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Research article

Estimation of generalized fractional integral operators with nonsingular function as a kernel

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Abstract: Bessel function has a significant role in fractional calculus having immense applications in physical and theoretical approach. Present work aims to introduce fractional integral operators in which generalized multi-index Bessel function as a kernel, and develop some important special cases which are connected with fractional operators in fractional calculus. Here, we construct important links to familiar findings from some individual occurrence with our key outcomes.

Keywords: beta function; fractional operators; generalized multi-index Bessel function; Wright's function
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1. Introduction

The Bessel function has immense applications in the field of engineering, physics, and applied mathematics. Baricz [1], *Generalized Bessel functions of the first kind* in (2010), which discussed the geometric properties, functional inequalities of generalized Bessel function and also the inequalities involving circular and hyperbolic functions to Bessel function and modified Bessel functions. Tumakov [2] investigated the numerical algorithms for fast computations of the Bessel functions of an integer order with the required accuracy. Choi and Agarwal [3], Abramowitz and Stegun [4], Heymans and Podlubny [5], Watson [6] and Purohit et al. [7] studied the following Bessel function (Bf) defined by

$$J_{\alpha}(y) = \sum_{n=0}^{\infty} \frac{(-1)^n (y/2)^{\alpha+2n}}{n!\Gamma(\alpha+n+1)}. \tag{1.1}$$

Edward Maitland Wright [8] introduced the generalized form of Bessel function with the name of Bessel-Maitland function (B-M1)

$$J_{\beta}^{\alpha}(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!\Gamma(\alpha n+\beta+1)}. \tag{1.2}$$

The properties of generalised Bessel function can be found in the work of Srivastava and Singh [9]. Suthar et al. [10, 11] discussed the various properties of Bessel-Maitland function. Ali et al. [12] established some fractional operators with the generalized Bessel-Maitland function.

Waseem et al. [13] established the generalized Bessel-Maitland function (B-M11) and discuss the numerous integral formulas for $y \in \mathbb{C}/(-\infty, 0]$; $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) \geq 0$, $\Re(\beta) \geq -1$, $\Re(\gamma) \geq 0$, $k \in (0, 1) \cup \mathbb{N}$ defined by

$$J_{\beta,k}^{\alpha,\gamma}(y) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-y)^n}{n!\Gamma(\alpha n+\beta+1)}. \tag{1.3}$$

Suthar et al. [14] studied the following generalized multi-index Bessel function (Gm-Bf) defined by

$$J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(y) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-y)^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n+\beta_j+1)}. \tag{1.4}$$

Recently, fractional integrals are widely applied in different branches of mathematics, physics, engineering due to their wide applications (see e.g., [5, 15–18]).
Riemann-Liouville fractional integral operators for $\Re(\rho) > 0$ are defined by

$$I_{a^+}^{\rho} h(u) = \frac{1}{\Gamma(\rho)} \int_a^y (y-u)^{\rho-1} h(u)du, a < y \tag{1.5}$$

$$I_{b}^{\rho} h(u) = \frac{1}{\Gamma(\rho)} \int_y^b (u-y)^{\rho-1} h(u)du, y < b. \tag{1.6}$$

Riemann-Liouville fractional differentials operators (RLDO) [12, 19] for $\Re(\rho) > 0$; $n = [\Re(n) - 1]$

$$D_{a^+}^{\rho} h(u) = (d/dy)^n I_{a^+}^{n-\rho} h(y) \tag{1.7}$$

$$D_{b}^{\rho} h(u) = (-d/dy)^n I_{b}^{n-\rho} h(y). \tag{1.8}$$

Srivastava and Singh [9] defined the following fractional integral operator for $\alpha_1, \beta_1, r \in \mathbb{C}$, $\Re(\alpha_1) > 0$, $\Re(\beta_1) \geq -1$ by

$$h(y) =^{def} \int_0^y (y-t)^{\beta_1} J_{\beta_1}^{\alpha_1}(r(y-t)^{\alpha_1}) h(u)du. \tag{1.9}$$

