_1(t) - u(t)| &\le |z_1(t) - N_1(t)| + |N_1(t) - u(t)|, \\ |z_2(t) - v(t)| &\le |z_2(t) - N_2(t)| + |N_2(t) - v(t)|. \end{aligned} \tag{4.15}$$

From Lemma 15, we get

$$|z_1(t) - u(t)| \le (\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 + L_5|z_1(t) - u(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ |x_{z_1}(s) - x_u(s)|\ ds,$$
$$|z_2(t) - v(t)| \le (\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_2 + L_6|z_2(t) - v(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ |y_{z_2}(s) - y_v(s)|\ ds, \tag{4.16}$$

where

$$\begin{aligned} x_{z_1}(t) &= f(t, z_1(t), y_{z_2}(t)) \quad , \quad x_u(t) = f(t, u(t), y_v(t)) \quad , \\ y_{z_2}(t) &= g(t, z_2(t), x_{z_1}(t)) \quad , \quad y_v(t) = g(t, v(t), x_u(t)) \quad . \end{aligned} \tag{4.17}$$

From (C1), we have

$$\begin{aligned} |x_{z_1}(t) - x_u(t)| &\le L_1|z_1(t) - u(t)| + L_2|y_{z_2}(t) - y_v(t)|, \\ |y_{z_2}(t) - y_v(t)| &\le L_3|z_2(t) - v(t)| + L_4|x_{z_1}(t) - x_u(t)|. \end{aligned} \tag{4.18}$$

Thus, we get

$$\begin{aligned} |x_{z_2}(t) - x_v(t)| &\le \frac{L_1}{1-L_2\ L_4}|z_1(t) - u(t)| + \frac{L_2\ L_3}{1-L_2\ L_4}|z_2(t) - v(t)|, \\ |y_{z_2}(t) - y_v(t)| &\le \frac{L_1\ L_4}{1-L_2\ L_4}|z_1(t) - u(t)| + (L_3 + \frac{L_2\ L_3\ L_4}{1-L_2\ L_4})|z_2(t) - v(t)|. \end{aligned} \tag{4.19}$$

Therefore, we obtain

$$|z_1(t) - u(t)| \le (\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 + (L_5 + \frac{L_1}{1-L_2\ L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})|z_1(t) - u(t)| + \frac{L_2\ L_3}{1-L_2\ L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})|z_2(t) - v(t)|, \tag{4.20}$$

and

$$|z_2(t) - v(t)| \leq \\ (\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\varepsilon_2 + (\frac{L_1\, L_4}{1-L_2\, L_4}(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})|z_1(t) - u(t)| + [L_6 + (L_3 + \frac{L_2\, L_3\, L_4}{1-L_2\, L_4})(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]|z_2(t) - v(t)|. \tag{4.21}$$

By Taking the maximum over $t \in I$ for (4.20) and (4.21), we get that

$$\|z_1 - u\|_C \leq (\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\varepsilon_1 + \gamma\ (\|z_1 - u\|_C + \|z_2 - v\|_C)), \tag{4.22}$$

and

$$\|z_2 - v\|_C \leq (\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\varepsilon_2 + \gamma\ (\|z_1 - u\|_C + \|z_2 - v\|_C))\ , \tag{4.23}$$

where

$$\gamma = \max\{(\frac{L_1\, L_4}{1-L_2\, L_4}(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}),\ \frac{L_2\, L_3}{1-L_2\, L_4}(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}),(\frac{L_1\, L_4}{1-L_2\, L_4}(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}),[L_6 + (L_3 + \frac{L_2\, L_3\, L_4}{1-L_2\, L_4})(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]\}. \tag{4.24}$$

Then by adding (4.22) and (4.23), we get

$$\|(z_1, z_2) - (u, v)\|_X \leq (\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\varepsilon + 2\gamma\ (\|(z_1, z_2) - (u, v)\|_X)). \tag{4.25}$$

