Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 5008–5019.
DOI: 10.3934/math.2021295
Received: 19 December 2020
Accepted: 23 February 2021
Published: 04 March 2021

Research article

Subclass of analytic functions defined by q-derivative operator associated with Pascal distribution series

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Abstract: The purpose of the present paper is to find the necessary and sufficient condition and inclusion relation for Pascal distribution series to be in the subclass $\mathcal{TC}_q(\lambda,\alpha)$ of analytic functions defined by q-derivative operator. Further, we consider an integral operator related to Pascal distribution series, and several corollaries and consequences of the main results are also considered.

Keywords: analytic functions; Hadamard product; q-starlike functions; q-convex functions; Pascal distribution series
Mathematics Subject Classification: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n, \qquad z \in \mathbb{U}. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau \in \mathbb{C}\backslash\{0\}$, $-1 \le B < A \le 1$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]}\right| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [13].

The theory of q-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, as well as geometric function theory of complex analysis. The application of q-calculus was initiated by Jackson [23]. Recently, many researchers studied q-calculus such as Srivastava et al. [52], Muhammad and Darus [31], Kanas and Răducanu [28], Aldweby and Darus [2–4] and Muhammad and Sokol [30]. For details on q-calculus one can refer [1, 5–7, 9, 20, 23, 25, 38, 39, 43, 44, 46, 48–51] and also the reference cited therein.

For $0 < q < 1$ the Jackson's q-derivative of a function $f \in \mathcal{A}$ is, by definition, given as follows [23]

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0, \end{cases} \tag{1.3}$$

and

$$D_q^2 f(z) = D_q(D_q f(z)).$$

From (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \tag{1.4}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \tag{1.5}$$

is sometimes called *the basic number n*. If $q \to 1-$, $[n]_q \to n$.

For a function $h(z) = z^n$, we obtain

$$D_q h(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

and

$$\lim_{q \to 1-} D_q h(z) = \lim_{q \to 1-} \left([n]_q z^{n-1}\right) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative.

Using the above defined q-calculus, several subclasses belonging to the class $\mathcal{A}$ have already been investigated in geometric function theory. Ismail et al. [26] were the first who used the q-derivative operator D_q to study the q-calculus analogous of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 1.1 below). However, a firm footing of the q-calculus in the context of geometric function theory was presented mainly and basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, ([45], p.347 et seq.); see also [46]).

For $0 < q < 1$, we define the class $\mathcal{S}_q^*(\alpha)$ of q- starlike functions and the class $C_q(\alpha)$ of q- convex functions of order $\alpha (0 \leq \alpha < 1)$ (see, [26, 40, 41]), as below:

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^*(\alpha)$ if it satisfies

$$\Re\left(\frac{z D_q f(z)}{f(z)}\right) > \alpha, \ (z \in \mathbb{U}).$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $C_q(\alpha)$ if it satisfies

$$\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) > \alpha,\ (z \in \mathbb{U}).$$

It is clear that $\lim\limits_{q\to 1-} \mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\lim\limits_{q\to 1-} C_q(\alpha) = C(\alpha)$, where $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ are, respectively, well-known starlike and convex functions of order α in $\mathbb{U}$.

We now introduce a new subclass of analytic functions defined by q-derivative operator D_q.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $C_q(\lambda, \alpha)$ if it satisfies

$$\Re\left(\frac{\lambda z^3(zD_qf(z))''' + (2\lambda+1)z^2(zD_qf(z))'' + z\left(zD_qf(z)\right)'}{\lambda z^2(zD_qf(z))'' + z\left(zD_qf(z)\right)'}\right) > \alpha,\ (z \in \mathbb{U}) \tag{1.6}$$

where $0 \le \alpha < 1,\ 0 \le \lambda \le 1$.

We write

$$\mathcal{T}C_q(\lambda, \alpha) = C_q(\lambda, \alpha) \cap \mathcal{T}.$$

A variable X is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \dots$ with probabilities $(1-s)^m$, $\frac{sm(1-s)^m}{1!}$, $\frac{s^2m(m+1)(1-s)^m}{2!}$, $\frac{s^3m(m+1)(m+2)(1-s)^m}{3!}, \dots$, respectively, where s and m are called the parameters, and thus

$$P(X = k) = \binom{k+m-1}{m-1} s^k(1-s)^m,\ k = 0, 1, 2, 3, \dots.$$

Very recently, El-Deeb et al. [15] (see also, [10, 34]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi_s^m(z) := z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m z^n,\ z \in \mathbb{U},$$

where $m \ge 1$, $0 \le s \le 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi_s^m(z) := 2z - \Psi_s^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m z^n,\ z \in \mathbb{U}. \tag{1.7}$$

