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Research article

New generalizations for Gronwall type inequalities involving a ψ–fractional operator and their applications

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Abstract: In this paper, we provide new generalizations for the Gronwall's inequality in terms of a ψ-fractional operator. The new forms of Gronwall's inequality are obtained within a general platform that includes several existing results as particular cases. To apply our results and examine their validity, we prove the existence and uniqueness of solutions for ψ-fractional initial value problem. Further, the Ulam-Hyers stability of solutions for ψ-fractional differential equations is discussed. For the sake of illustrating the proposed results, we give some particular examples.

Keywords: ψ-fractional operators; generalized Gronwall's inequality; ψ-fractional initial value problem; existence and uniqueness; Ulam–Hyers stability
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1. Background and preliminaries

The fractional calculus is nowadays considered as an important branch of mathematics, with a positive impact on several applied sciences; see, for example, the classical monograph by Samko et al. [1] and Kilbas et al. [2]. In [3], Kiryakova proposed a theory of a generalized fractional calculus (generalizations of fractional integrals and derivatives) and their applications. One of the proposed generalizations of the fractional calculus operators which has wide applications is the ψ–fractional operator. This notion is referred to as the fractional derivative and integral of a function with respect to another function ψ. Several properties of this operator could be found in [1, 2, 4–7]. For some new developments on this topic; see [8–12] and references therein.

Inequalities play a vital role in both pure and applied mathematics. In particular, inequalities involving the derivative and integral of functions are very captivating for researchers [13]. Integral inequalities have many applications in the theory of differential equations, theory of approximations, transform theory, probability, and statistical problems and many others. Therefore, in the literature we found several extensions and significant developments for the forms of classical integral inequalities. Furthermore, the study of qualitative and quantitative properties of solution of fractional differential and integral equations requires the use of various types of integral inequalities.

As our concern is Gronwall's inequality, we state its classical form as follows.

Theorem 1.1. *[14] Let $u(t), g(t)$ be nonnegative functions for any $t \in [a, T]$ and a, T and v be nonnegative constants such that*

$$u(t) \leq v + \int_a^t g(\tau) u(\tau) \, \mathrm{d}\tau, \tag{1.1}$$

then

$$u(t) \leq v \exp\left(\int_a^t g(\tau) \, \mathrm{d}\tau \right). \tag{1.2}$$

We review some recent results for the sake of comparison. In [15], Bellman generalized Theorem 1.1 by letting v be a nonnegative and nondecreasing function, which is stated in many references such as [13, 16]. In [17], Pachpatte also established the following inequality

$$u(t) \leq v(t) + \int_a^t g_1(\tau) u(\tau) \, \mathrm{d}\tau + \int_a^T g_2(\tau) u(\tau) \, \mathrm{d}\tau. \tag{1.3}$$

In [18], Kender et al. proved the following further generalizations of inequality (1.3) by replacing the linear term of the unknown function u by the nonlinear term u^p in both sides of the inequality and obtain the following

$$u^p(t) \leq v(t) + \int_a^t g_1(\tau) u(\tau) \, \mathrm{d}\tau + \int_a^T g_2(\tau) u(\tau)^p \, \mathrm{d}\tau, \quad p > 0. \tag{1.4}$$

In [19], Jiang and Meng discussed the following integral inequality

$$u^r(t) \leq v(t) + g_1(t) \int_a^t g_2(\tau) u(\tau)^p \, \mathrm{d}\tau + g_1(t) \int_a^t g_3(\tau) u(\tau)^q \, \mathrm{d}\tau, \quad r, p, q > 0, \tag{1.5}$$

under the same initial condition. For further detail on Gronwall–type inequalities involving the Riemann–Liouville fractional integrals [20, 21], for the Hadamard fractional integrals [22, 23] and for the Katugampola fractional integrals [24, 25], where other formulations of the Gronwall's inequality can be found via fractional integrals [26, 27].

As one of the objectives of this article is to propose a generalized Gronwall's inequality, we state the inequality of Gronwall which was first introduced in fractional settings in [28]

$$u(t) \leq v(t) + g(t) \int_a^t (t - \tau)^{\alpha - 1} u(\tau) \, \mathrm{d}\tau, \quad \alpha > 0, \tag{1.6}$$

where $u(t), v(t)$ are nonnegative functions and $g(t)$ is a nonnegative and nondecreasing function for $t \in [0, T]$. In [12], the Gronwall's inequality (1.6) was generalized as under

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) \, d\tau, \tag{1.7}$$

where $\psi \in C^1[a, T]$ is an increasing function such that $\psi'(t) \neq 0$, $\forall\, t \in [a, T]$. Further in [29], Willett discussed the linear inequality

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t h_i(\tau) u(\tau) \, d\tau, \quad h_i \in C^1[a, T]. \tag{1.8}$$

The following generalizations of the Gronwall type inequality were given in [30, 31]

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t (t - \tau)^{\alpha_i - 1} u(\tau) \, d\tau \tag{1.9}$$

and

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t (t - \tau)^{\alpha_i - 1} u^{p_i}(\tau) \, d\tau, \quad p_i > 0. \tag{1.10}$$

Oriented by above discussion, some other generalizations for the inequalities (1.1) and (1.6) have been elaborated. For relevant results; see [32–37] and the references cited therein.

The main objective of this paper is to extend Theorem 1.1, Gronwall–type inequalities (1.6), (1.7), (1.9) and (1.10) to the general case by the implementation of ψ–fractional operator. We claim that the results of this paper are obtained within a general platform that includes all previous forms as particular cases. As applications, we prove the existence and uniqueness of solutions for ψ–fractional initial value problem and study the Ulam–Hyers stability of solutions for ψ–fractional differential equations. Particular examples are given to confirm the proposed results.

We continue with the definitions and properties of the fractional derivative and integral of a function u with respect to given function ψ. These definitions are referred to as ψ–fractional operators.

The standard Riemann–Liouville fractional integral of order $\alpha > 0$, namely

$$\left(J_{a+,t}^{\alpha}\right)[u] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} u(\tau) \, d\tau, \quad t > a. \tag{1.11}$$

The left–sided factional integrals and fractional derivatives of a function u with respect to another function ψ in the sense of Riemann–Liouville are defined as follows [2]

$$\left(J_{a+,t}^{\alpha,\psi}\right)[u] = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) \, d\tau \tag{1.12}$$

and

$$\left(D_{a+,t}^{\alpha,\psi}\right)[u] = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \left(J_{a+,t}^{n-\alpha,\psi}\right)[u], \tag{1.13}$$

respectively, where $n = [\alpha] + 1$ and $u, \psi \in C^n[a, T]$ are two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a, T]$.

We propose the remarkable paper [38] in which some generalizations using ψ–fractional integrals and derivatives are described. In particular, we have

$$\begin{cases} \text{if} \quad \psi(t) \longrightarrow t, & \text{then} \quad J^{\alpha,\psi}_{a+,t} \longrightarrow J^{\alpha}_{a+,t}, \\ \text{if} \quad \psi(t) \longrightarrow \ln t, & \text{then} \quad J^{\alpha,\psi}_{a+,t} \longrightarrow {}^{H}J^{\alpha}_{a+,t}, \\ \text{if} \quad \psi(t) \longrightarrow t^{\rho}, & \text{then} \quad J^{\alpha,\psi}_{a+,t} \longrightarrow {}^{\rho}J^{\alpha}_{a+,t},\ \rho > 0, \end{cases} \tag{1.14}$$

where $J^{\alpha}_{a+,t}$, ${}^{H}J^{\alpha}_{a+,t}$ and ${}^{\rho}J^{\alpha}_{a+,t}$ are the classical Riemann–Liouville, Hadamard and Katugampola fractional operators, respectively.

Lemma 1.1. *[2] Let $\alpha, \beta > 0$. Then, we have the following*

$$\left(J^{\alpha,\psi}_{a+,t}\right)\left[\mathcal{K}(\tau; a)^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\mathcal{K}(t; a)^{\alpha+\beta-1} \tag{1.15}$$

and

$$\left(D^{\alpha,\psi}_{a+,t}\right)\left[\mathcal{K}(\tau; a)^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\mathcal{K}(t; a)^{\beta-\alpha-1}, \tag{1.16}$$

where

$$\mathcal{K}(t; \tau) = \psi(t) - \psi(\tau). \tag{1.17}$$

Lemma 1.2. *[6, 11] Given a function $u \in C^{n}[a, T]$ and $\alpha \in (0, 1)$. Then, we have*

$$J^{\alpha,\psi}_{a+,t}\left(D^{\alpha,\psi}_{a+,\tau}\right)[u] = u(t) - \frac{\left(J^{1-\alpha,\psi}_{a+,\tau}\right)[u]\Big|_{\tau=a}}{\Gamma(\alpha)}\left(\mathcal{K}(t; a)\right)^{\alpha-1}. \tag{1.18}$$

For $\alpha, \beta > 0$, the following properties are valid

$$\left(D^{\alpha,\psi}_{a+,t}\right)\left(J^{\beta,\psi}_{a+,\tau}\right)[u] = \left(J^{\beta-\alpha,\psi}_{a+,t}\right)[u]. \tag{1.19}$$

and

$$\left(J^{\alpha,\psi}_{a+,t}\right)\left(J^{\beta,\psi}_{a+,\tau}\right)[u] = \left(J^{\alpha+\beta,\psi}_{a+,t}\right)[u] \ \text{and}\ \left(D^{\alpha,\psi}_{a+,t}\right)\left(D^{\beta,\psi}_{a+,\tau}\right)[u] = \left(D^{\alpha+\beta,\psi}_{a+,t}\right)[u]. \tag{1.20}$$

The next result is helpful for the investigation obtained subsequently.

