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Research article

New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operators

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Abstract: In this work, a new strategy to derive inequalities by employing newly proposed fractional operators, known as a Hilfer generalized proportional fractional integral operator ($\widehat{\mathcal{GPFIO}}$). The presented work establishes a relationship between weighted extended Čebyšev version and Pólya-Szegö type inequalities, which can be directly used in fractional differential equations and statistical theory. In addition, the proposed technique is also compared with the existing results. This work is important and timely for evaluating fractional operators and predicting the production of numerous real-world problems in varying nature.

Keywords: integral inequality; Λ-generalized proportional fractional integral; Čebyšev inequality; Pólya-Szegö inequality
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1. Introduction

Fractional calculus has been extraordinarily improved as a result of the innovation and application of classical mathematics. Several analysts have demonstrated that the image processing with newly proposed calculus can depict the model more precisely rather than the classical images with fractional operators [1–3]. Resultantly, fractional calculus has been broadly utilized in the scientific displaying of issues in different scientific areas [4] and technology [5–7]. Several definitions/approaches, for example, Riemann-Liouville, Hadamard, Katugampola, Riesz, Caputo-Fabrizio, Grunwald-Letnikov and Atangana-Baleanu analytics, are presented and examined in a wide assortment of theory, see [1–3, 6–20]. Many significant methodologies have been utilized to attain the analytical solutions of fractional-order differential equations, for example, the Laplace, Mellin, Fourier, and Hankel transforms are acquainted. However, the fractional-order differential equations established from natural are regularly nonlinear and incredibly complicated, and many of them cannot attain the exact analytic solutions. Consequently, various fractional calculus has dominating features of depicting dynamic framework, moreover, they have few impediments. For instance, they can tackle smoothly differentiable and integral operators. Recently, another methodology, which was initially proposed by Jarad et al. [7], to determine nondifferentiable issues in a fractional Schrödinger equation, and its significant properties were created. Later on, Rashid et al. [21] proposed more general version of $\widehat{\mathcal{GPFIO}}$ has become progressively famous and attained significant progression due predominantly to its remarkable properties in demonstrating complex nonlinear dynamical frameworks in various parts of scientific material science, such as integrodifferential equations, heat transforms, probability density functions, and others.

Fractional integral inequalities have been found in the fields of engineering and physics. Fractional integral variants perform an imperative role in understanding the universe, and there are many direct approaches to find the uniqueness and existence of the linear and nonlinear differential equations, [22, 23]. Based on fractional operators, one derives several generalizations of the Hermite-Hadamard, Hardy, Salter, Ostrowski, Čebyšev, and Pólya-Szegö have taken an important place in pure and applied mathematics [24–34]. Furthermore, Rashid et al. [35, 36], Zaheer et al. [48], Chu et al. [49] and Set et al. [50, 51] contributed significantly in this field. For more information about inequalities on the fractional operators, we referred to the interested readers, see [21, 37–47].
Čebyšev [52] introduced the well-known celebrated functional for two integrable functions is stated as

$$\mathfrak{T}(\mathcal{F},\mathcal{G}) = \frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{F}(\varrho)\mathcal{G}(\varrho)d\varrho - \Big(\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{F}(\varrho)d\varrho\Big)\Big(\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{G}(\varrho)d\varrho\Big), \tag{1.1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are two integrable functions on $[q_1, q_2]$. If $\mathcal{F}$ and $\mathcal{G}$ are synchronous, i.e.,

$$(\mathcal{F}(\varrho) - \mathcal{F}(\omega))(\mathcal{G}(\varrho) - \mathcal{G}(\omega)) \geq 0,$$

for any $\varrho, \omega \in [q_1, q_2]$, then $\mathfrak{T}(\mathcal{F},\mathcal{G}) \geq 0$. The functional (1.1) has attracted many researchers attention due mainly to its revealed presentations in statistical theory, numerical analysis, transform theory and in decision-making analysis.

Besides aspects with abundant utilities, the functional (1.1) has been expanded plenteous of concentration to produce a diversity of essential variants (see, for example, [53, 54]). Various illustrious kinds stated in the literature are direct effects of the abundant tenders in optimizations and transform theory. In this concern, Pólya-Szegö integral inequality is one of the most celebrated inequality. In [55], Pólya-Szegö contemplated the following variant as follows:

$$\frac{\int_{q_1}^{q_2}\mathcal{F}^2(\varrho)d\varrho\int_{q_1}^{q_2}\mathcal{G}^2(\varrho)d\varrho}{\Big(\int_{q_1}^{q_2}\mathcal{F}(\varrho)\mathcal{G}(\varrho)d\varrho\Big)^2} \leq \frac{1}{4}\Bigg(\sqrt{\frac{QR}{qr}} + \sqrt{\frac{qr}{QR}}\Bigg)^2. \tag{1.2}$$

The constant $\frac{1}{4}$ is best feasible in (1.2) make the experience it cannot get replaced by a smaller constant.

It is extensively identified that the aforesaid variants in both continuous and discrete forms show a substantial job in inspecting the qualitative demeanor of differential/difference equations, respectively, further to numerous new branches of mathematics. Motivated by [52, 55], our intention is to demonstrate more wide description of Pólya-Szegö and Čebyšev type variants via Hilfer-$\widehat{\mathcal{GPFIO}}$.

In this paper, motivated and inspired by the ongoing research in this field, some novel weighted extensions of Čebyšev and Pólya-Szegö type inequalities are governed in the frame of Hilfer-$\widehat{\mathcal{GPFIO}}$ are developed. Several new generalizations are introduced which plays a crucial role in our investigations. More precisely, under some working assumptions and using extended Čebyšev functional, the $\widehat{\mathcal{GPFIO}}$ for the considered variants are studied. Here, characterization results are formulated and proved. Future research should focus on Hilfer-$\widehat{\mathcal{GPFIO}}$ through novel outcomes and extension of the existing results with permeable fields of science.