Srivastava and Tomovski [20] established the fractional integral operator (FIO) having Mittag-Leffler function as a kernel, discuss its boundedness and convergence of integral and also derive the product of FIO with Riemann-Liouville fractional integral operator defined for $r, \gamma \in \mathbb{C}$, $\Re(\alpha_1) > \max\{0, \Re(k) - 1\}$; $\min\{\Re(\beta_1), \Re(k)\} > 0$

$$(\mathscr{E}^{r;\gamma,k}_{a^+;\alpha_1,\beta_1} h)(y) = \int_a^y (y-t)^{\beta_1 - 1} E^{\gamma,k}_{\alpha_1,\beta_1}(r(y-t)^{\alpha_1}) h(t) dt. \tag{1.10}$$

Prabhakar fractional integral operators for $\gamma, \beta_1 \in \mathbb{C}$, $\Re(\alpha_1) > 0$ are defined in [21] by

$$\mathfrak{E}^*(\alpha_1, \beta_1; \gamma; r) h(y) =^\circ h(y) = \int_a^y (y-t)^{\beta_1 - 1} E^{\gamma}_{\alpha_1,\beta_1}(r(y-t)^{\alpha_1}) h(t) dt, a < y, \tag{1.11}$$

$$\mathfrak{E}^*(\alpha_1, \beta_1; \gamma; r) h(y) = \int_y^b (t-y)^{\beta_1 - 1} E^{\gamma}_{\alpha_1,\beta_1}(r(t-y)^{\alpha_1}) h(t) dt, y < b. \tag{1.12}$$

Tilahun et al. [22] derived the generalized FIO for $\Re(\beta_1) > 0, \Re(\alpha_1) > 0$ and $r, \gamma \in \mathbb{C}$ as

$$\left(\Im^{r;\alpha_1,\gamma}_{a^+,\beta_1,k} h\right)(y) = \int_a^y (y-t)^{\beta_1} J^{\gamma,k}_{\alpha_1,\beta_1}(r(y-t)^{\alpha_1}; p) h(t) dt, a < y \tag{1.13}$$

and

$$\left(\Im^{r;\alpha_1,\gamma}_{a^+,\beta_1,k} h\right)(y) = \int_y^b (y-t)^{\beta_1} J^{\gamma,k}_{\alpha_1,\beta_1}(r(y-t)^{\alpha_1}; p) h(t) dt, \tag{1.14}$$

where $y < b$.

Definition 1.1. *(FIO)Fractional integral operator with generalized multi-index Bessel function (Gm-Bf) kernel for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^m \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$.

$$\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h(y) = \prod_{j=1}^m \int_a^y (y-u)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(y-u)^{\alpha_j}) h(u) du, a < y \tag{1.15}$$

$$\mathcal{I}^{r;\gamma,k}_{b;(\alpha_j,\beta_j)_m} h(y) = \prod_{j=1}^m \int_y^b (u-y)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(u-y)^{\alpha_j}) h(u) du, y < b. \tag{1.16}$$

Dirichlet formula (Fubini's theorem) Samko et al. in [23] and Kelelaw et al. in [22] is defined as

$$\int_b^d dy \int_b^y h(y,z) dz = \int_b^d dz \int_z^d h(y,z) dy. \tag{1.17}$$

Kilbas [24] analyzed the generalized Wright function for $\xi_i, \zeta_j \in \mathbb{R}$, $(i = 1, 2 \cdots r), (j = 1, 2 \cdots s)$ and $b_i, c_j \in \mathbb{C}$ as

$${}_r\psi_s(y) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r \Gamma(b_i + \xi_i n)}{\prod_{j=1}^s \Gamma(c_j + \zeta_j n)} \frac{y^n}{n!}$$

$$= {}_r\psi_s \left[\begin{array}{c} (b_i, \xi_i)_{1,r} \\ (c_j, \zeta_j)_{1,s} \end{array} \middle| y \right]. \tag{1.18}$$

The integral representation of beta function [25, 26] for $\Re(y) > 0$, $\Re(z) > 0$ and also in gamma form appearance of beta function is defined as follows

$$B(y,z) = \int_0^1 u^{y-1}(1-u)^{z-1}du = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}. \tag{1.19}$$

Pochhammer symbol and its properties can be found [25–27] as

$$(\gamma)_n = \begin{cases} \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1), & \text{for } n \geq 1 \\ 1, & \text{for } n = 0, \gamma \neq 0 \end{cases} \tag{1.20}$$

$$= \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \qquad \textit{and} \qquad (\gamma)_{kn} = \frac{\Gamma(\gamma+kn)}{\Gamma(\gamma)} \qquad (k > 0). \tag{1.21}$$