Lemma 1.3. *(Young's Inequality) [39, page 622] For any $A, B > 0$ and $1 < p, q < +\infty, 1/p + 1/q = 1, \varepsilon > 0$, we get*

$$AB \leq \varepsilon A^{p} + C(\varepsilon) B^{q}, \tag{1.21}$$

where

$$C(\varepsilon) = (\varepsilon p)^{-q/p} q^{-1}.$$

Definition 1.1. *[40] The Mittag–Leffler function is given by the series*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + 1)}, \tag{1.22}$$

where $Re(\alpha) > 0$ and $\Gamma(z)$ is a Gamma function. In particular

$$E_{1}(z) = \exp(z), E_{2}\left(z^{2}\right) = \cosh(z), E_{1/2}\left(z^{1/2}\right) = \exp(z)\left[1 + erf\left(z^{1/2}\right)\right],$$

where $erf(z)$ error function.

We outline the structure of the paper as follows: Section 2 is devoted to the new generalizations for the ψ–Gronwall–type inequality. Meanwhile, two remarks are addressed to show that the obtained forms of Gronwall–type inequality include other results as particular cases. Section 3 provides applications for the proposed results. Firstly, we demonstrate that the new inequalities can be used as handy tools in the study of existence and uniqueness of solutions of ψ–fractional initial value problem. Secondly, we use the the new inequalities to investigate the Ulam–Hyers stability of ψ–fractional differential equations. We also give some interesting examples to illustrate the effectiveness of our main results in Section 4. At last, the paper is concluded in Section 5.

2. New generalized ψ–Gronwall's inequality

By the same arguments of [30, Lemma 2.1], we can easily obtain the following result, which plays a very important role in proving the main results.

Lemma 2.1. *For any* $t \in [a, T)$*,*

$$U(t) \geq \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i - 1} U(\tau)\,\mathrm{d}\tau, \tag{2.1}$$

where all functions are continuous. The constants $\alpha_i > 0$. g_i $(i = 0, 1, \ldots, n)$ *are bounded, nonnegative, and monotonic increasing functions on* $[a, T)$*, then* $U(t) \geq 0$*,* $t \in [a, T)$*.*

Proof. Clearly, $U(a) \geq 0$. If the proposition is false, that is

$$\{t : t \in [a, T),\ U(t) < 0\} \neq \phi, \tag{2.2}$$

where ϕ is an empty set, then there exists a point t_0 on $[a, T)$ which satisfies $U|_{[a,t_0]} \geq 0,\ U(t_0) = 0$. The function U is a strictly monotonic decreasing function on $(t_0, t_0 + \varepsilon) \subset [a, T)$. Let $\varepsilon > 0$. Hence, for each $t \in (t_0, t_0 + \varepsilon)$, we have $U(t) < 0$ and

$$\begin{aligned} U(t) &\geq \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i - 1} U(\tau)\,\mathrm{d}\tau \\ &\geq \sum_{i=1}^{n} g_i(t) \int_{t_0}^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i - 1} U(\tau)\,\mathrm{d}\tau \\ &\geq U(t)\sum_{i=1}^{n} g_i(t) \int_{t_0}^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i - 1}\,\mathrm{d}\tau \\ &= U(t)\sum_{i=1}^{n} g_i(t) \frac{\mathcal{K}(t;t_0)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \end{aligned}$$

which implies that

$$\sum_{i=1}^{n} g_i(t) \frac{\mathcal{K}(t;t_0)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \geq 1.$$

Let $t \longrightarrow t_0$, then we have a contradiction, that is, $0 \geq 1$. The proof of Lemma 2.1 is completed. □

In light of the approach introduced in [30], we generalize Gronwall's inequality as follows.

Theorem 2.1. *Let $\psi \in C^1[a,T]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall\ t \in [a,T]$. Assume that*

- (H_1) *$u(t)$ and $v(t)$ are nonnegative functions locally integrable on $[a,T)$;*
- (H_2) *The functions $(g_i)_{i=1,\dots,n}$ are the bounded and monotonic increasing functions on $[a,T)$;*
- (H_3) *The constants $\alpha_i > 0$, $(i = 1,2,\dots,n)$.*

If

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)\,(\mathcal{K}(t;\tau))^{\alpha_i-1}\,u(\tau)\,\mathrm{d}\tau, \tag{2.3}$$

then

$$u(t) \leq v(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',3',\dots,k'=1}^{n} \frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)\,(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\right] v(\tau)\,\mathrm{d}\tau \right). \tag{2.4}$$

Proof. Suppose that

$$w(t) = v(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',3',ldots,k'=1}^{n} \left[\frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \right] \int_a^t \left[\psi'(\tau)\,(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\right] v(\tau)\,\mathrm{d}\tau \right).$$

By Dirichlet's formula and using the definition of Beta function, the following equality is given

$$\begin{aligned} &\int_a^t \int_a^s \psi'(s)\,\psi'(\tau)\,(\mathcal{K}(t;s))^{\alpha_j-1}\,(\mathcal{K}(s;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\,v(\tau)\,\mathrm{d}\tau\mathrm{d}s \\ = &\frac{\Gamma\left(\alpha_j\right)\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)}{\Gamma\left(\alpha_j+\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \psi'(s)\,(\mathcal{K}(t;s))^{\alpha_j+\sum_{i=1}^{k}\alpha_{i'}-1}\,v(s)\,\mathrm{d}s. \end{aligned} \tag{2.5}$$

From the fact that g_i $(i = 0,1,\dots,n)$ are monotonic increasing functions on $[a,T)$ and $g_{i'}(s) \leq g_{i'}(t)$, for all $s \leq t$, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)\,(\mathcal{K}(t;\tau))^{\alpha_i-1}\,w(\tau)\,\mathrm{d}\tau \\ \leq &\sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{1',2',3',\dots k'=1}^{n} g_j(t) \int_a^t \int_a^s \frac{\prod_{i=1}^{k}\left(g_{i'}(s)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \psi'(s)\,(\mathcal{K}(t;s))^{\alpha_j-1}\,(\mathcal{K}(s;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\,v(\tau)\,\mathrm{d}\tau\mathrm{d}s \\ &+ \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)\,(\mathcal{K}(t;\tau))^{\alpha_i-1}\,v(\tau)\,\mathrm{d}\tau \\ \leq &\sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{1',2',3',\dots k'=1}^{n} g_j(t) \int_a^t \int_a^s \frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \psi'(s)\,(\mathcal{K}(t;s))^{\alpha_j-1}\,(\mathcal{K}(s;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\,v(\tau)\,\mathrm{d}\tau\mathrm{d}s \\ &+ \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)\,(\mathcal{K}(t;\tau))^{\alpha_i-1}\,v(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{2.6}$$

By using (2.5), the inequality (2.6) can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1} w(\tau)\,\mathrm{d}\tau \\
\leq\ & \sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{1',2',3',\dots k'=1}^{n} g_j(t)\int_a^t \frac{\Gamma(\alpha_j)\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\alpha_j+\sum_{i=1}^{k}\alpha_{i'}\right)}\psi'(s)(\mathcal{K}(t;s))^{\alpha_j+\sum_{i=1}^{k}\alpha_{i'}-1} v(s)\,\mathrm{d}s. \qquad (2.7)
\end{aligned}$$

Let $j=(k+1)'$ then, from (2.7), we have

$$\begin{aligned}
&\sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1} w(\tau)\,\mathrm{d}\tau \\
\leq\ & \sum_{k=1}^{\infty}\sum_{(k+1)'=1}^{n}\sum_{1',2',3',\dots k'=1}^{n} g_{(k+1)'}(t)\int_a^t \frac{\Gamma\left(\alpha_{(k+1)'}\right)\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\alpha_{(k+1)'}+\sum_{i=1}^{k}\alpha_{i'}\right)}\psi'(s)(\mathcal{K}(t;s))^{\alpha_{(k+1)'}+\sum_{i=1}^{k}\alpha_{i'}-1} v(s)\,\mathrm{d}s \\
\leq\ & \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\dots,k',(k+1)'=1}^{n} \frac{\prod_{i=1}^{k+1}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k+1}\alpha_{i'}\right)}\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k+1}\alpha_{i'}-1} v(\tau)\,\mathrm{d}\tau\right) \\
\leq\ & \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\dots,k'=1}^{n} \frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)}\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1} v(\tau)\,\mathrm{d}\tau\right) \\
=\ & w(t)-v(t),
\end{aligned}$$

which implies that

$$\begin{aligned}
&u(t)-\sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1} u(\tau)\,\mathrm{d}\tau \\
\leq\ & v(t) \\
\leq\ & w(t)-\sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1} w(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Let $U(t)=w(t)-u(t)$, then we have

$$U(t)\geq\sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1} U(\tau)\,\mathrm{d}\tau.$$

According to Lemma 2.1, $U(t)\geq 0$. That is, $u(t)\leq w(t)$ and $t\in[a,T)$. The proof of Theorem 2.1 is completed. □

Corollary 2.1. *Under the hypotheses of Theorem 2.1, assume further that $u(t)$ is a nondecreasing function for $t\in[a,T)$, then*

$$u(t)\leq v(T)\sum_{i=1}^{n} E_{\alpha_i}\left(g_i(t)\Gamma(\alpha_i)(\mathcal{K}(T;a))^{\alpha_i}\right),\quad t\in[a,T). \qquad (2.8)$$

where E_{α_i} is the Mittag–Leffler function.