In this section, we demonstrate the basic notions and related preliminaries concerning to fractional calculus [10].

Now, we describe a new fractional operator which is known as the the Λ-generalized proportional fractional integral which is proposed by Rashid et al. [21].

Definition 1.1. ([21]) Let $(q_1, q_2)(-\infty \leq q_1 < q_2 \leq \infty)$ be a finite or infinite real interval with $\varphi > 0$. Let a positive monotone and increasing function $\Lambda(\upsilon)$ defined on $(q_1, q_2]$ such that $\Lambda(0) = 0$ and $\Lambda'(\upsilon)$ is continuous on $[q_1, q_2)$. Then the left and right-sided Hilfer-$\widehat{\mathcal{GPFIO}}$ of a function $\mathcal{F}$ are presented as follows:

$$({}^{\Lambda}\mathcal{J}_{q_1}^{\varphi,\epsilon}\mathcal{F})(\varrho) = \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{q_1}^{\varrho}\frac{\exp[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))]\Lambda'(\upsilon)}{(\Lambda(\varrho)-\Lambda(\upsilon))^{1-\varphi}}\mathcal{F}(\upsilon)d\upsilon, \quad q_1 < \varrho \tag{1.3}$$

and

$$({}^{\Lambda}\mathcal{J}_{q_2}^{\varphi,\epsilon}\mathcal{F})(\varrho) = \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{\varrho}^{q_2}\frac{\exp[\frac{\epsilon-1}{\epsilon}(\Lambda(\upsilon)-\Lambda(\varrho))]\Lambda'(\upsilon)}{(\Lambda(\upsilon)-\Lambda(\varrho))^{1-\varphi}}\mathcal{F}(\upsilon)d\upsilon, \quad \varrho < q_2, \tag{1.4}$$

where the proportionality index $\epsilon \in (0, 1], \varphi \in \mathbf{C}, \Re(\varphi) > 0$, and $\Gamma(\varrho) = \int_0^{\infty} x^{\varrho-1}e^{-x}d\upsilon$ is the Gamma function.

Remark 1. In Definition 1.2:

(1) If we consider $\Lambda(\upsilon) = \upsilon$, then we will attain both sided generalized proportional fractional integral operator in [7].

(2) If we consider $\epsilon = 1$, then we will attain both sided generalized Riemann-Liouville fractional integral operator in [56].

(3) If we consider $\epsilon = 1$, along $\Lambda(\upsilon) = \upsilon$, then we will attain both sided Riemann-Liouville fractional integral operator in [10].

(4) If we consider $\Lambda(\upsilon) = \ln \upsilon$, then we will attain both sided generalized proportional Hadamard fractional integral operator in [54].

(5) If we consider $\Lambda(\upsilon) = \ln \upsilon$ along with $\epsilon = 1$, then we attain both sided Hadamard fractional integral operator [56].

Next, we present the one-sided definition of the Hilfer-$\widehat{\mathcal{GPFIO}}$ proposed by Rashid et al. [21].

Definition 1.2. ([21]) Let $(q_1, q_2)(-\infty \leq q_1 < q_2 \leq \infty)$ be a finite or infinite real interval with $\varphi > 0$. Let a positive monotone and increasing function $\Lambda(\upsilon)$ defined on $(q_1, q_2]$ such that $\Lambda(0) = 0$, and $\Lambda'(\upsilon)$ is continuous on $[q_1, q_2)$. Then the one sided Hilfer-$\widehat{\mathcal{GPFIO}}$ of a function $\mathcal{F}$ are presented as follows:

$$({}^{\Lambda}\mathcal{J}_{\upsilon_1}^{\varphi,\epsilon}\mathcal{F})(\varrho) = \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\frac{\exp[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))]\Lambda'(\upsilon)}{(\Lambda(\varrho)-\Lambda(\upsilon))^{1-\varphi}}\mathcal{F}(\upsilon)d\upsilon, \quad 0 < \varrho. \tag{1.5}$$

2. Weighted extensions of Čebyšev functionals via Hilfer-$\widehat{\mathcal{GPFIO}}$

In the sequel, we derive some refinements for the weighted extensions of Čebyšev functionals via Hilfer-$\widehat{\mathcal{GPFIO}}$. In this continuation, we assume that $\Lambda(\upsilon)$ is a strictly increasing function on $(0, \infty)$ and $\Lambda'(\upsilon)$ is continuous, $0 \leq q_1 < q_2$ with the assumption that at any point $q_3 \in [q_1, q_2]$, we have $\Lambda(q_3) = 0$.

Theorem 2.1. *For $\epsilon \in (0, 1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0, \infty)$. Also, assume that there is a positive integrable function $\mathcal{W}_1$ defined on $[0, \infty)$ such that $\mathcal{F}' \in L^s([0, \infty)), \mathcal{G}' \in L^r([0, \infty))$ for $s, r, u > 1$ having $\frac{1}{s} + \frac{1}{s_1} = 1$, $\frac{1}{r} + \frac{1}{r_1} = 1$, and $\frac{1}{u} + \frac{1}{u_1} = 1$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0, \infty)$ having $\Lambda(0) = 0$. Then the following variant grips for all $\varrho > 0$*

$$\begin{aligned}
&2\Big|({}^{\Lambda}\mathcal{J}_{0^+,\varrho}^{\epsilon,\varphi}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}_{0^+,\varrho}^{\epsilon,\varphi}\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}_{0^+,\varrho}^{\epsilon,\varphi}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}_{0^+,\varrho}^{\epsilon,\varphi}\mathcal{W}_1\mathcal{G})(\varrho)\Big| \\
&\leq \Big(\frac{\|\mathcal{F}'\|_s^u}{\epsilon^{u\varphi}\Gamma^u(\varphi)}\int_0^{\varrho}\int_0^{\varrho}\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big] \\
&\quad\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u}} \\
&\times\Big(\frac{\|\mathcal{G}'\|_r^{u_1}}{\epsilon^{u_1\varphi}\Gamma^{u_1}(\varphi)}\int_0^{\varrho}\int_0^{\varrho}\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big] \\
&\quad\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u_1}} \\
&\leq \frac{\|\mathcal{F}'\|_s^u\|\mathcal{G}'\|_r^{u_1}}{(\epsilon^{\varphi}\Gamma(\varphi))^2}\Big(\int_0^{\varrho}\int_0^{\varrho}\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big] \\
&\quad\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big).
\end{aligned} \tag{2.1}$$