Let $\varepsilon = \varepsilon_1 + \varepsilon_2$, and $E = (\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})$. Hence, if $\gamma < \frac{1}{2}$, we get that

$$\|(z_1, z_2) - (u, v)\|_X \leq \hat{q}\ \varepsilon, \tag{4.26}$$

where

$$\hat{q} = \frac{E}{1 - 2\gamma}.$$

Thus, we have

$$\begin{aligned} \|z_1 - u\|_C &\leq \hat{q}\ \varepsilon, \\ \|z_2 - v\|_C &\leq \hat{q}\ \varepsilon. \end{aligned} \tag{4.27}$$

Thus the system (1.1) is UHS. Therefore, if we put $\varphi_1(\varepsilon) = \varphi_2(\varepsilon) = \hat{q}\ \varepsilon$, we have that $\varphi_1(0) = \varphi_2(0) = 0$ and

$$\begin{aligned} \|z_1 - u\|_C &\leq \varphi_1(\varepsilon), \\ \|z_2 - v\|_C &\leq \varphi_2(\varepsilon). \end{aligned} \tag{4.28}$$

Thus, the system (1.1) is GUHS. □

Lemma 17. *Suppose the following condition holds:*

(C5) If $\phi_1, \phi_2 \in C(I, \mathbb{R})$ are increasing. Then, there exists $\mu_{\phi_1},\ \mu_{\phi_2} \in \mathbb{R}^+$ such that for every $t \in I$, the following inequalities hold

$$J_{0^+}^{\alpha,\psi}\phi_1(t) \leq \mu_{\phi_1}\ \phi_1(t) \quad , \quad J_{0^+}^{\alpha,\psi}\phi_2(t) \leq \mu_{\phi_2}\ \phi_2(t).$$

If $(z_1, z_2) \in X$ is the solution of the inequalities (4.5), then (z_1, z_2) are the solution of the following inequalities

$$\begin{aligned} |z_1(t) - N_1(t)| &\leq \mu_{\phi_1}\phi_1(t)\ \varepsilon_1, \\ |z_2(t) - N_2(t)| &\leq \mu_{\phi_2}\phi_2(t)\ \varepsilon_2. \end{aligned} \tag{4.29}$$

Proof. From Lemma 15, we get

$$\begin{aligned}|z_1(t) - N_1(t)| &\leq \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \,|\Theta_1(s)|\, ds,\\ |z_2(t) - N_2(t)| &\leq \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \,|\Theta_2(s)|\, ds.\end{aligned} \tag{4.30}$$

From (C5), we have that

$$\begin{aligned}|z_1(t) - N_1(t)| &\leq \mu_{\phi_1}\phi_1(t)\, \varepsilon_1,\\ |z_2(t) - N_2(t)| &\leq \mu_{\phi_2}\phi_2(t)\, \varepsilon_2.\end{aligned} \tag{4.31}$$

□

Theorem 18. *Consider the Conditions (C1), (C2) and (C5) hold. Then the system (1.1) is UHRS and GUHRS under the following condition*

$$\max\{(\tfrac{L_1\, L_4}{1-L_2\, L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}), \tfrac{L_2\, L_3}{1-L_2\, L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}), (\tfrac{L_1\, L_4}{1-L_2\, L_4}(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}), [L_6 + (L_3 + \tfrac{L_2\, L_3\, L_4}{1-L_2\, L_4})(\tfrac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)})]\} < \tfrac{1}{2}. \tag{4.32}$$

Proof. Let $(z_1, z_2) \in X$ be any solution of the system of inequalities (4.5) and (u, v) be the unique solution of the system (1.1). By using Lemma 17 and by doing the same steps as in the proof of Theorem 16, we get

$$\|z_1 - u\|_C \leq \mu_{\phi_1}\phi_1(t)\, \varepsilon_1 + \gamma\, (\|z_1 - u\|_C + \|z_2 - v\|_C), \tag{4.33}$$

and

$$\|z_2 - v\|_C \leq \mu_{\phi_2}\phi_2(t)\, \varepsilon_2 + \gamma\, (\|z_1 - u\|_C + \|z_2 - v\|_C). \tag{4.34}$$