Proof. From (2.4) and $v(t)$ is a nondecreasing function for $t \in [a,T)$, we have

$$u(t) \leq v(t)\left[1+\sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n}\left[\frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma\left(\alpha_{i'}\right)\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)}\right]\int_a^t\left[\psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\sum_{i=1}^{k}\alpha_{i'}-1}\right]\mathrm{d}\tau\right)\right].$$

Then, with the help of (1.12) and Lemma 1.1, it follows that

$$\begin{aligned} u(t) &\leq v(t)\left[\sum_{k=0}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n}\left[\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma\left(\alpha_{i'}\right)\right)\right]\left(J_{a+,t}^{\sum_{i=1}^{k}\alpha_{i'},\psi}\right)[1]\right)\right] \\ &\leq v(t)\left[\sum_{k=0}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n}\left[\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma\left(\alpha_{i'}\right)\right)\right]\frac{\left(\mathcal{K}(t;a)\right)^{\sum_{i=1}^{k}\alpha_{i'}}}{\Gamma\left(1+\sum_{i=1}^{k}\alpha_{i'}\right)}\right)\right] \\ &\leq v(t)\sum_{i=1}^{n}E_{\alpha_i}\left(g_i(t)\Gamma\left(\alpha_i\right)\left(\mathcal{K}(t;a)\right)^{\alpha_i}\right). \end{aligned}$$

The proof is completed. □

Remark 2.1. *From Theorem 2.1, we have the following particular cases as follows:*

- • *If $n = 1$, then Theorem 2.1 reduces to inequality (1.7) which itself contains, as a special case, the inequalities (1.1) and (1.6).*
  - *If $n = 2$, then Theorem 2.1 reduces to the inequality given by [28, Theorem 2].*

Remark 2.2. *From Theorem 2.1, we have the following particular cases in the general forms of Gronwall's inequality as follows:*

- • *If $\psi(t) = t$, then the inequality given by [30, Theorem 1.4] reduces to the Gronwall's inequality for Riemann–Liouville fractional integral operator.*
  - *If $\psi(t) = \ln t$, then the inequality given by [30, Theorem 1.5] reduces to the Gronwall's inequality for Hadamard fractional integral operator.*
  - *If $\psi(t) = t^{\rho}$, then the inequality given by [24, Theorem 2.1. with $n = 1$] reduces to the Gronwall's inequality for Katugampola fractional integral operator.*

With the help of this Theorem 2.1, we have the following results.

Theorem 2.2. *Let $\psi \in C^1[a,T]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall\ t \in [a,T]$. Assume that (H_1) holds and*

- •(H_4) *The functions $(g_i)_{i=1,\ldots,n}$ and $(c_i)_{i=1,\ldots,n}$ are the bounded and monotonic increasing functions on $[a,T)$;*
- (H_5) *The constants $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 1$ and $0 < \lambda_i < 1$, $(i = 1,2,\ldots,n)$.*

If

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1} c_i(\tau)\, u^{\lambda_i}(\tau)\,\mathrm{d}\tau, \tag{2.9}$$

then

$$u(t) \leq \tilde{v}(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots,k'=1}^{n} \frac{\prod_{i=1}^{k}\left(\tilde{g}_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\sum_{i=1}^{k}\alpha_{i'}-1}\right]\tilde{v}(\tau)\,\mathrm{d}\tau\right), \tag{2.10}$$

where

$$\tilde{v}(t) = v(t) + \sum_{i=1}^{n} C_i(\varepsilon)\,g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1}\left[c_i(\tau)\right]^{1/(1-\lambda_i)}\mathrm{d}\tau \tag{2.11}$$

$$\tilde{g}_{i'}(t) = \varepsilon g_i(t),\ i = 1,\ldots,n. \tag{2.12}$$

and

$$C_i(\varepsilon) = (1-\lambda_i)\left(\frac{\lambda_i}{\varepsilon}\right)^{\frac{\lambda_i}{1-\lambda_i}}, \tag{2.13}$$

Here ε is an arbitrary given positive number.

By Young's inequality (Lemma 1.3), we have

$$c_i(t)\,u^{\lambda_i}(t) \leq \varepsilon\left[u^{\lambda_i}(t)\right]^{1/\lambda_i} + C_i(\varepsilon)\left[c_i(t)\right]^{1/(1-\lambda_i)}, \quad t \in [a,T],$$

which implies that, for any $t \in [a,T]$

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1} c_i(\tau)\,u^{\lambda_i}(\tau)\,\mathrm{d}\tau.$$

Hence, we have

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1}\left[\varepsilon u(\tau) + C_i(\varepsilon)\left[c_i(\tau)\right]^{1/(1-\lambda_i)}\right]\mathrm{d}\tau.$$

Consequently,

$$\begin{aligned} u(t) \leq{}& v(t) + \sum_{i=1}^{n} C_i(\varepsilon)\,g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1}\left[c_i(\tau)\right]^{1/(1-\lambda_i)}\mathrm{d}\tau \\ &+ \sum_{i=1}^{n} \varepsilon g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1} u(\tau)\,\mathrm{d}\tau. \end{aligned}$$

The proof is completed.

Corollary 2.2. *Let $\psi \in C^1[a,T]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall\ t \in [a,T]$. Assume that (H_1), (H_4) hold and*

- (H_6) *The constants $0 < \alpha_1 = \alpha_2 = \alpha_n = 1$ and $0 < \lambda_i < 1$ $(i = 1,2,\ldots,n)$.*

If

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau) c_i(\tau) u^{\lambda_i}(\tau) \mathrm{d}\tau, \tag{2.14}$$

then

$$u(t) \leq \tilde{v}(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',3',\ldots k'=1}^{n} \frac{\prod_{i=1}^{k} \left(\tilde{g}_{i'}(t)\right)}{k!} \int_a^t \left[\psi'(\tau) \left(\mathcal{K}(t;\tau)\right)^{k-1}\right] \tilde{v}(\tau) \mathrm{d}\tau \right), \tag{2.15}$$

where

$$\tilde{v}(t) = v(t) + \sum_{i=1}^{n} C_i(\varepsilon) g_i(t) \int_a^t \psi'(\tau) \left[c_i(\tau)\right]^{1/(1-\lambda_i)} \mathrm{d}\tau \tag{2.16}$$

and $\tilde{g}_{i'}(t) = \varepsilon g_i(t)$ *is defined by* (2.12) *for* $i = 1, \ldots, n$.

Remark 2.3. *From Theorem 2.2, we have the following particular cases:*

-
  - *If* $n = 1$, *then Theorem 2.2 reduces to one of the well-known Gronwall's inequality.*
  - *If* $\lambda_i = 1$ *or* $\lambda_i = 0$ $(i = 1, \ldots, n)$, *then it reduces to one of the well-known Gronwall's inequality.*
  - *If* $\psi(t) = t^{\rho}$, *then*
    - *If* $\rho \to 1$, *then the inequality given by [31, Theorem 4.] and [36, Theorem 2.1] reduces to the Gronwall's inequality for Riemann–Liouville fractional integral operator.*
    - *If* $\rho \to 0^{+}$, *then the inequality given by [31, Theorem 5] reduces to the Gronwall's inequality for Hadamard fractional integral operator.*

We conclude that Theorem 2.2 is more general than [31, Theorem 4 or Theorem 5].

3. Some applications

In this section, we present some applications of Theorem 2.1 and Theorem 2.2 to obtain the existence and uniqueness of the solution for ψ–fractional initial value problem. Further, we apply the main results of this work to study the stability of the ψ–fractional differential equations.

3.1. Existence and uniqueness

Consider the initial value problems with the ψ–fractional derivative

$$\begin{cases} \sum_{i=1}^{n} \left(D_{a+,t}^{\alpha_i,\psi}\right)[u] = f(t, u(t)) \\ \sum_{i=1}^{n} \left(J_{a+,t}^{1-\alpha_i,\psi}\right)[u]\Big|_{t=a} = \delta, \end{cases} \tag{3.1}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $D_{a+,t}^{\alpha_i,\psi}$, $J_{a+,t}^{\alpha_i,\psi}$ denote the left-sided of fractional derivative and fractional integral operators of a function u with respect to another function ψ in the sense of Riemann–Liouville, $f \in C([a, T] \times \mathbb{R}, \mathbb{R})$ and $\delta \in \mathbb{R}$.