The space of Lebesgue measurable for complex and real valued functions defined by Kelelaw et al. [22] as follows

$$L(a,y) = \left\{ h : \|h\|_1 := \int_a^y \left| h(u) \right| du < \infty \right\}. \tag{1.22}$$

1.1. Conditions of fractional integral operator

The following some conditions of fractional integral operators can be obtained by setting the integrals according to requirements:

1). Setting $r = 0$, $j = 1 = m$ and $\beta_1 = \beta_1 - 1$ in (FIO) defined in Eqs (1.15) and (1.16), we get the Riemann-Liouville fractional integral operator defined in [28] as

$$\mathcal{I}^{0;\gamma,k}_{a^+;(\alpha_1,\beta_1-1)_m} h(y) = I^{\beta_1}_{a^+} h(y) \tag{1.23}$$

$$\mathcal{I}^{0;\gamma,k}_{b;(\alpha_1,\beta_1-1)_m} h(y) = I^{\beta_1}_{b} h(y). \tag{1.24}$$

2). Setting $j = m = 1, \beta_1 = \beta_1 - 1$ in Eq (1.15), we have a fractional integral defined in Eq (1.10) as

$$\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_1,\beta_1-1)_m} h(y) = (\mathscr{E}^{r;\gamma,k}_{a^+;\alpha_1,\beta_1} h)(y). \tag{1.25}$$

3). Setting $j = m = 1$, $k = 1$, $\beta_1 = \beta_1 - 1$, in Eqs (1.15) and (1.16), we get the FIO defined in Eqs (1.11) and (1.12) respectively

$$\mathcal{I}^{r;\gamma,1}_{a^+;(\alpha_1,\beta_1-1)_m} h(y) = \mathfrak{E}^*(\alpha_1,\beta_1;\gamma;r)h(y) =^\circ h(y) \tag{1.26}$$

$$\mathcal{I}^{r;\gamma,1}_{b;(\alpha_1,\beta_1-1)_m} h(y) = \mathfrak{E}^*(\alpha_1,\beta_1;\gamma;r)h(y). \tag{1.27}$$

4). Setting $j = m = 1$, $k = 0$ and limits from $[0, y]$ in Eq (1.15), we get a fractional integral defined in Eq (1.9) as

$$\mathcal{I}^{r;\gamma,0}_{a^+;(\alpha_1,\beta_1)_m} h(y) = \int_0^y (y-t)^{\beta_1} J^{\alpha_1}_{\beta_1}(r(y-t)^{\alpha_1}) h(u)du = h(y). \tag{1.28}$$

5). Setting $j = m = 1$ in Eq (1.15) then, we get the generalized fractional integral operator defined in Eq (1.13) as

$$\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_1,\beta_1)}h(y) = \left(\Im^{r;\alpha_1,\gamma}_{a^+,\beta_1,k}h\right)(y). \tag{1.29}$$

Lemma 1.1. *Consider the Riemann-Liouville fractional integral operator with multi-index power function for* $\alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$ *and* $\Re(\rho) > 0$ *as*

$$\prod_{j=1}^{m} I^{\rho}_{a^+}[(u-a)^{\beta_j+\alpha_j n}](y) = \prod_{j=1}^{m}\Big[\frac{\Gamma(\beta_j+\alpha_j n+1)}{\Gamma(\rho+\beta_j+\alpha_j n+1)}(y-a)^{\rho+\beta_j+\alpha_j n}\Big]. \tag{1.30}$$

Remark 1.1. *Setting* $j = m = 1$ *in lemma 1.1 then we obtain the result that defined the Mathai Haubold [29] and Kelelaw et al. [22] as*

$$I^{\rho}_{a^+}[(u-a)^{\beta_1+\alpha_1 n}](y) = (y-a)^{\rho+\beta_1+\alpha_1 n}\frac{\Gamma(\beta_1+\alpha_1 n+1)}{\Gamma(\rho+\beta_1+\alpha_1 b+1)}. \tag{1.31}$$

2. Some properties of generalized multi-index Bessel function

The preliminary results for generalized multi-index Bessel function which used to proceed the new results is given in this section. We calculate the *nth*-differential and also develop some results with the coordination of Riemann-Liouville fractional operator and (Gm-Bf).