Let consider the linear operator $\mathcal{I}_s^m : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}_s^m f(z) := \Psi_s^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m a_n z^n,\ z \in \mathbb{U},$$

where $m \ge 1$ and $0 \le s \le 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [8, 11, 21, 29, 42, 47]), generalized Bessel functions (see for example, [18, 22, 33, 36]), Struve functions (see for example, [12, 24]), Poisson distribution series (see for example, [14, 16, 19, 32, 35, 37]) and Pascal distribution series (see for example, [10, 15, 17, 34]), in this paper we determine the necessary and sufficient condition for Φ_s^m to be in the class $\mathcal{TC}_q(\lambda,\alpha)$. Furthermore, we give sufficient condition for $\mathcal{I}_s^m(\mathcal{R}^\tau(A,B)) \subset \mathcal{TC}_q(\lambda,\alpha)$ and finally, we give necessary and sufficient condition for the function f such that its image by the integral operator $\mathcal{G}_s^m f(z) = \int_0^z \frac{\Phi_s^m(t)}{t}dt$ belongs to the class $\mathcal{TC}_q(\lambda,\alpha)$.

To establish our main results, we need the following Lemmas.

Lemma 1.4. *A function f of the form (1.2) is in $\mathcal{TC}_q(\lambda,\alpha)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}[n]_q n(n-\alpha)(\lambda n-\lambda+1)\,|a_n| \le 1-\alpha, \tag{1.8}$$

where $0 \le \alpha < 1,\ 0 \le \lambda \le 1$ *and* $z \in \mathbb{U}$.

Lemma 1.4 can be proved using the same technique as in [27].

Lemma 1.5. *[13] If $f \in \mathcal{R}^\tau(A,B)$ is of the form (1.1) , then*

$$|a_n| \le (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N}\backslash\{1\}.$$

The result is sharp.

2. Necessary and sufficient condition for $\Phi_s^m \in \mathcal{TC}_q(\lambda,\alpha)$

For convenience throughout in the sequel, we use the following identities that hold for $m \ge 1$ and $0 \le s < 1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}s^n = \frac{1}{(1-s)^m}, \quad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}s^n = \frac{1}{(1-s)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}s^n = \frac{1}{(1-s)^{m+1}}, \quad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}s^n = \frac{1}{(1-s)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}s^{n-1} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}s^n - 1 = \frac{1}{(1-s)^m} - 1, \tag{2.1}$$

$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}s^{n-1} = sm\sum_{n=0}^{\infty}\binom{n+m}{m}s^n = s\frac{\binom{m}{m-1}}{(1-s)^{m+1}}, \tag{2.2}$$

$$\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}s^{n-1} = 2s^2\frac{\binom{m+1}{m-1}}{(1-s)^{m+2}} \tag{2.3}$$

$$\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1}s^{n-1} = 6s^3\frac{\binom{m+2}{m-1}}{(1-s)^{m+3}} \tag{2.4}$$

and

$$\sum_{n=5}^{\infty}(n-1)(n-2)(n-3)(n-4)\binom{n+m-2}{m-1}s^{n-1} = 24s^4\frac{\binom{m+3}{m-1}}{(1-s)^{m+4}}. \tag{2.5}$$

Unless otherwise mentioned, we shall assume in this paper that $0 \le \alpha < 1$ and $0 \le \lambda \le 1, 0 < q < 1$ and $0 \le s < 1$.

Firstly, we obtain the necessary and sufficient conditions for Φ_s^m to be in the class $\mathcal{TC}_q(\lambda, \alpha)$.

Theorem 2.1. *Let $m \ge 1$ and $q \to 1-$.Then $\Phi_s^m \in \mathcal{TC}_q(\lambda, \alpha)$ if and only if*

$$\begin{aligned} & 24\lambda\frac{\binom{m+3}{m-1}s^4}{(1-s)^{m+4}} + 6(\lambda(9-\alpha)+1)\frac{\binom{m+2}{m-1}s^3}{(1-s)^{m+3}} + 2(4\lambda(2-\alpha)+7-3\alpha)\frac{\binom{m+1}{m-1}s^2}{(1-s)^{m+2}} \\ & (4\lambda(2-\alpha)+7-3\alpha)\frac{\binom{m}{m-1}s}{(1-s)^{m+1}} \\ \le\ & 1-\alpha. \end{aligned} \tag{2.6}$$

Proof. Since Φ_s^m is defined by (1.7), in view of Lemma 1.4 it is sufficient to show that

$$P_q := \sum_{n=2}^{\infty}[n]_q n(n-\alpha)(\lambda n-\lambda+1)\binom{n+m-2}{m-1}s^{n-1}(1-s)^m \le 1-\alpha.$$