Proof. Let us suppose the function

$$\mathcal{H}(\upsilon,\bar{\upsilon}) = (\mathcal{F}(\upsilon)-\mathcal{F}(\bar{\upsilon}))(\mathcal{G}(\upsilon)-\mathcal{G}(\bar{\upsilon})); \quad \upsilon,\bar{\upsilon}\in(0,\varrho), \tag{2.2}$$

which can be written as

$$\mathcal{H}(\upsilon,\bar{\upsilon}) = \mathcal{F}(\upsilon)\mathcal{G}(\upsilon)-\mathcal{F}(\upsilon)\mathcal{G}(\bar{\upsilon})-\mathcal{F}(\bar{\upsilon})\mathcal{G}(\upsilon)-\mathcal{G}(\bar{\upsilon})\mathcal{F}(\bar{\upsilon}). \tag{2.3}$$

Conducting produt on both sides of (2.2) by $\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)$ and then integrating the estimates with respect to υ over $(0,\varrho)$, we have

$$\begin{aligned}
&\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{H}(\upsilon,\bar{\upsilon})d\upsilon\\
&=\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{F}(\upsilon)\mathcal{G}(\upsilon)d\upsilon\\
&-\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{F}(\upsilon)\mathcal{G}(\bar{\upsilon})d\upsilon\\
&-\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{F}(\bar{\upsilon})\mathcal{G}(\upsilon)d\upsilon\\
&-\mathcal{G}(\bar{\upsilon})\mathcal{F}(\bar{\upsilon})\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)d\upsilon,
\end{aligned} \tag{2.4}$$

arrives at

$$\begin{aligned}
&\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{H}(\upsilon,\bar{\upsilon})d\upsilon\\
&=({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho)-\mathcal{G}(\bar{\upsilon})({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)\\
&\quad-\mathcal{F}(\bar{\upsilon})({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)+\mathcal{F}(\bar{\upsilon})\mathcal{G}(\bar{\upsilon})({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho).
\end{aligned} \tag{2.5}$$

Again, taking product both sides of (2.5) by $\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\right](\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\mathcal{W}_1(\bar{\upsilon})$ and then performing integration for the variable $\bar{\upsilon}$ over $(0,\varrho)$, we have

$$\begin{aligned}
&\frac{1}{(\epsilon^{\varphi}\Gamma(\varphi))^2}\int_0^{\varrho}\int_0^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right]\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\right]\\
&\quad\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})\mathcal{H}(\upsilon,\bar{\upsilon})d\upsilon d\bar{\upsilon}\\
&=2\Big(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho)-({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big).
\end{aligned} \tag{2.6}$$

Moreover, alternately, we have

$$\mathcal{H}(\upsilon,\bar{\upsilon}) = \int_x^y \int_x^y \mathcal{F}'(\theta)\mathcal{G}'(\vartheta)d\theta d\vartheta. \tag{2.7}$$

Taking into account the Hölder inequality, we ahve

$$|\mathcal{F}(\upsilon) - \mathcal{F}(\bar{\upsilon})| \leq |\upsilon - \bar{\upsilon}|^{\frac{1}{s_1}} \Big| \int_x^y |\mathcal{F}'(\theta)|^s d\theta \Big|^{\frac{1}{s}} \tag{2.8}$$

and

$$|\mathcal{G}(\upsilon) - \mathcal{G}(\bar{\upsilon})| \leq |\upsilon - \bar{\upsilon}|^{\frac{1}{r_1}} \Big| \int_x^y |\mathcal{G}'(\vartheta)|^r d\vartheta \Big|^{\frac{1}{r}}. \tag{2.9}$$

Conducting product between (2.8) and (2.9), we get

$$\begin{aligned} |\mathcal{G}(\upsilon,\bar{\upsilon})| \quad & \leq |(\mathcal{F}(\upsilon) - \mathcal{F}(\bar{\upsilon}))(\mathcal{G}(\upsilon) - \mathcal{G}(\bar{\upsilon}))| \\ & \leq |\upsilon - \bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}} \Big| \int_x^y |\mathcal{F}'(\theta)|^s d\theta \Big|^{\frac{1}{s}} \Big| \int_x^y |\mathcal{G}'(\vartheta)|^r d\vartheta \Big|^{\frac{1}{r}}. \end{aligned} \tag{2.10}$$

Thus, from (2.6) and (2.10), we have

$$\begin{aligned} & 2\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big| \\ & = \frac{1}{(\epsilon^{\varphi}\Gamma(\varphi))^2} \int_0^{\varrho}\int_0^{\varrho} \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big] \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\ & \quad \times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\mathcal{G}(\upsilon,\bar{\upsilon})|d\upsilon d\bar{\upsilon} \\ & \leq \frac{1}{(\epsilon^{\varphi}\Gamma(\varphi))^2} \int_0^{\varrho}\int_0^{\varrho} \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big] \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\ & \quad \times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon}) \\ & \quad \times|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}} \Big| \int_x^y |\mathcal{F}'(\theta)|^s d\theta \Big|^{\frac{1}{s}} \Big| \int_x^y |\mathcal{G}'(\vartheta)|^r d\vartheta \Big|^{\frac{1}{r}} d\upsilon d\bar{\upsilon}. \end{aligned} \tag{2.11}$$