The following lemma presents the uniqueness of solution for the initial value problem (3.1). For simplicity of presentation, we set $f_u \equiv f(t, u(t))$.

Lemma 3.1. *For each $t \in [a,T)$, suppose that $\gamma(t) \geq 0$ is a bounded and monotonic increasing function and*

$$|f(t,u_2) - f(t,u_1)| \leq \gamma(t)\,|u_2 - u_1|\,, \text{ for all } u_1, u_2 \in \mathbb{R}. \tag{3.2}$$

If the initial value problem (3.1) *has a solution, then the solution is unique.*

Proof. The proof will be given in two claims.

Claim 1. Since $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < 1$, then according to Lemma 1.2, we get

$$\begin{aligned} \left(J^{\alpha_n,\psi}_{a+,t}\right)[f_u] &= \sum_{i=1}^{n} \left(J^{\alpha_n,\psi}_{a+,t}\right)\left(D^{\alpha_i,\psi}_{a+,\tau}\right)[u] \\ &= \sum_{i=1}^{n} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)\left(\left(J^{\alpha_i,\psi}_{a+,s}\right)D^{\alpha_i,\psi}_{a+,\tau}\right)[u] \\ &= \sum_{i=1}^{n} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)\left(u(t) - c_i \mathcal{K}(t;a)^{\alpha_i-1}\right), \end{aligned} \tag{3.3}$$

where c_i, $(i = 1,2,\ldots,n)$ are some real numbers. By (3.3) and (1.15) we also have

$$\begin{aligned} \left(J^{\alpha_n,\psi}_{a+,t}\right)[f_u] &= \sum_{i=1}^{n} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)[u] - \sum_{i=1}^{n} c_i \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)\mathcal{K}(t;a)^{\alpha_i-1} \\ &= \sum_{i=1}^{n} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)[u] - \frac{\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)} \sum_{i=1}^{n} c_i \Gamma(\alpha_i)\,. \end{aligned} \tag{3.4}$$

Applying the fractional integral operator $\left(J^{1-\alpha_n,\psi}_{a+,t}\right)$ to both sides of (3.4), we get

$$\left(J^{1,\psi}_{a+,t}\right)[f_u] = \sum_{i=1}^{n} \left(J^{1-\alpha_i,\psi}_{a+,t}\right)[u] - \left[\left(J^{1-\alpha_n,\psi}_{a+,t}\right)\frac{\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)}\right] \sum_{i=1}^{n} c_i \Gamma(\alpha_i)\,.$$

Hence, we have

$$0 = \left(J^{1,\psi}_{a+,t}\right)[f_u]\Big|_{t=a} = \sum_{i=1}^{n} \left(J^{1-\alpha_i,\psi}_{a+,t}\right)[u]\Bigg|_{t=a} - \sum_{i=1}^{n} c_i \Gamma(\alpha_i)\,.$$

We obtain

$$\sum_{i=1}^{n} c_i \Gamma(\alpha_i) = \sum_{i=1}^{n} \left(J^{1-\alpha_i,\psi}_{a+,t}\right)[u]\Bigg|_{t=a} = \delta. \tag{3.5}$$

By (3.5), we have

$$\begin{aligned} \left(J^{\alpha_n,\psi}_{a+,t}\right)[f_u] &= \sum_{i=1}^{n} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)[u] - \delta \frac{\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)} \\ &= u(t) + \sum_{i=1}^{n-1} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)[u] - \delta \frac{\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)}. \end{aligned}$$

Since

$$u(t) = \delta \frac{\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)} - \sum_{i=1}^{n-1} \left(J^{\alpha_n-\alpha_i,\psi}_{a+,t}\right)[u] + \left(J^{\alpha_n,\psi}_{a+,t}\right)[f_u]\,. \tag{3.6}$$

Claim 2. Let u_1 and u_2 be two solutions of (3.1). Then from (3.6) and (3.2), we get

$$\begin{aligned} |u_2(t) - u_1(t)| &= \left| \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n-\alpha_i,\psi} \right) [u_2 - u_1] - \left(J_{a+,t}^{\alpha_n,\psi} \right) [f_{u_2} - f_{u_1}] \right| \\ &\leq \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n-\alpha_i,\psi} \right) [|u_2 - u_1|] + \left(J_{a+,t}^{\alpha_n,\psi} \right) \left[\left| f_{u_2} - f_{u_1} \right| \right] \\ &\leq \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n-\alpha_i,\psi} \right) [|u_2 - u_1|] + \gamma(t) \left(J_{a+,t}^{\alpha_n,\psi} \right) [|u_2 - u_1|] \\ &\leq 0, \end{aligned}$$

which yields

$$|u_2(t) - u_1(t)| \leq 0. \tag{3.7}$$

Therefore, we can conclude that

$$u_2(t) = u_1(t). \quad t \in [a, T). \tag{3.8}$$

Then the initial value problem (3.1) has at most one solution. The proof is completed. □

Consider the following fractional system with the ψ–fractional derivative

$$\begin{cases} \sum_{i=1}^{n} \left(D_{a+,t}^{\alpha_i,\psi} \right) [u] = f(t, u(t)) \\ \sum_{i=1}^{n} \left(J_{a+,t}^{1-\alpha_i,\psi} \right) [u] \Big|_{t=0} = \delta_1 \end{cases} \quad \text{and} \quad \begin{cases} \sum_{i=1}^{n} \left(D_{a+,t}^{\alpha_i,\psi} \right) [v] = g(t, v(t)) \\ \sum_{i=1}^{n} \left(J_{a+,t}^{1-\alpha_i,\psi} \right) [v] \Big|_{t=0} = \delta_2. \end{cases} \tag{3.9}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $D_{a+,t}^{\alpha_i,\psi}$, $J_{a+,t}^{\alpha_i,\psi}$ denote the left-sided of fractional derivative and fractional integral operators of a function u with respect to another function ψ in the sense of Riemann–Liouville, $f, g \in C([a, T] \times \mathbb{R}, \mathbb{R})$ and $\delta_1, \delta_2 \in \mathbb{R}$.

We have the following lemma.

Lemma 3.2. *Let $f, g : [a, T] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and let u, v be solutions of the two systems (3.9). Assume that the following assumptions hold:*

- (A_1) *There exists a positive constant c such that*

$$|g(t, v_2(t)) - g(t, v_1(t))| \leq c\,|v_2(t) - v_1(t)|, \quad c > 0, \quad \forall\, t \in [a, T], \quad \forall\, v_1, v_2 \in \mathbb{R}.$$

(A_2) *There exists a continuous function $\chi : [a, T] \to \mathbb{R}_0^+$ such that*

$$|f(t, u(t)) - g(t, u(t))| \leq \chi(t), \quad \forall\, t \in [a, T].$$

Then, for all $t \in [a, T]$, we have the following inequality:

$$|u(t) - v(t)| \leq w(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',3',\ldots k'=1}^{n} \frac{\prod_{i=1}^{k} \left(g_{i'}(t) \Gamma(\alpha_{i'}) \right)}{\Gamma\left(\sum_{i=1}^{k} \alpha_{i'} \right)} \int_a^t \left[\psi'(\tau) \left(\mathcal{K}(t; \tau) \right)^{\sum_{i=1}^{k} \alpha_{i'} - 1} \right] w(\tau)\, d\tau \right), \tag{3.10}$$

where the function $w : [a, T] \to \mathbb{R}$ is defined by

$$w(t) = |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \left(J_{a+,t}^{\alpha_n,\psi}\right)[\chi]. \tag{3.11}$$

Proof. With the help of (A_1) and (A_2), it follows that

$$\begin{aligned}
&|u(t) - v(t)| \\
= \; & \left|(\delta_1 - \delta_2) \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n - \alpha_i,\psi}\right)[u - v] - \left(J_{a+,t}^{\alpha_n,\psi}\right)[f_u - g_v]\right| \\
\leq \; & |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n - \alpha_i,\psi}\right)[|u - v|] + \left(J_{a+,t}^{\alpha_n,\psi}\right)[|f_u - g_v|] \\
\leq \; & |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n - \alpha_i,\psi}\right)[|u - v|] + \left(J_{a+,t}^{\alpha_n,\psi}\right)[|f_u - g_u + g_u - g_v|] \\
\leq \; & |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n - \alpha_i,\psi}\right)[|u - v|] + \left(J_{a+,t}^{\alpha_n,\psi}\right)[\chi] + c\left(J_{a+,t}^{\alpha_n,\psi}\right)[|u - v|] \\
\leq \; & |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \left(J_{a+,t}^{\alpha_n,\psi}\right)[\chi] + c\left(J_{a+,t}^{\alpha_n,\psi}\right)[|u - v|] + \sum_{i=1}^{n-1} \left(J_{a+,t}^{\alpha_n - \alpha_i,\psi}\right)[|u - v|].
\end{aligned} \tag{3.12}$$

Setting

$$w(t) = |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)} + \left(J_{a+,t}^{\alpha_n,\psi}\right)[\chi]. \tag{3.13}$$

By applying Theorem 2.1 to (3.12), the desired inequality (3.10) is obtained. This completes the proof. □

Remark 3.1. *In particular, when $f = g$ then $\chi(t) \equiv 0$, we obtain a simpler formula (3.10) with*

$$w(t) = |\delta_1 - \delta_2| \frac{\mathcal{K}(t;a)^{\alpha_n - 1}}{\Gamma(\alpha_n)}. \tag{3.14}$$

In view of inequality (3.10) with (3.14), we see that the solution of system (3.9) is unique.