Since $[n]_q \to n$, when $q \to 1-$, we get

$$\begin{aligned} P_1 &= \sum_{n=2}^{\infty} n^2(n-\alpha)(\lambda n-\lambda+1)\binom{n+m-2}{m-1}s^{n-1}(1-s)^m \\ &= \sum_{n=2}^{\infty}\left[\lambda n^4 + (1-\lambda-\alpha\lambda)\,n^3 + \alpha(\lambda-1)n^2\right]\binom{n+m-2}{m-1}s^{n-1}(1-s)^m. \end{aligned}$$

Writing

$$n^2 = (n-1)(n-2)+3(n-1)+1, \tag{2.7}$$

$$n^3 = (n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1, \tag{2.8}$$

$$\begin{aligned} n^4 &= (n-1)(n-2)(n-3)(n-4)+10(n-1)(n-2)(n-3) \\ &\quad +25(n-1)(n-2)+15(n-1)+1, \end{aligned} \tag{2.9}$$

and using (2.2)–(2.5), we have

$$
\begin{aligned}
P_1 &= \sum_{n=2}^{\infty}\left[\lambda n^4 + (1-\lambda-\alpha\lambda)\, n^3 + \alpha(\lambda-1)n^2\right]\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&= \lambda\sum_{n=5}^{\infty}(n-1)(n-2)(n-3)(n-4)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&\quad +(\lambda(9-\alpha)+1)\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&\quad +(\lambda(19-5\alpha)+6-\alpha)\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&\quad +(4\lambda(2-\alpha)+7-3\alpha)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&\quad +(1-\alpha)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\
&= 24\lambda\frac{\binom{m+3}{m-1}s^4}{(1-s)^4} + 6(\lambda(9-\alpha)+1)\frac{\binom{m+2}{m-1}s^3}{(1-s)^3} + 2(4\lambda(2-\alpha)+7-3\alpha)\frac{\binom{m+1}{m-1}s^2}{(1-s)^2} \\
&\quad (4\lambda(2-\alpha)+7-3\alpha)\frac{\binom{m}{m-1}s}{1-s} + (1-\alpha)(1-(1-s)^m).
\end{aligned}
$$

but this last expression is upper bounded by $1-\alpha$ if and only if (2.6) holds. □

3. Sufficient condition for $\mathcal{I}_s^m(\mathcal{R}^\tau(A,B)) \subset \mathcal{TC}_q(\lambda,\alpha)$

Making use of Lemma 1.5, we will study the action of the Pascal distribution series on the class $\mathcal{TC}_q(\lambda,\alpha)$.

Theorem 3.1. *Let $m \geq 1$ and $q \to 1-$ .If $f \in \mathcal{R}^\tau(A,B)$ and the inequality*

$$
\begin{aligned}
&(A-B)|\tau|\Bigg[6\lambda s^3\frac{\binom{m+2}{m-1}}{(1-s)^3} + 2(\lambda(5-\alpha)+1)s^2\frac{\binom{m+1}{m-1}}{(1-s)^2} \\
&\quad +(2\lambda(2-\alpha)+3-\alpha)\frac{\binom{m}{m-1}s}{1-s} + (1-\alpha)(1-(1-s)^m)\Bigg] \\
&\leq 1-\alpha. \qquad (3.1)
\end{aligned}
$$

is satisfied then $\mathcal{I}_s^m f \in \mathcal{TC}_q(\lambda,\alpha)$.

Proof. According to Lemma 1.4 it is sufficient to show that

$$
Q_q := \sum_{n=2}^{\infty}[n]_q n(n-\alpha)(\lambda n-\lambda+1)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m|a_n| \leq 1-\alpha. \qquad (3.2)
$$

Since $f \in \mathcal{R}^{\tau}(A, B)$, using Lemma 1.5 we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n},\ n \in \mathbb{N} \setminus \{1\},$$

therefore

$$\begin{aligned} Q_1 &\leq (A-B)|\tau| \left[\sum_{n=2}^{\infty} n(n-\alpha)(\lambda n - \lambda + 1)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \right] \\ &= (A-B)|\tau| \left[\sum_{n=2}^{\infty} \left[\lambda n^3 + (1-\lambda-\alpha\lambda) n^2 + \alpha(\lambda-1)n \right] \binom{n+m-2}{m-1} s^{n-1}(1-s)^m \right]. \end{aligned}$$

Writing n^2, n^3 as given in (2.7) and (2.8), $n = n-1+1$, and making use of (2.2)–(2.5), we get

$$\begin{aligned} Q_1 \leq\ & (A-B)|\tau| \left[\lambda \sum_{n=4}^{\infty} (n-1)(n-2)(n-3)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \right. \\ & + (\lambda(5-\alpha)+1) \sum_{n=3}^{\infty} (n-1)(n-2)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\ & + (2\lambda(2-\alpha)+3-\alpha) \sum_{n=2}^{\infty} (n-1)\binom{n+m-2}{m-1} s^{n-1}(1-s)^m \\ & \left. + (1-\alpha) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m \right] \\ =\ & (A-B)|\tau| \left[6\lambda s^3 \frac{\binom{m+2}{m-1}}{(1-s)^3} + 2(\lambda(5-\alpha)+1)s^2 \frac{\binom{m+1}{m-1}}{(1-s)^2} \right. \\ & \left. + (2\lambda(2-\alpha)+3-\alpha)\frac{\binom{m}{m-1}s}{1-s} + (1-\alpha)(1-(1-s)^m) \right]. \end{aligned}$$

but this last expression is upper bounded by $1-\alpha$ if and only if (3.1) holds. □