Theorem 2.1. *Consider the nth-differential of generalized multi-index Bessel function with power function for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$, $n \in \mathbb{N}$ *as*

$$(d/dy)^n[(y-u)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(y-u)^{\alpha_j})] = (y-u)^{\beta_j-n} J^{(\alpha_j,\beta_j-n)_m}_{\gamma,k}(r(y-u)^{\alpha_j}). \tag{2.1}$$

Proof. Let the *nth*-differential of generalized multi-index Bessel function with power function as

$$(d/dy)^n[(y-u)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(y-u)^{\alpha_j})], \tag{2.2}$$

using the behavior of (1.4), we take as

$$\begin{aligned}
&(d/dy)^n[(y-u)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(y-u)^{\alpha_j})] \\
&= (d/dy)^n\Big[(y-u)^{\beta_j}\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r(y-u)^{\alpha_j})^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}\Big] \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}(d/dy)^n\Big[(y-u)^{\beta_j+\alpha_j n}\Big] \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}(d/dy)^n\Big[(y-u)^{(\beta_1+\beta_2+\cdots+\beta_m)+(\alpha_1 n+\alpha_2 n+\cdots+\alpha_m n)}\Big] \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}(d/dy)^n\Big[(y-u)^{(\beta_1+\alpha_1 n)+(\beta_2+\alpha_2 n)+\cdots+(\beta_m+\alpha_m n)}\Big]
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + 1)}$$

$$\times (d/dy)^n \Big[(y-u)^{(\beta_1+\alpha_1 n)} (y-u)^{(\beta_2+\alpha_2 n)} \cdots (y-u)^{(\beta_m+\alpha_m n)} \Big]. \tag{2.3}$$

Using the identity result for simplification of (2.3), we get

$$(d/dy)^n y^\theta = \frac{\Gamma(\theta+1)}{\Gamma(\theta-n+1)} y^{\theta-n}, \theta \geq n \tag{2.4}$$

$$(d/dy)^n [(y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j})] = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + 1)}$$

$$\times \frac{\Gamma(\beta_1 + \alpha_1 n + 1)\Gamma(\beta_2 + \alpha_2 n + 1) \cdots \Gamma(\beta_m + \alpha_m n + 1)(y-u)^{\alpha_j n + \beta_j - n}}{\Gamma(\beta_1 + \alpha_1 n - n + 1)\Gamma(\beta_2 + \alpha_2 n - n + 1) \cdots \Gamma(\beta_m + \alpha_m n - n + 1)}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + 1)} \frac{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j + 1)}{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j - n + 1)} (y-u)^{\alpha_j n + \beta_j - n}$$

$$= (y-u)^{\beta_j - n} \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r(y-u)^{\alpha_j})^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j - n + 1)}$$

$$= (y-u)^{\beta_j - n} J_{\gamma,k}^{(\alpha_j,\beta_j-n)_m}(r(y-u)^{\alpha_j}). \tag{2.5}$$

□

Corollary 2.1. *Suppose that* $\alpha_j, \beta_j, \gamma \in \mathbb{C}$*,* $(j = 1, 2 \cdots m)$*,* $\Re(\alpha_j) > 0$*,* $\Re(\beta_j) > -1$*,* $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$*,* $k > 0$*,* $n \in \mathbb{N}$ *then theorem 2.1 can be expressed as*

$$(d/dy)^n [(y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j})]$$

$$= \frac{1}{\Gamma(\gamma)} (y-u)^{\beta_j - n} {}_2\psi_{m+1} \left\{ \begin{array}{c} (\gamma, k)(\beta_j + 1, \alpha_j) \\ (\beta_j - n + 1, \alpha_j)(\beta_j + 1, \alpha_j)|_{j=1}^{m} \end{array} \middle| r(y-u)^{\alpha_j} \right\}. \tag{2.6}$$