Consider the following fractional system

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \left(D_{a+,t}^{\alpha_{n+1} - \alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] + \left(D_{a+,t}^{\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right] = f(t, u(t)) \\ \displaystyle\sum_{i=1}^{n} \left(J_{a+,t}^{1 - \alpha_{n+1} + \alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] + \left(J_{a+,t}^{1 - \alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right]\Bigg|_{t=a} = \delta, \end{cases} \tag{3.15}$$

where all functions are continuous. Moreover, $h_i(t) > 0$ and the constants λ_i, $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$. Consider $\lambda_0 > 0$, $\delta \in \mathbb{R}$ and $\max\{\alpha_i : i = 1, 2, \ldots, n\} < \alpha_{n+1} < 1$.

By applying similar arguments to the technique used in Lin [31], we can conclude the following result.

Lemma 3.3. *Suppose that*

- (A_3) *The function* $h_{n+1}(t) > 0$ *is continuous and the constant* $\lambda_{n+1} \in (0,1)$*, such that*

$$\left|f(t,u_2(t)) - f(t,u_1(t))\right| \le h_{n+1}(t)\left|u_2^{\lambda_{n+1}}(t) - u_1^{\lambda_{n+1}}(t)\right|, \tag{3.16}$$

for any $t \in [a,T]$ *and for all* $u_1, u_2 \in \mathbb{R}$.

(i) If $\max\{\lambda_i : i = 1,2,\ldots,n+1\} < \lambda_0$*, then for any solution* $u(t)$ *of the problem (3.15), we get*

$$|u(t)|^{\lambda_0} \le \tilde{v}(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n+1} \frac{\prod_{i=1}^{k}\left(\tilde{g}_{i'}(t)\Gamma\left(\alpha_{i'}\right)\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\sum_{i=1}^{k}\alpha_{i'}-1}\right]\tilde{v}(\tau)\,\mathrm{d}\tau\right), \tag{3.17}$$

where

$$\tilde{v}(t) = v(t) + \sum_{i=1}^{n+1} C_i(\varepsilon)\,g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i-1}\left[h_i(\tau)\right]^{1/\left(1-\frac{\lambda_i}{\lambda_0}\right)}\mathrm{d}\tau, \tag{3.18}$$

and

$$C_i(\varepsilon) = \frac{1}{\lambda_0}(\lambda_0-\lambda_i)\left(\frac{\lambda_i}{\varepsilon\lambda_0}\right)^{\frac{\lambda_i}{\lambda_0-\lambda_i}}, \quad \tilde{g}_{i'}(t) = \varepsilon g_i(t), \quad g_i(t) = \frac{1}{\Gamma(\alpha_i)},\ i = 1,\ldots,n+1.$$

(ii) If $\min\{\lambda_i : i = 1,2,\ldots,n+1\} \ge \lambda_0$*, then the continuous solution of problem (3.15) is unique.*

(i) Since $\max\{\alpha_i : i = 1,2,\ldots,n\} < \alpha_{n+1} < 1$, by using Lemma 1.1 and Lemma 1.2, we obtain

$$\begin{aligned}
\left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[f_u] &= \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left(D_{a+,\tau}^{\alpha_{n+1}-\alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] + \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left(D_{a+,\tau}^{\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right] \\
&= \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left(J_{a+,t}^{-\alpha_i,\psi}\right)\left(D_{a+,\tau}^{\alpha_{n+1}-\alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] + \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left(D_{a+,\tau}^{\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right] \\
&= u^{\lambda_0} - c_{n+1}\mathcal{K}(t;a)^{\alpha_{n+1}-1} + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i u^{\lambda_i} - c_i\mathcal{K}(t;a)^{\alpha_{n+1}-\alpha_i-1}\right].
\end{aligned}$$

It follows from (3.16) and (1.15) that

$$\left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[f_u] = u^{\lambda_0} + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] - \left(c_{n+1} + \sum_{i=1}^{n} c_i\frac{\Gamma(\alpha_{n+1}-\alpha_i)}{\Gamma(\alpha_{n+1})}\right)\mathcal{K}(t;a)^{\alpha_{n+1}-1},$$

which, together with (1.15) and (1.20), imply that

$$\begin{aligned}
\left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left(J_{a+,\tau}^{\alpha_{n+1},\psi}\right)[f_u] &= \left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right] + \sum_{i=1}^{n}\left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left(J_{a+,\tau}^{\alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] \\
&\quad - \left(c_{n+1} + \sum_{i=1}^{n} c_i\frac{\Gamma(\alpha_{n+1}-\alpha_i)}{\Gamma(\alpha_{n+1})}\right)\left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left[\mathcal{K}(t;a)^{\alpha_{n+1}-1}\right].
\end{aligned}$$

From Lemma 1.2, we have

$$\begin{aligned}\left(J_{a+,t}^{1,\psi}\right)[f_u] &= \left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right] + \sum_{i=1}^{n}\left(J_{a+,t}^{1+\alpha_i-\alpha_{n+1},\psi}\right)\left[h_i u^{\lambda_i}\right] \\ &\quad -\left(c_{n+1} + \sum_{i=1}^{n} c_i \frac{\Gamma(\alpha_{n+1}-\alpha_i)}{\Gamma(\alpha_{n+1})}\right)\Gamma(\alpha_{n+1}). \qquad (3.19)\end{aligned}$$

From (3.19), we have

$$\begin{aligned}\left(J_{a+,t}^{1,\psi}\right)[f_u]\Big|_{t=a} &= \left(J_{a+,t}^{1-\alpha_{n+1},\psi}\right)\left[u^{\lambda_0}\right]\Big|_{t=a} + \sum_{i=1}^{n}\left(J_{a+,t}^{1+\alpha_i-\alpha_{n+1},\psi}\right)\left[h_i u^{\lambda_i}\right]\Big|_{t=a} \\ &\quad -\left(c_{n+1} + \sum_{i=1}^{n} c_i \frac{\Gamma(\alpha_{n+1}-\alpha_i)}{\Gamma(\alpha_{n+1})}\right)\Gamma(\alpha_{n+1}). \qquad (3.20)\end{aligned}$$

From the condition (3.15) and (3.20), we have

$$c_{n+1}\Gamma(\alpha_{n+1}) + \sum_{i=1}^{n} c_i\Gamma(\alpha_{n+1}-\alpha_i) = \delta.$$

and

$$u^{\lambda_0}(t) = \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[f_u] - \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i u^{\lambda_i}\right] + \frac{\delta}{\Gamma(\alpha_{n+1})}\mathcal{K}(t;a)^{\alpha_{n+1}-1}. \qquad (3.21)$$

Let $w(t) = u^{\lambda_0}(t)$ and applying (3.16) and (3.21), given the fact that

$$|f_u| \le |f_u - f_0| + |f_0| \le h_{n+1}(t)\,|u(t)|^{\lambda_{n+1}} + |f_0|$$

obtains the following

$$\begin{aligned}|w(t)| &= \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_u|] + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[\left|h_i w^{\frac{\lambda_i}{\lambda_0}}\right|\right] + \frac{|\delta|}{\Gamma(\alpha_{n+1})}\mathcal{K}(t;a)^{\alpha_{n+1}-1} \\ &\le \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_0|] + \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_u - f_0|] \\ &\quad + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[\left|h_i w^{\frac{\lambda_i}{\lambda_0}}\right|\right] + \frac{|\delta|}{\Gamma(\alpha_{n+1})}\mathcal{K}(t;a)^{\alpha_{n+1}-1}. \qquad (3.22)\end{aligned}$$

Rearranging the terms of (3.22), and by using condition (3.16), we obtain

$$|w(t)| \le \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_0|] + \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left[\left|h_i w^{\frac{\lambda_{n+1}}{\lambda_0}}\right|\right] + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[\left|h_i w^{\frac{\lambda_i}{\lambda_0}}\right|\right] + \frac{|\delta|\,\mathcal{K}(t;a)^{\alpha_{n+1}-1}}{\Gamma(\alpha_{n+1})},$$

which gives

$$|w(t)| \le \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_0|] + \frac{|\delta|\,\mathcal{K}(t;a)^{\alpha_{n+1}-1}}{\Gamma(\alpha_{n+1})} + \sum_{i=1}^{n+1}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[\left|h_i w^{\frac{\lambda_i}{\lambda_0}}\right|\right].$$

If $\max\{\lambda_i : i = 1,2,\dots,n+1\} < \lambda_0$, then, for any $(i = 1,2,\dots,n+1)$, $\frac{\lambda_i}{\lambda_0} < 1$. Setting, for any $t \in [a,T]$,

$$v(t) = \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_0|] + \frac{|\delta| \mathcal{K}(t;a)^{\alpha_{n+1}-1}}{\Gamma(\alpha_{n+1})}, \quad g_i(t) = \frac{1}{\Gamma(\alpha_i)}.$$

By virtue of Theorem 2.2, we obtain

$$w(t) \le \tilde{v}(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\dots k'=1}^{n+1} \frac{\prod_{i=1}^{k}\left(\tilde{g}_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)(\mathcal{K}(t;\tau))^{\sum_{i=1}^{k}\alpha_{i'}-1}\right]\tilde{v}(\tau)\, \mathrm{d}\tau\right).$$

where the expression of $\tilde{v}(t)$ is shown in (3.18). Hence, the conclusion of (i) is derived.