Thus, we have

$$\|(z_1, z_2) - (u, v)\|_X \leq \mu_\phi \phi(t)\varepsilon + 2\, \gamma\, (\|z_1 - u\|_C + \|z_2 - v\|_C), \tag{4.35}$$

where

$$\varepsilon = \varepsilon_1 + \varepsilon_2,$$

and

$$\phi(t) = \max\{\phi_1(t), \phi_2(t)\},\ \forall\, t \in I,\ \mu_\phi = \max\{\mu_{\phi_1}, \mu_{\phi_2}\}.$$

If $\gamma < \frac{1}{2}$, we get

$$\|(z_1, z_2) - (u, v)\|_X \leq \tilde{q}\, \phi(t)\, \varepsilon. \tag{4.36}$$

Thus, we have

$$\begin{aligned}\|z_1 - u\|_C &\leq \tilde{q}\, \phi(t)\, \varepsilon,\\ \|z_2 - v\|_C &\leq \tilde{q}\, \phi(t)\, \varepsilon.\end{aligned} \tag{4.37}$$

Thus, the system (1.1) is UHRS. Therefore, if we put $\varepsilon = 1$, then the system (1) is GUHRS. □

5. An application

The next example is an application to the previous theoretical results. Consider the following ψ-Caputo system of FDEs

$$\begin{cases} {}^*D_{0^+}^{\frac{3}{5},\psi} u(t) = \frac{2+|u(t)|+|{}^*D_{0^+}^{\frac{3}{5},\psi} v(t)|}{70e^{t+10}(1+|u(t)|+|{}^*D_{0^+}^{\frac{3}{5},\psi} v(t)|)}; \\ {}^*D_{0^+}^{\frac{3}{5},\psi} v(t) = \frac{t}{50} + \frac{1}{100}(t\ \cos(v(t)) - v(t)\sin(t)) + \frac{1}{50}\sin({}^*D_{0^+}^{\frac{3}{5},\psi} u(t)); \\ t \in I = [0,1], \\ \text{subject to the nonlocal conditions} \\ u(0) = \frac{1}{10}u(\frac{3}{5}); \\ v(0) = \frac{1}{10}v(\frac{3}{5}), \end{cases} \tag{5.1}$$

where $\psi(t) = \sqrt{1+t}$, for all $t \in [0,1]$. It it clear that ψ is increasing and differentiable function on $[0,1]$. Therefore,

$$f(t,x,y) = \frac{2+|x|+|y)|}{70e^{t+10}(1+|x|+|y|)},$$

also

$$g(t,x,y) = \frac{t}{50} + \frac{1}{100}(t\ \cos(x) - x\sin(t)) + \frac{1}{50}\sin(y).$$

It is clear that, f and g are continuous. Then the condition (C1) holds. Therefore, we have

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{70e^{10}}(|x_1-x_2| + |y_1-y_2|),$$

$$|g(t,x_1,y_1)| \le \frac{1}{50}(|x_1-x_2| + |y_1-y_2|),$$

$$|f(t,x,y)| \le \frac{1}{35e^{10}} + \frac{1}{70e^{10}}|x| + \frac{1}{70e^{10}}|y|,$$

$$|g(t,x,y)| \le \frac{1}{50}(1+|x|+|y|),$$

for all $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $t \in [0,1]$. Thus, the conditions (C2) and (C3) hold with

$$L_1 = L_2 = \frac{1}{70e^{10}},\ L_3 = L_4 = \frac{1}{50},$$

therefore

$$A_f^* = \frac{1}{35e^{10}},\quad B_f^* = \frac{1}{70e^{10}},\quad C_f^* = B_f^* = \frac{1}{70e^{10}},\quad A_g^* = B_g^* = C_g^* = \frac{1}{50}.$$