Further, taking into consideration the Hölder inequality for bivariate integral, we have

$$\begin{aligned} & 2\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big| \\ & \leq \frac{1}{(\epsilon^{\varphi}\Gamma(\varphi))^2}\Big(\int_0^{\varrho}\int_0^{\varrho} \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big] \exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \end{aligned}$$

$$\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})$$
$$\times|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}\Big|\int_x^y|\mathcal{F}'(\theta)|^s d\theta\Big|^{\frac{u}{s}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u}}$$
$$\times\Big(\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big]$$
$$\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})$$
$$\times|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}\Big|\int_x^y|\mathcal{G}'(\vartheta)|^r d\vartheta\Big|^{\frac{u_1}{r}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u_1}}. \tag{2.12}$$

Now, using the following properties

$$\Big|\int_x^y|\mathcal{F}'(\theta)|^s d\theta\Big|^{\frac{1}{s}}\le\|\mathcal{F}'\|_s \qquad \textit{and} \qquad \Big|\int_x^y|\mathcal{G}'(\vartheta)|^r d\vartheta\Big|^{\frac{1}{r}}\le\|\mathcal{G}'\|_r. \tag{2.13}$$

From (2.12), we have

$$2\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho)-({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big|$$
$$\le\Big(\frac{\|\mathcal{F}'\|_s^u}{\epsilon^{u\varphi}\Gamma^u(\varphi)}\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big]$$
$$\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u}}$$
$$\times\Big(\frac{\|\mathcal{G}'\|_r^{u_1}}{\epsilon^{u_1\varphi}\Gamma^{u_1}(\varphi)}\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big]$$
$$\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u_1}}. \tag{2.14}$$

Therefore, we conclude that

$$2\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho)-({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big|$$
$$\le\frac{\|\mathcal{F}'\|_s^u\|\mathcal{G}'\|_r^{u_1}}{(\epsilon^{\varphi}\Gamma(\varphi))^2}\Big(\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big]$$
$$\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big), \tag{2.15}$$

this is the desired inequality (2.16). □

Several notable special cases of Theorem 2 are discussed as follows.

(I) If we take $\Lambda(\upsilon)=\upsilon$ in Theorem 2, then we attain a new result for generalized proportional fractional integral operator.

Corollary 1. *For $\epsilon \in (0,1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Also, assume that there is a positive integrable function $\mathcal{W}_1$ defined on $[0,\infty)$ such that $\mathcal{F}' \in L^s([0,\infty)), \mathcal{G}' \in L^r([0,\infty))$ for $s, r, u > 1$ having $\frac{1}{s} + \frac{1}{s_1} = 1$, $\frac{1}{r} + \frac{1}{r_1} = 1$, and $\frac{1}{u} + \frac{1}{u_1} = 1$. Then the following varinat grips for all $\varrho > 0$*

$$
\begin{aligned}
&2\Big|(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho) - (\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big| \\
&\leq \Big(\frac{\|\mathcal{F}'\|_s^u}{\epsilon^{u\varphi}\Gamma^u(\varphi)}\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\upsilon)\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\bar{\upsilon})\big](\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1} \\
&\quad\times\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u}} \\
&\times\Big(\frac{\|\mathcal{G}'\|_r^{u_1}}{\epsilon^{u_1\varphi}\Gamma^{u_1}(\varphi)}\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\upsilon)\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\bar{\upsilon})\big](\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1} \\
&\quad\times\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u_1}} \\
&\leq \frac{\|\mathcal{F}'\|_s^u\|\mathcal{G}'\|_r^{u_1}}{(\epsilon^\varphi\Gamma(\varphi))^2}\Big(\int_0^\varrho\int_0^\varrho \exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\upsilon)\big]\exp\big[\frac{\epsilon-1}{\epsilon}(\varrho-\bar{\upsilon})\big](\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1} \\
&\quad\times\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big).
\end{aligned}
$$

(II) If we take $\Lambda(\upsilon) = \upsilon$ along with $\epsilon = 1$ in Theorem 2, then we get the new result for Riemann-Liouville fractional integral operator.

Corollary 2. *For $\varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Also, assume that there is a positive integrable function $\mathcal{W}_1$ defined on $[0,\infty)$ such that $\mathcal{F}' \in L^s([0,\infty)), \mathcal{G}' \in L^r([0,\infty))$ for $s, r, u > 1$ having $\frac{1}{s} + \frac{1}{s_1} = 1$, $\frac{1}{r} + \frac{1}{r_1} = 1$, and $\frac{1}{u} + \frac{1}{u_1} = 1$. Then the following varinat grips for all $\varrho > 0$*

$$
\begin{aligned}
&2\Big|(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho) - (\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)\Big| \\
&\leq \Big(\frac{\|\mathcal{F}'\|_s^u}{\Gamma^u(\varphi)}\int_0^\varrho\int_0^\varrho (\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1}\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u}} \\
&\quad\times\Big(\frac{\|\mathcal{G}'\|_r^{u_1}}{\Gamma^{u_1}(\varphi)}\int_0^\varrho\int_0^\varrho (\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1}\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big)^{\frac{1}{u_1}} \\
&\leq \frac{\|\mathcal{F}'\|_s^u\|\mathcal{G}'\|_r^{u_1}}{(\Gamma(\varphi))^2}\Big(\int_0^\varrho\int_0^\varrho (\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\varphi-1}\mathcal{W}_1(\upsilon)\mathcal{W}_1(\bar{\upsilon})|\upsilon-\bar{\upsilon}|^{\frac{1}{s_1}+\frac{1}{r_1}}d\upsilon d\bar{\upsilon}\Big),
\end{aligned}
$$

which is proposed by Dahmani et al. [32]

Remark 2. In Theorem 2:
(1) If we choose $\Lambda(\upsilon) = \frac{x^\alpha}{\alpha}$ along with $\varphi = 1$, then we get Theorem 3.1 of Tassaddiq et al. [57].