(ii) We assume that problem (3.15) has two continuous solutions u_1 and u_2. Combining with the fact that $h_i(t) \in C([a,T],\mathbb{R})$ for any $i = 1,\dots,n+1$ and the boundedness of the continuous function on a closed interval, there exists a finite number M which satisfies that, for any $t \in [a,T]$,

$$\max\left\{|u_1(t)|, |u_2(t)|, \max_{1\le i\le n+1}|h_i(t)|\right\} < M. \tag{3.23}$$

Cauchy's mean value theorem provides the following inequality

$$\left|u_2^{\lambda_i}(t) - u_1^{\lambda_i}(t)\right| = \left|u_2^{\lambda_0}(t) - u_1^{\lambda_0}(t)\right| \left|\frac{\lambda_i \xi_i^{\lambda_i-1}}{\lambda_0 \xi_i^{\lambda_0-1}}\right| = \frac{\lambda_i\left|\xi_i^{\lambda_i-\lambda_0}\right|}{\lambda_0}\left|u_2^{\lambda_0}(t) - u_1^{\lambda_0}(t)\right|,$$

where ξ_i, $i = 1,\dots,n+1$, are the numbers between $u_1(t)$and $u_2(t)$. The following estimation is deduced by applying (3.23) and the hypothesis of $\lambda_0 \le \min\{\lambda_i : i = 1,2,\dots,n+1\}$ in (ii)

$$\left|u_2^{\lambda_i}(t) - u_1^{\lambda_i}(t)\right| = \frac{\lambda_i M^{\lambda_i-\lambda_0}}{\lambda_0}\left|u_2^{\lambda_0}(t) - u_1^{\lambda_0}(t)\right|, \tag{3.24}$$

holds for any $t \in [a,T]$ and $i = 1,\dots,n+1$. Therefore, (3.21), (3.23), and (3.24) give

$$\begin{aligned}
\left|u_2^{\lambda_0}(t) - u_1^{\lambda_0}(t)\right| &= \left|\left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left[f_{u_2} - f_{u_1}\right] - \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i\left(u_2^{\lambda_i} - u_1^{\lambda_i}\right)\right]\right| \\
&\le \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)\left[h_{n+1}\left|u_2^{\lambda_i} - u_1^{\lambda_i}\right|\right] + \sum_{i=1}^{n}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i\left|u_2^{\lambda_i} - u_1^{\lambda_i}\right|\right] \\
&\le \sum_{i=1}^{n+1}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i\left|u_2^{\lambda_i} - u_1^{\lambda_i}\right|\right] \\
&\le \sum_{i=1}^{n+1}\frac{\lambda_i M^{\lambda_i-\lambda_0}}{\lambda_0}\left(J_{a+,t}^{\alpha_i,\psi}\right)\left[h_i\left(\left|u_2^{\lambda_0} - u_1^{\lambda_0}\right|\right)\right].
\end{aligned}$$

According to Theorem 2.2, we have

$$\left|u_2^{\lambda_0}(t) - u_1^{\lambda_0}(t)\right| \le 0,$$

which means that $u_2^{\lambda_0}(t) = u_1^{\lambda_0}(t)$. This completes the proof of (ii).

3.2. Ulam–Hyers stability

In this subsection, we study the Ulam–Hyers stability of the initial value problem (3.1).

Remark 3.2. *For every $\epsilon > 0$, a function $u_\varepsilon \in C([a,T],\mathbb{R})$ is a solution of the inequality*

$$\left|\sum_{i=1}^{n}\left(D_{a+,t}^{\alpha_i,\psi}\right)[u_\varepsilon] - f\left(t,u_\varepsilon(t)\right)\right| \le \varepsilon, \tag{3.25}$$

if and only if there exists a function $h \in C([a,T],\mathbb{R})$, (which depends on u_ε) such that

(i) $|h(t)| \le \epsilon,\ \forall t \in [a,T]$.
(ii) $\sum_{i=1}^{n}\left(D_{a+,t}^{\alpha_i,\psi}\right)[u] = f(t,u(t)) + h(t)$.

Lemma 3.4. *Assume that f_{u_ε} is a continuous function that satisfies (3.2). The Eq (3.1) is Ulam-Hyers stable with respect to ψ if there exists a real number $C_\epsilon > 0$ such that for each $\epsilon > 0$ and for each solution $u_\varepsilon \in C^1([a,T],\mathbb{R})$ of the inequality (3.25), there exists a solution $u^* \in C^1([a,T],\mathbb{R})$ of (3.1) with*

$$|u_\varepsilon(t) - u^*(t)| \le \epsilon C_\epsilon. \tag{3.26}$$

Proof. If u_ε is a solution to (3.25), then u_ε is a solution to the problem

$$u_\varepsilon(t) = \frac{\delta\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)} - \sum_{i=1}^{n-1}\left(J_{a+,t}^{\alpha_n-\alpha_i,\psi}\right)[u_\varepsilon] + \left(J_{a+,t}^{\alpha_n,\psi}\right)\left[f_{u_\varepsilon} + h(t)\right].$$

For each $t \in [a,T]$, one has

$$\left|u_\varepsilon(t) - \frac{\delta\mathcal{K}(t;a)^{\alpha_n-1}}{\Gamma(\alpha_n)} + \sum_{i=1}^{n-1}\left(J_{a+,t}^{\alpha_n-\alpha_i,\psi}\right)[u_\varepsilon] - \left(J_{a+,t}^{\alpha_n,\psi}\right)[f_{u_\varepsilon}]\right| \le \left(J_{a+,t}^{\alpha_n,\psi}\right)[h].$$

Then, it follows that

$$\begin{aligned}
|u_\varepsilon(t) - u^*(t)| &\le \sum_{i=1}^{n-1}\left(J_{a+,t}^{\alpha_n-\alpha_i,\psi}\right)[|u_\varepsilon - u^*|] + \left(J_{a+,t}^{\alpha_n,\psi}\right)\left[\left|f_{u_\varepsilon} + h - f_{u^*}\right|\right] \\
&\le \sum_{i=1}^{n-1}\left(J_{a+,t}^{\alpha_n-\alpha_i,\psi}\right)[|u_\varepsilon - u^*|] + \left(J_{a+,t}^{\alpha_n,\psi}\right)\left[\left|f_{u_\varepsilon} - f_{u^*}\right|\right] + \left(J_{a+,t}^{\alpha_n,\psi}\right)[h] \\
&\le \left(J_{a+,t}^{\alpha_n,\psi}\right)[h] + \sum_{i=1}^{n-1}\left(J_{a+,t}^{\alpha_n-\alpha_i,\psi}\right)[|u_\varepsilon - u^*|] + L\left(J_{a+,t}^{\alpha_n,\psi}\right)[|u_\varepsilon - u^*|].
\end{aligned}$$

In virtue of (2.3), one has

$$w(t) \le v(t) + \sum_{i=1}^{n} g_i(t)\int_a^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\beta_i-1} w(\tau)\,\mathrm{d}\tau,$$

where

$$w(t) = |u_\varepsilon(t) - u^*(t)|, \quad v(t) = \varepsilon\left(J_{a+,t}^{\alpha_n,\psi}\right)[1].$$

Setting

$$g_n(t) = \frac{L}{\Gamma(\beta_n)}, \quad \beta_n = \alpha_n$$
$$g_i(t) = \frac{1}{\Gamma(\beta_i)}, \quad \beta_i = \alpha_n - \alpha_i, \ i = 1, \ldots, n-1.$$

By using Corollary 2.1, we obtain

$$w(t) \le v(t)\left[\sum_{i=1}^{n-1} E_{\beta_i}\left(g_i(t)\Gamma(\beta_i)(\mathcal{K}(t;a))^{\beta_i}\right) + E_{\beta_n}\left(Lg_n(t)\Gamma(\beta_n)(\mathcal{K}(t;a))^{\beta_n}\right)\right].$$

Therefore

$$|u_\varepsilon(t) - u^*(t)| \le \frac{\varepsilon(\mathcal{K}(T;a))^{\alpha_n}}{\Gamma(1+\alpha_n)}\left[\sum_{i=1}^{n-1} E_{\alpha_n-\alpha_i}\left((\mathcal{K}(T;a))^{\alpha_n-\alpha_i}\right) + E_{\alpha_n}\left(L(\mathcal{K}(T;a))^{\alpha_n}\right)\right].$$

This shows that (3.26) holds. The proof is completed. □

4. Particular examples

In this section, we provide some particular examples that validate and confirm the proposed theorems.