Corollary 2.2. *Suppose that* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$*,* $(j = 1, 2 \cdots m)$*,* $\Re(\alpha_j) > 0$*,* $\Re(\beta_j) > -1$*,* $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$*,* $k > 0$*,* $n \in \mathbb{N}$ *and setting that* $\beta_j = \beta_j - 1$*,* $(r(y-u)^{\alpha_j}) = (-r(y-u)^{\alpha_j})$ *in theorem 2.1 we see that*

$$(d/dy)^n [(y-u)^{\beta_j - 1} J_{\gamma,k}^{(\alpha_j,\beta_j-1)_m}(r(y-u)^{\alpha_j})]$$

$$= (y-u)^{\beta_j - 1 - n} E_{\gamma,k}^{(\alpha_j,\beta_j-n)_m}(r(y-u)^{\alpha_j}), \tag{2.7}$$

where $E_{\gamma,k}^{(\alpha_j,\beta_j-n)_m}(.)$ *is generalized multi-index Mittag-Leffler function.*

Theorem 2.2. *Consider the Riemann-Liouville fractional integral operator defined in Eq* (1.5) *with generalized multi-index Bessel function for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$*,* $(j = 1, 2 \cdots m)$*,* $\Re(\alpha_j) > 0$*,* $\Re(\beta_j) > -1$*,* $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$*,* $k > 0$ *and* $y > a$*,* $a \in \Re_+(0, \infty)$*,* $\Re(\rho) > 0$ *as*

$$\prod_{j=1}^{m} I_{a^+}^{\rho} [(u-a)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(u-a)^{\alpha_j}) \Big](y)$$

$$= \prod_{j=1}^{m} (y-a)^{\beta_j + \rho} J_{\gamma,k}^{(\alpha_j,\beta_j+\rho)_m}(r(y-a)^{\alpha_j}). \tag{2.8}$$

Proof. Let (RLIO) with (Gm-Bf) is defined in Eq (1.4), we have

$$\begin{aligned}
&\prod_{j=1}^{m} I^{\rho}_{a^+}[(u-a)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(u-a)^{\alpha_j})\Big](y) \\
&= \prod_{j=1}^{m} I^{\rho}_{a^+}[(u-a)^{\beta_j} \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r(u-a)^{\alpha_j})^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}] \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)} \prod_{j=1}^{m} I^{\rho}_{a^+}[(u-a)^{\beta_j+\alpha_j n}]. \qquad (2.9)
\end{aligned}$$

By using Lemma 1.1 in Eq (2.9) then we attain the equation as

$$\begin{aligned}
&\prod_{j=1}^{m} I^{\rho}_{a^+}[(u-a)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(u-a)^{\alpha_j})](y) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)} \prod_{j=1}^{m} \frac{\Gamma(\alpha_j n+\beta_j+1)(y-a)^{\alpha_j n+\rho+\beta_j}}{\Gamma(\alpha_j n+\beta_j+\rho+1)} \\
&= \prod_{j=1}^{m}(y-a)^{\rho+\beta_j} \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r(y-a)^{\alpha_j})^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\rho+1)} \\
&= \prod_{j=1}^{m}(y-a)^{\beta_j+\rho} J^{(\alpha_j,\beta_j+\rho)_m}_{\gamma,k}(r(y-a)^{\alpha_j}). \qquad (2.10)
\end{aligned}$$

□

Corollary 2.3. *Consider the right-sided Riemann-Liouville fractional integral operator with generalized multi-index Bessel function for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2\cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m}\Re(\alpha)_j > \max\{0; \Re(k)-1\}$, $k > 0$ *and* $y > a$, $a \in \Re_+(0,\infty)$, $\Re(\rho) > 0$ *as*

$$\begin{aligned}
&\prod_{j=1}^{m} I^{\rho}_{b}[(b-u)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(b-u)^{\alpha_j})](y) \\
&= \prod_{j=1}^{m}(b-y)^{\beta_j+\rho} J^{(\alpha_j,\beta_j+\rho)_m}_{\gamma,k}(r(b-y)^{\alpha_j}). \qquad (2.11)
\end{aligned}$$

Corollary 2.4. *Consider the Riemann-Liouville fractional differential operator defined in Eq* (1.7) *with generalized multi-index Bessel function for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2\cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m}\Re(\alpha)_j > \max\{0; \Re(k)-1\}$, $k > 0$ *and* $y > a$, $a \in \Re_+(0,\infty)$, $\Re(\rho) > 0$ *as*

$$\begin{aligned}
&\prod_{j=1}^{m} D^{\rho}_{a^+}[(u-a)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(u-a)^{\alpha_j})](y) \\
&= \prod_{j=1}^{m}(y-a)^{\beta_j-\rho} J^{(\alpha_j,\beta_j-\rho)_m}_{\gamma,k}(r(y-a)^{\alpha_j}). \qquad (2.12)
\end{aligned}$$