4. Integral operator

Theorem 4.1. *Let $m \geq 1$ and $q \to 1-$ .If the integral operator $\mathcal{G}_s^m$ is given by*

$$\mathcal{G}_s^m(z) := \int_0^z \frac{\Phi_s^m(t)}{t} dt,\ z \in \mathbb{U}, \tag{4.1}$$

then $\mathcal{G}_s^m \in \mathcal{TC}_q(\lambda, \alpha)$ if and only

$$\begin{aligned} & 6\lambda s^3 \frac{\binom{m+2}{m-1}}{(1-s)^{m+3}} + 2(\lambda(5-\alpha)+1)s^2 \frac{\binom{m+1}{m-1}}{(1-s)^{m+2}} \\ & + (2\lambda(2-\alpha)+3-\alpha)\frac{\binom{m}{m-1}s}{(1-s)^{m+1}} \\ \leq\ & 1-\alpha. \end{aligned} \tag{4.2}$$

Proof. According to (1.7) it follows that

$$\mathcal{G}_s^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m \frac{z^n}{n},\ z \in \mathbb{U}.$$

Using Lemma 1.4, the function $\mathcal{G}_q^m(z)$ belongs to $\mathcal{TC}_q(\lambda, \alpha)$ if and only if

$$R_q := \sum_{n=2}^{\infty} [n]_q n(n-\alpha)(\lambda n - \lambda + 1) \times \frac{1}{n} \binom{n+m-2}{m-1} s^{n-1}(1-s)^m \leq 1-\alpha,$$

Now,

$$R_1 = \sum_{n=2}^{\infty} \left[\lambda n^3 + (1-\lambda-\alpha\lambda)\, n^2 + \alpha(\lambda-1)n\right] \binom{n+m-2}{m-1} s^{n-1}(1-s)^m$$

By a similar proof like those of Theorem 3.1 we get that $\mathcal{G}_s^m f \in \mathcal{TC}_q(\lambda, \alpha)$ if and only if (4.2) holds. □

5. Corollaries and consequences

Corollary 5.1. *Let* $m \geq 1$ *and* $q \to 1-$ *.Then* $\Phi_s^m \in \mathcal{TC}_q(0, \alpha)$*, if and only if*

$$6\frac{\binom{m+2}{m-1} s^3}{(1-s)^{m+3}} + 2(7-3\alpha)\frac{\binom{m+1}{m-1} s^2}{(1-s)^{m+2}} + (7-3\alpha)\frac{\binom{m}{m-1} s}{(1-s)^{m+1}} \leq 1-\alpha.$$

Corollary 5.2. *Let* $m \geq 1$ *and* $q \to 1-$ *. If* $f \in \mathcal{R}^{\tau}(A, B)$ *and the inequality*

$$(A-B)|\tau| \left[2\frac{\binom{m+1}{m-1} s^2}{(1-s)^2} + (3-\alpha)\frac{\binom{m}{m-1} s}{1-s} + (1-\alpha)(1-(1-s)^m)\right] \leq 1-\alpha.$$

is satisfied then $\mathcal{I}_s^m f \in \mathcal{TC}_q(0, \alpha)$.

Corollary 5.3. *Let* $m \geq 1$ *and* $q \to 1-$ *. If the integral operator* $\mathcal{G}_s^m$ *is given by (4.1), then* $\mathcal{G}_s^m \in \mathcal{TC}_q(0, \alpha)$ *if and only*

$$2s^2 \frac{\binom{m+1}{m-1}}{(1-s)^{m+2}} + (3-\alpha)\frac{\binom{m}{m-1} s}{(1-s)^{m+1}} \leq 1-\alpha.$$

6. Conclusions

In this paper, we find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in a subclass of analytic functions defined by q-derivative operator. Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [[45], pp.350–351] and [[44], p.328]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [44], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [[44], p.340]). This observation by Srivastava [44] will indeed apply also to any attempt to produce the rather straightforward (p, q)-variations of the results which we have presented in this paper.

Acknowledgements

This research was funded by Universiti Kebangsaan Malaysia, grant number GUP-2019-032. The authors would like to thank the referees for their helpful comments and suggestions.

Conflicts of interest

The authors declare no conflict of interest.

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