From the nonlocal condition, we have $u_0 = v_0 = 0$ and $h_1(u) = \frac{1}{10}u(\frac{3}{5})$, $h_2(v) = \frac{1}{10}v(\frac{3}{5})$. Hence, we have that

$$|h_1(u_1) - h_1(u_2)| \le \frac{1}{10}|u_1-u_2|,$$

$$|h_2(u_1) - h_2(u_2)| \le \frac{1}{10}|u_1-u_2|,$$

for all $u_1, u_2 \in C([0,1],\mathbb{R})$. Hence the condition (C4) holds with $L_5 = L_6 = 0.1$. By using the above conditions, we get that $\mathbb{L}^* \cong 0.126191 < \frac{1}{2}$. As a result, the system (5.1) has a unique solution. Therefore

$$\max\{(\frac{L_1\,L_4}{1-L_2\,L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}), \frac{L_2\,L_3}{1-L_2\,L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}),$$
$$(\frac{L_1\,L_4}{1-L_2\,L_4}(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}), [L_6+(L_3+\frac{L_2\,L_3\,L_4}{1-L_2\,L_4})(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})]\} \cong 0.136191 < \frac{1}{2}.$$

From Theorem 18, the system (5.1) is UHS, GUHS, UHRS and GUHRS.

6. Conclusions

In this article, we investigated the solvability of a nonlinear coupled system of implicit fractional differential equations including ψ-Caputo fractional operator under nonlocal conditions. The proposed system is more general than many of the systems proposed in pervious literature. We applied Schaefers and Banach fixedpoint theorems to study the solvability of the proposed system. Furthermore, we extended the results to investigate several of Ulam stability for the proposed system. Finally, we provided an example to illustrate the abstract results. The results given in this article extended and developed some previous works.

Conflict of interest

The author declares that he has no conflicts of interest.

References

1. M. F. Elettreby, A. A. Al-Raezah, T. Nabil, Fractional-order model of two-prey one-predator system, *Math. Probl. Eng.*, **2017** (2017), 6714538.
2. D. N. Tien, Fractional stochastic differential equations with applications to finance, *J. Math. Anal. Appl.*, **397** (2013), 338–348.
3. G. M. Bahaa, Optimal control problem for variable-order fractional differential systems with time delay involving Atangana-Baleanu derivatives, *Chaos, Solitons Fractals*, **122** (2019), 129–142.
4. G. Q. Zeng, J. Chen, Y. X. Dai, L. M. Li, C. W. Zheng, M. R. Chen, Design of fractional order PID controller for automatic regulator voltage system based on multi-objective extremal optimization, *Neurocomputing*, **160** (2015), 173–184.
5. F. Dong, Q. Ma, Single image blind deblurring based on the fractional-order differential, *Comput. Math. Appl.*, **78** (2019), 1960–1977.
6. E. Hashemizadeh, A. Ebrahimzadeh, An efficient numerical scheme to solve fractional diffusion-wave and fractional Klein-Gordon equations in fluid mechanics, *Physica A*, **503** (2018), 1189–1203.
7. V. E. Tarasov, E. C. Aifantis, On fractional and fractal formulations of gradient linear and nonlinear elasticity, *Acta Mech.*, **230** (2019), 2043–2070.

8. G. Ali, K. Shah, T. Abdeljawad, H. Khan, G. U. Rahman, A. Khan, On existence and stability results to a class of boundary value problems under Mittag-Leffler power law, *Adv. Differ. Equations*, **2020** (2020), 1–13.

9. H. Khan, Z. A. Khan, H. Tajadodi, A. Khan, Existence and data-dependence theorems for fractional impulsive integro-differential system, *Adv. Differ. Equations*, **2020** (2020), 1–11.

10. H. Khan, J. F. Gomez-Aguilar, T. Abdeljawad, A. Khan, Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation, *Fractals*, **28** (2020), 1–9.

11. A. Shah, R. A. Khan, A. Khan, H. Khan, J. F. Gomez-Aguilar, Investigation of a system of nonlinear fractional order hybrid differential equations under usual boundary conditions for existence of solution, *Math. Methods Appl. Sci.*, **44** (2021), 1628–1638.