Corollary 2.5. *Consider the right-sided Riemann-Liouville fractional differential operator with generalized multi-index Bessel function for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$ *and* $y > a$, $a \in \Re_+(0, \infty)$, $\Re(\rho) > 0$ *as*

$$\prod_{j=1}^{m} D_b^{\rho}[(b-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(b-u)^{\alpha_j})](y)$$
$$= \prod_{j=1}^{m}(b-y)^{\beta_j-\rho} J_{\gamma,k}^{(\alpha_j,\beta_j-\rho)_m}(r(b-y)^{\alpha_j}). \tag{2.13}$$

3. Properties of generalized fractional integral with non singular function as a kernel

In this section, we discuss some properties of the generalized fractional integrals with non singular function as a kernel.

Theorem 3.1. *Fractional integral operator having generalized multi-index Bessel function (Gm-Bf) kernel for* $\vartheta, \rho, \sigma, r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$, *when* $h(u) = (u-a)^{\frac{\vartheta+\rho}{\sigma}-1}$, *then*

$$[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}(u-a)^{\frac{\vartheta+\rho}{\sigma}-1}](y)$$
$$= \prod_{j=1}^{m}(y-a)^{\frac{\vartheta+\rho}{\sigma}+\beta_j}\Gamma(\frac{\vartheta+\rho}{\sigma}) J_{\gamma,k}^{(\alpha_j,\beta_j+\frac{\vartheta+\rho}{\sigma})_m}(r(y-a)^{\alpha_j}). \tag{3.1}$$

Proof. (FIO) defined in Eq (1.15) with Eq (1.4), we obtain as

$$[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}(u-a)^{\frac{\vartheta+\rho}{\sigma}-1}](y)$$
$$= \prod_{j=1}^{m}\int_a^y (y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j})(u-a)^{\frac{\vartheta+\rho}{\sigma}-1} du$$
$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r)^n}{n!\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}\prod_{j=1}^{m}\int_a^y (u-a)^{\frac{\vartheta+\rho}{\sigma}-1}(y-u)^{\beta_j+\alpha_j n} du$$
$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r)^n}{n!}\frac{1}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}\prod_{j=1}^{m}\int_a^y (u-a)^{\frac{\vartheta+\rho}{\sigma}-1}(y-u)^{\beta_j+\alpha_j n} du \tag{3.2}$$

Substituting $u = y - z(u-a)$ and using the definition of beta function and the following relation in Eq (3.2), we get

$$\mathcal{I}_{a^+}^{\lambda}[(y-a)^{u-1}](y) = \frac{\Gamma(u)}{\Gamma(\lambda+u)}(y-a)^{\lambda+u-1}.$$

$$[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}(u-a)^{\frac{\vartheta+\rho}{\sigma}}](y)$$

$$= \prod_{j=1}^{m} \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}(-r)^n}{n!} \frac{\Gamma(\frac{\vartheta+\rho}{\sigma})}{\prod_{j=1}^{m}\Gamma(\frac{\vartheta+\rho}{\sigma}+\alpha_j n+\beta_j+1)}(y-a)^{\frac{\vartheta+\rho}{\sigma}+\beta_j+\alpha_j n}$$
$$= \prod_{j=1}^{m}(y-a)^{\frac{\vartheta+\rho}{\sigma}+\beta_j}\Gamma(\frac{\vartheta+\rho}{\sigma})\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-r(y-a)^{\alpha_j})^n}{n!\prod_{j=1}^{m}\Gamma(\frac{\vartheta+\rho}{\sigma}+\alpha_j n+\beta_j+1)}$$
$$= \prod_{j=1}^{m}(y-a)^{\frac{\vartheta+\rho}{\sigma}+\beta_j}\Gamma(\frac{\vartheta+\rho}{\sigma})J_{\gamma,k}^{(\alpha_j,\beta_j+\frac{\vartheta+\rho}{\sigma})_m}(r(y-a)^{\alpha_j}). \tag{3.3}$$