(2) If we choose $\Lambda(\upsilon) = \upsilon$ along with $\epsilon = \varphi = 1$, then we get result of Elezovic et al. [58].

Theorem 2.2. *For $\epsilon \in (0, 1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0, \infty)$. Also, assume that there are two positive integrable function $\mathcal{W}_1$ and $\mathcal{W}_2$ defined on $[0, \infty)$ such that $\mathcal{F}' \in L^s([0, \infty)), \mathcal{G}' \in L^r([0, \infty))$ for $s, r, u > 1$ having $\frac{1}{s} + \frac{1}{s_1} = 1$, $\frac{1}{r} + \frac{1}{r_1} = 1$, and $\frac{1}{u} + \frac{1}{u_1} = 1$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0, \infty)$ having $\Lambda(0) = 0$. Then the following variant grips for all $\varrho > 0$*

$$\begin{aligned}
&\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho) \\
&\quad -({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho) + ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F}\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)\Big| \\
&\leq \frac{\|\mathcal{F}\|_s\|\mathcal{G}\|_r}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\
&\times \quad (\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\Big|\upsilon-\bar{\upsilon}\Big|^{\frac{1}{s_1}+\frac{1}{r_1}}\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})d\upsilon d\bar{\upsilon}. \qquad (2.16)
\end{aligned}$$

Proof. Conducting product on both sides of (2.5) by $\frac{1}{\epsilon^{\zeta}\Gamma(\zeta)}\exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\big](\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\bar{\upsilon})\mathcal{W}_2(\bar{\upsilon})$ and integrating the estimates with respect to $\bar{\upsilon}$ over $(0, \varrho)$, we have

$$\begin{aligned}
&\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\
&\times \quad (\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})\mathcal{H}(\upsilon,\bar{\upsilon})d\upsilon d\bar{\upsilon} \\
&= ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho) \\
&\quad -({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho) + ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F}\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho). \qquad (2.17)
\end{aligned}$$

Taking modulas on both sides of (2.20), one obtains

$$\begin{aligned}
&\Big|({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho) - ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho) \\
&\quad -({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho) + ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F}\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)\Big| \\
&= \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\
&\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})\Big|\mathcal{H}(\upsilon,\bar{\upsilon})\Big|d\upsilon d\bar{\upsilon} \\
&\leq \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big] \\
&\times(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\Big|\upsilon-\bar{\upsilon}\Big|^{\frac{1}{s_1}+\frac{1}{r_1}}\Big| \\
&\times\int_x^y|\mathcal{F}'(\theta)|^s d\theta\Big|^{\frac{1}{s}}\Big|\int_x^y|\mathcal{G}'(\vartheta)|^r d\vartheta\Big|^{\frac{1}{r}}\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})d\upsilon d\bar{\upsilon}
\end{aligned}$$

$$= \frac{\|\mathcal{F}\|_s\|\mathcal{G}\|_r}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\Big]$$
$$\times\quad(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\zeta-1}\Lambda'(\upsilon)\Lambda'(\upsilon)\big|\upsilon-\bar{\upsilon}\big|^{\frac{1}{s_1}+\frac{1}{r_1}}\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})d\upsilon d\bar{\upsilon}. \quad (2.18)$$

□

Some special cases of Theorem 2.2 are stated as follows.
(I) If we choose $\Lambda(\upsilon)=\upsilon$, then we get a new result for $\widehat{\mathcal{GPFIO}}$.

Corollary 3. *For $\epsilon\in(0,1],\varphi\in\mathbf{C}$ with $\Re(\varphi)>0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Also, assume that there are two positive integrable function $\mathcal{W}_1$ and $\mathcal{W}_2$ defined on $[0,\infty)$ such that $\mathcal{F}'\in L^s([0,\infty)),\mathcal{G}'\in L^r([0,\infty))$ for $s,r,u>1$ having $\frac{1}{s}+\frac{1}{s_1}=1$, $\frac{1}{r}+\frac{1}{r_1}=1$, and $\frac{1}{u}+\frac{1}{u_1}=1$. Then the following varinat grips for all $\varrho>0$*

$$\Big|(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2)(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho)-(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)$$
$$-(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)+(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F}\mathcal{G})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)\Big|$$
$$\leq\frac{\|\mathcal{F}\|_s\|\mathcal{G}\|_r}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}\exp\Big[\frac{\epsilon-1}{\epsilon}(\varrho-\upsilon)\Big]\exp\Big[\frac{\epsilon-1}{\epsilon}(\varrho-\bar{\upsilon})\Big](\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\zeta-1}$$
$$\times\big|\upsilon-\bar{\upsilon}\big|^{\frac{1}{s_1}+\frac{1}{r_1}}\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})d\upsilon d\bar{\upsilon}. \quad (2.19)$$

(II) If we choose $\Lambda(\upsilon)=\upsilon$ along with $\epsilon=1$, then we get a result for Riemann-Liouville fractional integral operator.