12. E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, New York: Springer, 1986.

13. S. M. Ulam, *A collection of mathematical problems*, New York: Interscience, 1960.

14. D. H. Hyers, On the stability of the linear functional equations, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224.

15. T. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.

16. D. H. Hyers, G. Isac, T. M. Rassias, *Stability of Functional Equations in Several Variables*, New York: Springer, 1998.

17. T. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta. Appl. Math.*, **62** (2000), 23–130.

18. A. Khan, H. Khan, J. F. Gomez-Aguilar, T. Abdeljawad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, *Chaos, Solitons Fractals*, **127** (2019), 422–427.

19. H. Khan, T. Abdeljawad, M. Aslam, R. Khan, A. Khan, Existence of positive solution and Hyers-Ulam stability for a nonlinear singular-delay-fractional differential equation, *Adv. Differ. Equations*, **2019** (2019), 104.

20. Y. Guo, X. Shu, Y. Li, F. Xu, The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, *Boundary Value Probl.*, 2019 (2019), 59. Available from: `https://doi.org/10.1186/s13661-019-1172-6`.

21. D. X. Cuong, On the Hyers-Ulam stability of Riemann-Liouville multi-order fractional differential equations, *Afr. Mat.*, **30** (2019), 1041–1047.

22. Y. Basci, S. Ogreki, A. Misir, On Hyers-Ulam stability for fractional differential equations including the new Caputo-Fabrizio fractional derivative, *Mediterr. J. Math.*, **16** (2019), 131.

23. A. Ali, B. Samet, K. Shah, R. Khan, Existence and stability of solution of a toppled systems of differential equations non-integer order, *Bound. Valu Probl.*, **2017** (2017), 16. Available from: `https://doi.org/10.1186/s13661-017-0749-1`.

24. H. Khan, Y. Li, W. Chen, D. Baleanu, A. Khan, Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator, *Bound. Value Probl.*, **2017** (2017), 157. Available from: `https://doi.org/10.1186/s13661-017-0878-6`.
25. Z. Ali, A. Zada, K. Shah, Ulam stability results for the solutions of nonlinear implicit fractional order differential equations, *Hacettepe J. Math. Stat.*, **48** (2019), 1092–1109.
26. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Amsterdam: Elsevier, 2006.
27. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44** (2017), 460–481.
28. R. Almeida, A. B. Malinowska, M. T. Monterio, Fractional differential equations with a Caputo derivative with respect to a kernel functions and their applications, *Math. Method. Appl. Sci.*, **41** (2018), 336–352.
29. M. S. Abdo, A. G. Ibrahim, S. K. Panchal, Nonlinear implicit fractional differential equations involving ψ-Caputo fractional derivative, *Proc. Jangieon Math. Soc.*, **22** (2019), 387–400.
30. R. Almeida, Functional differential equations involving the ψ-Caputo fractional derivative, fractal and fractional, **4** (2020), 1–8.
31. D. B. Pachpatte, On some ψ-Caputo fractional Cebysev like inequalities for functions of two and three variables, *AIMS Math.*, **5** (2020), 2244–2260.
32. C. Derbazi, Z. Baitiche, M. S. Abdo, T. Abdeljawad, Qualitative analysis of fractional relaxation equation and coupled system with ψ-Caputo fractional derivative in Banach spaces, *AIMS Math.*, **6** (2021), 2486–2509.
33. T. Nabil, Existence results for nonlinear coupled system of integral equations of Urysohn Volterra-Chandrasekhar mixed type, *Demonstratio Mathematica*, **53** (2020), 236–248.
34. T. Nabil, Solvability of Fractional differential inclusion with a generalized Caputo derivative, *J. Funct. Spaces*, **2020** (2020), 1–11.
35. T. Nabil, On nonlinear fractional neutral differential equation with the ψ-Caputo fractional derivative, *J. Math. Appl.*, **43** (2020), 99.
36. A. Granas, J. Dugundji, *Fixed Point Theory*, New York: Springer-Verlag, 2003.
37. J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations*, New York: Springer-Verlag, 1993.
38. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.*, **26** (2010), 103–107.

AIMS Press