Example 4.1. *we consider the linear inequality as follows*

$$u(t) \le v(t) + \sum_{i=1}^{n} g_i(t)\int_0^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1}u(\tau)\,\mathrm{d}\tau, \quad t \in [0,1] \tag{4.1}$$

Here $n = 3$, $\alpha_1 = 1/2$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\psi(t) = t$, $g_i(t) = 1/2\Gamma(\alpha_i)$,$i = 1, 2, 3$ *and*

$$v(t) = t^2 - \frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{t^3}{6} - \frac{t^4}{24}. \tag{4.2}$$

By equality (4.2), we derive that $v(t)$ *is nonnegative and increasing on* $[0,1]$. *According to Corollary 2.1, we have*

$$\begin{aligned} u(t) &\le \left(t^2 - \frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{t^3}{6} - \frac{t^4}{24}\right)\left[E_{\frac{1}{2}}\left(\frac{1}{2}t^{\frac{1}{2}}\right) + E_1\left(\frac{1}{2}t\right) + E_2\left(\frac{1}{2}t^2\right)\right] \\ &\le \left(t^2 - \frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{t^3}{6} - \frac{t^4}{24}\right)\left[e^{\frac{t}{4}}\left(1 + erf\left(\frac{\sqrt{t}}{2}\right)\right) + e^{\frac{t}{2}} + \cosh\left(\frac{\sqrt{2}t}{2}\right)\right] = w(t). \end{aligned} \tag{4.3}$$

it can be seen that the values of exact solution the linear integral equation

$$u(t) = v(t) + \sum_{i=1}^{n} g_i(t)\int_0^t \psi'(\tau)(\mathcal{K}(t;\tau))^{\alpha_i-1}u(\tau)\,\mathrm{d}\tau, \quad t \in [0,1], \tag{4.4}$$

is $u(t) = t^2$. *In Figure 1, we plot the graphs of estimated bound of* $w(t)$, $u(t)$ *and* $w(t) - u(t)$ *for* $t \in [0,1)$.

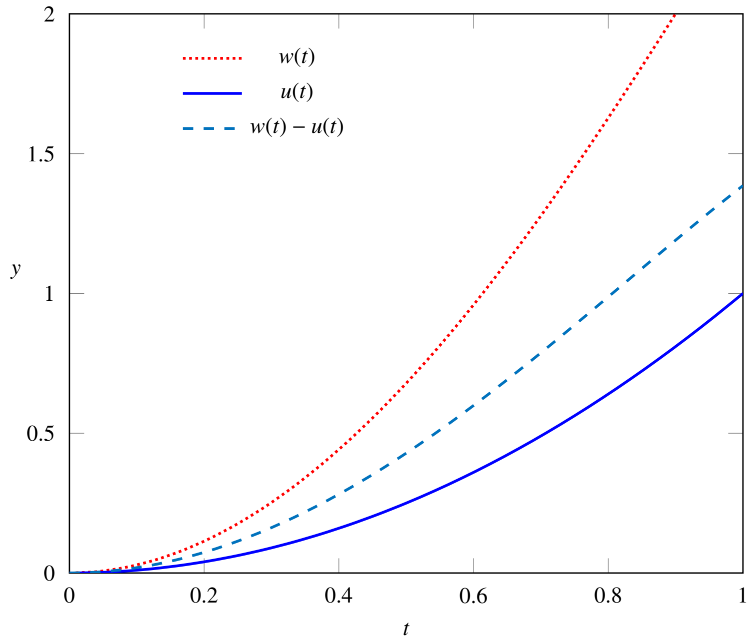

Figure 1. Graphs of estimated bound of $w(t)$, $u(t)$ and $w(t) - u(t)$ for $t \in [0, 1)$.

Example 4.2. *Consider the following initial value problems with the ψ–fractional derivative of the form:*

$$\begin{cases} \sum_{i=1}^{n} \left(D_{0+,t}^{\alpha_i,\psi}\right)[u] = 2t\left(\frac{t^{\frac{7}{8}}}{\Gamma(\frac{23}{8})} + \frac{t^{\frac{3}{4}}}{\Gamma(\frac{11}{4})} + \frac{4t^{\frac{1}{2}}}{3\sqrt{\pi}} - \frac{t^2}{4}\right) + \frac{t}{2}u(t), \quad t \in [0,1] \\ \sum_{i=1}^{n} \left(J_{0+,t}^{1-\alpha_i,\psi}\right)[u]\Big|_{t=0} = 0, \end{cases} \tag{4.5}$$

Here $n = 3$, $\alpha_1 = 1/8$, $\alpha_2 = 1/4$, $\alpha_3 = 1/2$ with $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$, $\psi(t) = t$, $\delta = 0$, and

$$f(t, u(t)) = 2t\left(\frac{t^{\frac{7}{8}}}{\Gamma(\frac{23}{8})} + \frac{t^{\frac{3}{4}}}{\Gamma(\frac{11}{4})} + \frac{4t^{\frac{1}{2}}}{3\sqrt{\pi}} - \frac{t^2}{4}\right) + \frac{t}{2}u(t). \tag{4.6}$$

It follows that the inequality

$$|f(t, u_2) - f(t, u_1)| \leq \frac{t}{2}|u_2 - u_1|, \quad \forall u_1, u_2 \in \mathbb{R}.$$

From (3.2) with the above inequality, we get the function $\gamma(t) = t/2$ is a bounded and monotonic increasing on $t \in [0, 1]$. It is easy to see that the function $u(t) = t^2$ is a solution of the initial value problem (4.5). Since all assumptions of Lemma 3.1 are satisfied, then the problem (4.5) has a unique solution on $[0, 1]$. Furthermore, we can also compute that the real number

$$C_\epsilon \leq \frac{2}{\sqrt{\pi}}\left[E_{\frac{3}{8}}(1) + E_{\frac{1}{4}}(1) + E_{\frac{1}{2}}(\frac{1}{2})\right].$$

Therefore, by Lemma 3.4, the problem (4.5) is Ulam-Hyers stable with respect to ψ on [0.1].

Example 4.3. *Consider the following fractional system with the ψ–fractional derivative of the form:*

$$\begin{cases} \left(D^{\frac{1}{2},t}_{0+,t}\right)[u] = \dfrac{1}{5} + \dfrac{t^2}{2}\cdot\dfrac{|u(t)|}{1+|u(t)|} \\ \left(J^{\frac{1}{2},t}_{0+,t}\right)[u]\Big|_{t=0} = \dfrac{1}{2} \end{cases} \quad \textit{and} \quad \begin{cases} \left(D^{\frac{1}{2},t}_{0+,t}\right)[v] = \dfrac{1}{5} + \dfrac{t^2}{4}\cdot\dfrac{|v(t)|}{1+|v(t)|} \\ \left(J^{\frac{1}{2},t}_{0+,t}\right)[v]\Big|_{t=0} = \dfrac{1}{3}. \end{cases} \tag{4.7}$$

Here $n = 1$, $\alpha_1 = 1/2$ with $\psi(t) = t$, $\delta_1 = 1/2$, $\delta_2 = 1/3$ and for $t \in [0, 1]$,

$$f(t, u(t)) = \frac{1}{5} + \frac{t^2}{2}\cdot\frac{|u(t)|}{1+|u(t)|} \quad \text{and} \quad g(t, v(t)) = \frac{1}{5} + \frac{t^2}{4}\cdot\frac{|v(t)|}{1+|v(t)|}.$$

It follows that the inequalities

$$\begin{aligned} |g(t, v_2(t)) - g(t, v_1(t))| &\le \frac{1}{4}|v_2(t) - v_1(t)|, \quad \forall t \in [0,1], \quad \forall v_1, v_2 \in \mathbb{R} \\ |f(t, u(t)) - g(t, u(t))| &\le \frac{t^2}{4}, \quad \forall t \in [0,1]. \end{aligned}$$

The assumption (A_1)–(A_2) of Lemma 3.2 are satisfied with the positive constant $c = 1/4 > 0$ and the continuous function $\chi(t) = t^2/4$. Then, for all $t \in [0, 1]$, we have the following inequality

$$|u(t) - v(t)| \le w(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\dots k'=1}^{n} \frac{\prod_{i=1}^{k}\left(g_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\sum_{i=1}^{k}\alpha_{i'}-1}\right] w(\tau)\, \mathrm{d}\tau\right),$$

where $w(t) = 1/(6\sqrt{t\pi}) + (4t^{5/2})/(15\sqrt{\pi})$.