Corollary 4. *For $\epsilon\in(0,1],\varphi\in\mathbf{C}$ with $\Re(\varphi)>0$ and let there are two differentiable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Also, assume that there are two positive integrable function $\mathcal{W}_1$ and $\mathcal{W}_2$ defined on $[0,\infty)$ such that $\mathcal{F}'\in L^s([0,\infty)),\mathcal{G}'\in L^r([0,\infty))$ for $s,r,u>1$ having $\frac{1}{s}+\frac{1}{s_1}=1$, $\frac{1}{r}+\frac{1}{r_1}=1$, and $\frac{1}{u}+\frac{1}{u_1}=1$. Then the following varinat grips for all $\varrho>0$*

$$\Big|(\mathcal{J}^{\zeta}_{0^+,\varrho}\mathcal{W}_2)(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F}\mathcal{G})(\varrho)-(\mathcal{J}^{\zeta}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{F})(\varrho)$$
$$-(\mathcal{J}^{\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F})(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1\mathcal{G})(\varrho)+(\mathcal{J}^{\zeta}_{0^+,\varrho}\mathcal{W}_2\mathcal{F}\mathcal{G})(\varrho)(\mathcal{J}^{\varphi}_{0^+,\varrho}\mathcal{W}_1)(\varrho)\Big|$$
$$\leq\frac{\|\mathcal{F}\|_s\|\mathcal{G}\|_r}{\Gamma(\varphi)\Gamma(\zeta)}\int_0^{\varrho}\int_0^{\varrho}(\varrho-\upsilon)^{\varphi-1}(\varrho-\bar{\upsilon})^{\zeta-1}\big|\upsilon-\bar{\upsilon}\big|^{\frac{1}{s_1}+\frac{1}{r_1}}\mathcal{W}_1(\upsilon)\mathcal{W}_2(\bar{\upsilon})d\upsilon d\bar{\upsilon}, \quad (2.20)$$

which is proposed by Dahmani et al. [32].

Remark 3. In Theorem 2.2:
(1) If we choose $\Lambda(\upsilon)=\frac{\upsilon^{\alpha}}{\alpha}$ along with $\varphi=1$, then we get Theorem 3.2 of Tassaddiq et al. [57].
(2) If we choose $\Lambda(\upsilon)=\upsilon$ along with $\epsilon=\varphi=1$, then we get result of Dahmani et al. [31].

3. Pólya-Szegö types inequalities involving the Hilfer-$\widehat{\mathcal{GPFIO}}$

In this section, we shall derive certain Pólya-Szegö type integral inequalities for real-valued integrable functions via Hilfer-$\widehat{\mathcal{GPFIO}}$ defined in (1.2).

Theorem 3.1. *For $\epsilon \in (0,1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0) = 0$. Assume that there exist four positive integrable functions $\Upsilon_1, \Upsilon_2, \chi_1$ and χ_2 on $[0,\infty)$ such that*

$$(I) \quad 0 \le \Upsilon_1(\upsilon) \le \mathcal{F}(\varrho) \le \Upsilon_2(\upsilon), \quad 0 \le \chi_1(\upsilon) \le \mathcal{G}(\varrho) \le \chi_2(\upsilon), \quad (x \in [0,\varrho], \varrho > 0).$$

then for $\varrho > 0$, the following inequality holds:

$$\frac{1}{4}\Big(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}(\Upsilon_1\chi_1 + \Upsilon_2\chi_2)\mathcal{F}\mathcal{G})(\varrho)\Big)^2 \ge ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\Upsilon_1\Upsilon_2\mathcal{G}^2)(\varrho). \tag{3.1}$$

Proof. From Condition (I), for $\upsilon \in [0,\varrho], \varrho > 0$, we have

$$\Big(\frac{\Upsilon_2(\upsilon)}{\chi_1(\upsilon)} - \frac{\mathcal{F}(\upsilon)}{\mathcal{G}(\upsilon)}\Big) \ge 0. \tag{3.2}$$

Analogously, we have

$$\Big(\frac{\mathcal{F}(\upsilon)}{\mathcal{G}(\upsilon)} - \frac{\Upsilon_1(\upsilon)}{\chi_2(\upsilon)}\Big) \ge 0. \tag{3.3}$$

Multiplying (3.2) and (3.3), we obtain

$$[\Upsilon_1(\upsilon)\chi_1(\upsilon) + \Upsilon_2(\upsilon)\chi_2(\upsilon)]\mathcal{F}(\upsilon)\mathcal{G}(\upsilon) \ge \chi_1(\upsilon)\chi_2(\upsilon)\mathcal{F}^2(\upsilon) + \Upsilon_1(\upsilon)\Upsilon_2(\upsilon)\mathcal{G}^2(\upsilon). \tag{3.4}$$

Conducting product on both sides of (3.4) by $\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)} \exp\big[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho) - \Lambda(\upsilon))\big](\Lambda(\varrho) - \Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)$ and integrating the estimates with respect to x over $(0,\varrho)$, we get

$$({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}[(\Upsilon_1\chi_1 + \Upsilon_2\chi_2)\mathcal{F}\mathcal{G}])(\varrho) \ge ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2\mathcal{F}^2)(\varrho) + ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\Upsilon_1\Upsilon_2\mathcal{G}^2)(\varrho).$$

Applying the $AM - GM$ inequality, i.e., $\mu + \nu \ge 2\sqrt{\mu\nu}, \quad \mu,\nu \in \mathbb{R}^+$, we have

$$({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}[(\Upsilon_1\chi_1 + \Upsilon_2\chi_2)\mathcal{F}\mathcal{G}])(\varrho) \ge 2\sqrt{({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\Upsilon_1\Upsilon_2\mathcal{G}^2)(\varrho)},$$

which leads to

$$\frac{1}{4}\Big(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}[(\Upsilon_1\chi_1 + \Upsilon_2\chi_2)\mathcal{F}\mathcal{G}])(\varrho)\Big)^2 \ge ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\Upsilon_1\Upsilon_2\mathcal{G}^2)(\varrho).$$

Therefore, we obtain the inequality (3.5) as required. □

Some special cases of Theorem 3.1 are satated as follows.

Corollary 5. *For $\epsilon \in (0,1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0) = 0$. Then*

$$(II) \quad 0 < q \le \mathcal{F}(\varrho) \le Q < \infty, \quad 0 < r \le \mathcal{G}(\varrho) \le R < \infty, \quad (x \in [0,\varrho], \varrho > 0).$$

Then for $\varrho > 0$, we have

$$\frac{({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G}^2)(\varrho)}{({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho))^2} \le \frac{1}{4}\Big(\sqrt{\frac{qr}{QR}} + \sqrt{\frac{QR}{qr}}\Big)^2.$$

(I) If we choose $\Lambda(\upsilon) = \upsilon$, then Theorem 3.1 reduces to a a new result for generalized proportional fractional integral.