Example 4.4. *Consider the following fractional system of the form:*

$$\begin{cases} \displaystyle\sum_{i=1}^{2}\left(D^{\alpha_3-\alpha_i,\psi}_{0+,t}\right)\left[(e^t + i - 1)^{(1-2\lambda_i)}u^{\lambda_i}\right] + \left(D^{\alpha_3,\psi}_{0+,t}\right)\left[u^{\lambda_0}\right] = \frac{1}{2} + \frac{1}{6}(e^t+2)^{(1-2\lambda_3)}u^{\frac{1}{6}}(t), \quad t \in [0,1], \\ \displaystyle\sum_{i=1}^{2}\left(J^{1-\alpha_3+\alpha_i,\psi}_{0+,t}\right)\left[(t+i)^2u^{\lambda_i}\right] + \left(J^{1-\alpha_3,\psi}_{0+,t}\right)\left[u^{\lambda_0}\right]\Big|_{t=0} = 0, \end{cases} \tag{4.8}$$

Here $n = 2$, $\alpha_1 = 1/2$, $\alpha_2 = 2/3$, $\alpha_3 = 3/4$, $\psi(t) = e^t$, $\delta = 0$, $h_i(t) = (e^t + i - 1)^{1-2\lambda_i}$, for $i = 1, 2$ and

$$f(t, u(t)) = \frac{1}{2} + \frac{1}{6}(e^t + 2)^{(1-2\lambda_3)}u^{\frac{1}{6}}(t).$$

For u_1, $u_2 \in \mathbb{R}$, we have

$$|f(t, u_2(t)) - f(t, u_1(t))| \le \frac{1}{6}(e^t + 2)^{(1-2\lambda_3)}|u_2^{\frac{1}{6}}(t) - u_1^{\frac{1}{6}}(t)|.$$

The assumption (A_3) of Lemma 3.3 is satisfied with the continuous function $h_3(t) = (1/6)(e^t + 2)^{(1-2\lambda_3)} > 0$ for $t \in [0, 1]$ and the constant $\lambda_3 = 1/6 \in (0, 1)$.

(i) If we set $\lambda_0 = 1/2$, $\lambda_1 = 1/3$ and $\lambda_2 = 1/4$, we get that $\max\{\lambda_1, \lambda_2, \lambda_3\} < \lambda_0$. Then, for any solution $u(t)$ of the problem (4.8), we can estimate that

$$|u(t)|^{\frac{1}{2}} \leq \tilde{v}(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',3',\ldots,k'=1}^{3} \frac{\prod_{i=1}^{k} \left(\tilde{g}_{i'}(t)\Gamma(\alpha_{i'})\right)}{\Gamma\left(\sum_{i=1}^{k}\alpha_{i'}\right)} \int_a^t \left[\psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\sum_{i=1}^{k}\alpha_{i'}-1}\right] \tilde{v}(\tau)\,\mathrm{d}\tau \right),$$

where

$$\tilde{v}(t) = v(t) + \sum_{i=1}^{3} C_i(\varepsilon)\, g_i(t) \int_0^t \psi'(\tau)\left(\mathcal{K}(t;\tau)\right)^{\alpha_i - 1} \left[h_i(\tau)\right]^{1/(1-2\lambda_i)} \mathrm{d}\tau,$$

and

$$v(t) = \left(J_{a+,t}^{\alpha_{n+1},\psi}\right)[|f_0|] = \frac{1}{2}\left(J_{0+,t}^{\alpha_3,e^t}\right)[1] = \frac{2}{3}\frac{e^t - 1}{\Gamma\left(\frac{3}{4}\right)}.$$

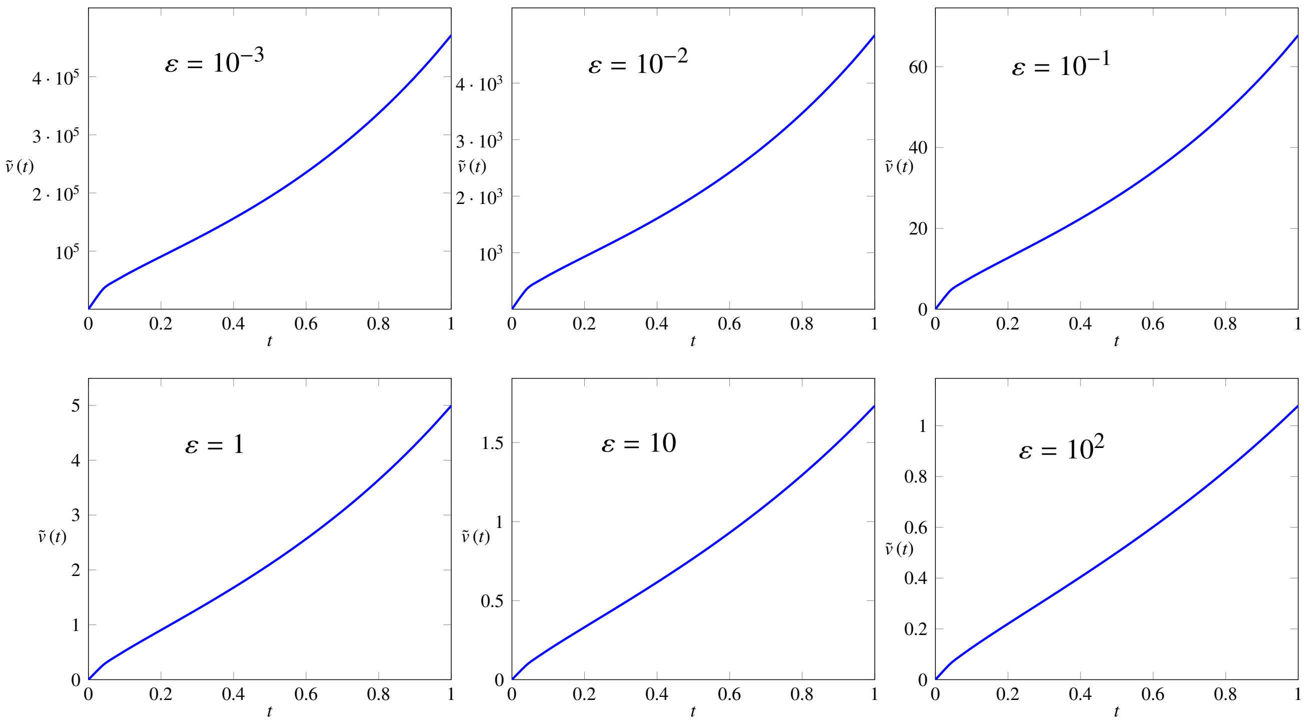

Figure 2. Graphs of the function $\tilde{v}(t) = \tilde{v}(t,\varepsilon)$ for $\varepsilon = 10^{-3}, 10^{-2}, 10^{-1}, 1, 10$ and 10^2.

Since $v(t)$ is nonnegative and increasing, $\tilde{v}(t)$ is also nonnegative and increasing, where

$$C_i(\varepsilon) = (1 - 2\lambda_i)\left(\frac{2\lambda_i}{\varepsilon}\right)^{\frac{2\lambda_i}{1-2\lambda_i}}, \quad C_1(\varepsilon) = \frac{4}{27\varepsilon^2}, \quad C_2(\varepsilon) = \frac{1}{4\varepsilon}, \quad C_3(\varepsilon) = \frac{2\sqrt{3}}{9\sqrt{\varepsilon}}$$

and

$$\tilde{g}_{i'}(t) = \varepsilon g_i(t), \quad g_i(t) = \frac{1}{\Gamma(\alpha_i)}, \quad i = 1, 2, 3.$$

$$\tilde{v}(t) = q_1 + \frac{q_2}{\sqrt{\varepsilon}} + \frac{q_3}{\varepsilon} + \frac{q_4}{\varepsilon^2}$$

and

$$
\begin{aligned}
q_1 &= \frac{2}{3\Gamma\left(\frac{3}{4}\right)}\left(e^t-1\right)^{\frac{3}{4}}, \quad q_2=\frac{32\sqrt{3}}{189\Gamma\left(\frac{3}{4}\right)}\left(e^t+\frac{17}{4}\right)\left(e^t-1\right)^{\frac{3}{4}},\\
q_3 &= \frac{9}{40\Gamma\left(\frac{2}{3}\right)}\left(e^t+\frac{7}{3}\right)\left(e^t-1\right)^{\frac{2}{3}}, \quad q_4=\frac{16}{81\sqrt{\pi}}\left(e^t+\frac{1}{2}\right)\left(e^t-1\right)^{\frac{1}{2}}.
\end{aligned}
$$

In Figure 2, we plotted the graph of $\tilde{v}(t)=\tilde{v}(t,\varepsilon)$ for $t\in[0,1)$ with $\varepsilon=10^{-3},10^{-2},10^{-1},1,10$ and 10^2.

(ii) If we set $\lambda_0=1/2$, $\lambda_1=1/2$ and $\lambda_2=3/4$, we get that $\min\{\lambda_1,\lambda_2,\lambda_3\}\geq\lambda_0$. Then the continuous solution of problem (4.8) is unique.

5. Conclusions

In this paper, we introduced new generalizations for Gronwall's inequality within the ψ–fractional integral operators. The results of this paper provide general forms of Gronwall's inequality that include the forms obtained in [12, 30, 31]. Furthermore, Gronwall's inequalities involving fractional integrals of Riemann-Liouville, Hadamard and Katugampola types as well as fractional integrals of a function with respect to another function are recovered for particular cases of function ψ.

To examine the validity and applicability of our results, we discussed the existence and uniqueness of solutions of ψ–fractional initial and boundary value problems which are an important and useful contributions to the existing theory. On the other hand, the stability of ψ–fractional differential equations was studied via the obtained generalized ψ–Gronwall's inequality. Interesting examples are discussed at the end for the sake of confirming the results.

Reported results in this paper can be considered as a promising contribution to the theory of fractional integral inequalities. These results can be used to study and develop further quantitative and qualitative properties of generalized fractional differential equations.

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Conflict of interest

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally and significantly to this paper. All authors have read and approved the final version of the manuscript.

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