Corollary 6. *For $\epsilon \in (0,1], \varphi \in \mathbf{C}$ with $\Re(\varphi) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Then*

$$\frac{1}{4}\Big((\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}[(\Upsilon_1\chi_1 + \Upsilon_2\chi_2)\mathcal{F}\mathcal{G}])(\varrho)\Big)^2 \ge (\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2\mathcal{F}^2)(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\Upsilon_1\Upsilon_2\mathcal{G}^2)(\varrho). \tag{3.5}$$

Remark 4. If we choose $\Lambda(\upsilon) = \upsilon$ along with $\epsilon = 1$, then Theorem 3.1 reduces to Lemma 3.1 in [53].

Theorem 3.2. *For $\epsilon \in (0,1], \varphi, \zeta \in \mathbf{C}$ with $\Re(\varphi) > 0, \Re(\zeta) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0) = 0$. Assume that there exist four positive integrable functions $\Upsilon_1, \Upsilon_2, \chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (I), then the following inequality holds:*

$$\frac{({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\Upsilon_2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G}^2)(\varrho)}{\Big(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\mathcal{G})(\varrho) + ({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_2\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_2\mathcal{G})(\varrho)\Big)^2} \le \frac{1}{4}. \tag{3.6}$$

Proof. Applying condition (I) to prove (3.9), we get

$$\Big(\frac{\Upsilon_2(\upsilon)}{\chi_1(\bar{\upsilon})} - \frac{\mathcal{F}(\upsilon)}{\mathcal{G}(\bar{\upsilon})}\Big) \ge 0$$

and

$$\Big(\frac{\mathcal{F}(\upsilon)}{\mathcal{G}(\bar{\upsilon})} - \frac{\Upsilon_1(\upsilon)}{\chi_2(\bar{\upsilon})}\Big) \ge 0,$$

which imply that

$$\Big(\frac{\Upsilon_1(\upsilon)}{\chi_2(\bar{\upsilon})} + \frac{\Upsilon_2(\upsilon)}{\chi_1(\bar{\upsilon})}\Big)\frac{\mathcal{F}(\upsilon)}{\mathcal{F}(\bar{\upsilon})} \ge \frac{\mathcal{F}^2(\upsilon)}{\mathcal{G}^2(\upsilon)} + \frac{\Upsilon_1(\upsilon)\Upsilon_2(\upsilon)}{\chi_1(\bar{\upsilon})\chi_2(\bar{\upsilon})}. \tag{3.7}$$

Multiplying both sides of (3.7) by $\chi_1(\bar{\upsilon})\chi_2(\bar{\upsilon})\mathcal{G}^2\bar{\upsilon}$, we have

$$\begin{aligned}&\Upsilon_1(\upsilon)\mathcal{F}(\upsilon)\chi_1(\bar{\upsilon})\mathcal{G}(\bar{\upsilon}) + \Upsilon_2(\upsilon)\mathcal{F}(\upsilon)\chi_2(\bar{\upsilon})\mathcal{G}(\bar{\upsilon})\\ &\ge \chi_1(\bar{\upsilon})\chi_2(\bar{\upsilon})\mathcal{F}^2(\upsilon) + \Upsilon_1(\upsilon)\Upsilon_2(\upsilon)\mathcal{G}^2(\bar{\upsilon}).\end{aligned} \tag{3.8}$$

Conducting product on both sides of (3.8) by $\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)\epsilon^{\zeta}\Gamma(\zeta)}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right]\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\right]$ $(\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\upsilon)\Lambda'(\bar{\upsilon})$ and integrating the estimates with respect to υ and $\bar{\upsilon}$ over $(0,\varrho)$, we get

$$\begin{aligned}&(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\mathcal{F})(\varrho)(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\mathcal{G})(\varrho)+(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_2\mathcal{F})(\varrho)(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_2\mathcal{G})(\varrho)\\&\geq(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F}^2)(\varrho)(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2)(\varrho)+(({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G}^2)(\varrho)(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\Upsilon_2)(\varrho).\end{aligned}$$

Applying the $AM-GM$ inequality, we get

$$\begin{aligned}&({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1 g)(\varrho)+({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_2\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_2\mathcal{G})(\varrho)\\&\geq 2\sqrt{({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2)(\varrho)+({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\Upsilon_2)(\varrho)},\end{aligned}$$

which leads to the desired inequality in (3.9). The proof is completed. □

Some special cases of Theorem 3.2 are stated as follows.

Corollary 7. *For $\epsilon\in(0,1],\varphi,\zeta\in\mathbf{C}$ with $\Re(\varphi)>0,\Re(\zeta)>0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0)=0$. Assume that there exist four positive integrable functions $\Upsilon_1,\Upsilon_2,\chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (II), then the following inequality holds:*

$$\begin{aligned}&rR({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho})(1)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2)(\varrho)+qQ({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho})(1)({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{G}^2)(\varrho)\\&\leq\frac{(qr+QR)^2}{4}(({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G})(\varrho))^2.\end{aligned}$$

(I) If we choose $\Lambda(\upsilon)=\upsilon$, then we have a new result for generalized proportional fractional integral operator.

Corollary 8. *For $\epsilon\in(0,1],\varphi,\zeta\in\mathbf{C}$ with $\Re(\varphi)>0,\Re(\zeta)>0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Assume that there exist four positive integrable functions $\Upsilon_1,\Upsilon_2,\chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (I), then the following inequality holds:*

$$\frac{(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\Upsilon_2)(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\chi_2)(\varrho)(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F}^2)(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{G}^2)(\varrho)}{\left((\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_1\mathcal{F})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_1\mathcal{G})(\varrho)+(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\Upsilon_2\mathcal{F})(\varrho)(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\chi_2\mathcal{G})(\varrho)\right)^2}\leq\frac{1}{4}. \tag{3.9}$$

Remark 5. If we choose $\Lambda(\upsilon)=\upsilon$ along with $\epsilon=1$, then Theorem 3.2 reduces to Lemma 3.3 in [53].

Theorem 3.3. *For $\epsilon\in(0,1],\varphi,\zeta\in\mathbf{C}$ with $\Re(\varphi)>0,\Re(\zeta)>0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0)=0$. Assume that there exist four positive integrable functions $\Upsilon_1,\Upsilon_2,\chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (I), then the following inequality holds:*

$$({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\frac{\Upsilon_2\mathcal{F}\mathcal{G}}{\chi_1})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\frac{\chi_2\mathcal{F}\mathcal{G}}{\Upsilon_1})(\varrho)\geq({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2)(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{G}^2)(\varrho). \tag{3.10}$$

Proof. Using condition (I), we have

$$\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{0}^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\frac{\Upsilon_2(\upsilon)}{\chi_1(\upsilon)}\mathcal{F}(\upsilon)\mathcal{G}(\upsilon)d\upsilon$$
$$\geq \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{0}^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\upsilon))\right](\Lambda(\varrho)-\Lambda(\upsilon))^{\varphi-1}\Lambda'(\upsilon)\mathcal{F}^2(\upsilon)d\upsilon,$$

which implies

$$\left({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\frac{\Upsilon_2\mathcal{F}\mathcal{G}}{\chi_1}\right)(\varrho) \geq \left({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2\right)(\varrho). \tag{3.11}$$

Analogously, we obtain

$$\frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{0}^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\right](\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\bar{\upsilon})\frac{\chi_2(\bar{\upsilon})}{\Upsilon_1(\bar{\upsilon})}\mathcal{F}(\bar{\upsilon})\mathcal{G}(\bar{\upsilon})d\bar{\upsilon}$$
$$\geq \frac{1}{\epsilon^{\varphi}\Gamma(\varphi)}\int_{0}^{\varrho}\exp\left[\frac{\epsilon-1}{\epsilon}(\Lambda(\varrho)-\Lambda(\bar{\upsilon}))\right](\Lambda(\varrho)-\Lambda(\bar{\upsilon}))^{\varphi-1}\Lambda'(\bar{\upsilon})\mathcal{G}^2(\bar{\upsilon})d\bar{\upsilon},$$

from which one has

$$\left({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\frac{\chi_2\mathcal{F}\mathcal{G}}{\Upsilon_1}\right)(\varrho) \geq \left({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{G}^2\right)(\varrho). \tag{3.12}$$

Multiplying (3.11) and (3.12), we get the desired inequality (3.10). □

Some special casesof Theorem 3.3 are presented as follows.

Corollary 9. *For $\epsilon \in (0,1], \varphi, \zeta \in \mathbf{C}$ with $\Re(\varphi) > 0, \Re(\zeta) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Suppose that a positive monotone function Λ with continuous derivative defined on $[0,\infty)$ having $\Lambda(0) = 0$. Assume that there exist four positive integrable functions $\Upsilon_1, \Upsilon_2, \chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (I), then the following inequality holds:*

$$\frac{({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2)(\varrho)\,({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{G}^2)(\varrho)}{({}^{\Lambda}\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho)({}^{\Lambda}\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}\mathcal{G})(\varrho)} \leq \frac{QR}{qr}.$$

(I) If we choose $\Lambda(\upsilon) = \upsilon$, then we have a new result for $\widehat{\mathcal{GPFIO}}$.

Corollary 10. *For $\epsilon \in (0,1], \varphi, \zeta \in \mathbf{C}$ with $\Re(\varphi) > 0, \Re(\zeta) > 0$ and let there are two positive integrable functions $\mathcal{F}$ and $\mathcal{G}$ defined on $[0,\infty)$. Assume that there exist four positive integrable functions $\Upsilon_1, \Upsilon_2, \chi_1$ and χ_2 on $[0,\infty)$ satisfying condition (I), then the following inequality holds:*

$$\left(\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\frac{\Upsilon_2\mathcal{F}\mathcal{G}}{\chi_1}\right)(\varrho)\left(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\frac{\chi_2\mathcal{F}\mathcal{G}}{\Upsilon_1}\right)(\varrho) \geq (\mathcal{J}^{\epsilon,\varphi}_{0^+,\varrho}\mathcal{F}^2)(\varrho)(\mathcal{J}^{\epsilon,\zeta}_{0^+,\varrho}\mathcal{G}^2)(\varrho).$$

Remark 6. If we choose $\Lambda(\upsilon) = \upsilon$ along with $\epsilon = 1$ then Theorem 3.3 reduces to Lemma 3.4 in [53].

4. Conclusions

The main objective of this paper determining weighted and extended Čebyšev functionals within the Hilfer-$\widehat{\mathcal{GPFIO}}$, which is quite useful in deriving nonlinear-differentiable problems in fractional calculus. We have derived several generalizations that are little different from the existing research results. Additionally, the newly proposed operator is the generalization of several existing operators such as generalized Riemann-Liouville, Riemann-Liouville, generalized proportional fractional, Hadamard and Conformable fractional integral operators, but they are unified when the proportionality index $\epsilon = 1$. To have a better understanding of the method, we discussed the earlier results proposed by Dhamani et al. [31, 32], Elezovic [58] and Ntouyas [53]. The findings demonstrate that the suggested scheme is enormously imperative and computationally attractive to deal with analogous types of differential equations. As a result, the innovative practices attained in the contemporary research can be extended to achieve analytical solutions of other image processing familiarized in diverse mechanism circulated presently associated with high-dimensional fractional equations [59, 60].

Conflict of interest

The authors declare that they have no competing interests.

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