□

Corollary 3.1. *Fractional integral operator having generalized multi-index Bessel function (Gm-Bf) kernel for* $\vartheta,\rho,\sigma,r,\alpha_j,\beta_j,\gamma\in\mathbb{C}$, $(j=1,2\cdots m)$, $\Re(\alpha_j)>0$, $\Re(\beta_j)>-1$, $\sum_{j=1}^{m}\Re(\alpha)_j>\max\{0;\Re(k)-1\}$, $k>0$, *when* $h(u)=(a-u)^{\frac{\vartheta+\rho}{\sigma}}$, *then*

$$[\mathcal{I}_{b;(\alpha_j,\beta_j)_m}^{r;\gamma,k}(b-u)^{\frac{\vartheta+\rho}{\sigma}}](y)$$
$$= \prod_{j=1}^{m}(b-y)^{\frac{\vartheta+\rho}{\sigma}+\beta_j+1}\Gamma(\frac{\vartheta+\rho}{\sigma}+1)J_{\gamma,k}^{(\alpha_j,\beta_j+\frac{\vartheta+\rho}{\sigma}+1)_m}(r(b-y)^{\alpha_j}). \tag{3.4}$$

Theorem 3.2. *Consider the composition of Riemann-Liouville fractional integral operator with (FIO) for* $r,\alpha_j,\beta_j,\gamma\in\mathbb{C}$, $(j=1,2\cdots m)$, $\Re(\alpha_j)>0$, $\Re(\beta_j)>-1$, $\sum_{j=1}^{m}\Re(\alpha)_j>\max\{0;\Re(k)-1\}$, $k>0$, $\Re(\rho)>0$ *then*

$$\Big\{I_{a^+}^{\rho}[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}h]\Big\}(y)=\Big\{\mathcal{I}_{a^+;(\alpha_j,\beta_j+\rho)_m}^{r;\gamma,k}\Big\}(y)=\Big\{\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}[I_{a^+}^{\rho}h]\Big\}(y). \tag{3.5}$$

Proof. Let the left side of Eq (3.5), we seen as

$$\Big\{I_{a^+}^{\rho}[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}h]\Big\}(y)$$
$$=\frac{1}{\Gamma(\rho)}\int_a^y(y-u)^{\rho-1}[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}h](u)du$$
$$=\frac{1}{\Gamma(\rho)}\int_a^y(y-u)^{\rho-1}\Big[\prod_{j=1}^{m}\int_a^u(u-t)^{\beta_j}J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(u-t)^{\alpha_j})h(t)dt\Big]du. \tag{3.6}$$

By interchanging the order of integrations and using the Eq (1.17) in Eq (3.6), we attain as

$$\Big\{I_{a^+}^{\rho}[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}h]\Big\}(y)$$
$$=\int_a^y\Big[\frac{1}{\Gamma(\rho)}\prod_{j=1}^{m}\int_t^y(y-u)^{\rho-1}(u-t)^{\beta_j}J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(u-t)^{\alpha_j})du\Big]\times h(t)dt. \tag{3.7}$$

Setting $u-t=\eta$, we have

$$\Big\{I_{a^+}^{\rho}[\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k}h]\Big\}(y)$$

$$= \int_a^y \Big[\prod_{j=1}^m \frac{1}{\Gamma(\rho)} \int_0^{y-t} (y-t-\eta)^{\rho-1} \eta^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(\eta)^{\alpha_j}) d\eta \Big] \times h(t) dt$$

$$= \int_a^y \prod_{j=1}^m I_{0^+}^{\rho} \Big[\eta^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(\eta)^{\alpha_j}) \Big](y-t) \times h(t) dt. \tag{3.8}$$

applying theorem 2.2, we see

$$\Big\{ I_{a^+}^{\rho} [\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} h] \Big\}(y)$$
$$= \prod_{j=1}^m \int_a^y \Big[(y-t)^{\beta_j+\rho} J_{\gamma,k}^{(\alpha_j,\beta_j+\rho)_m}(r(y-t)^{\alpha_j}) \Big] h(t) dt$$
$$= \Big\{ \mathcal{I}_{a^+;(\alpha_j,\beta_j+\rho)_m}^{r;\gamma,k} \Big\}(y). \tag{3.9}$$

We start the right side to determine the second part of (3.5) with (FIO) defined in Eq (1.15) as

$$\Big\{ \mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} [I_{a^+}^{\rho} h] \Big\}(y)$$
$$= \prod_{j=1}^m \int_a^y (y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j}) [I_{a^+}^{\rho} h](u) du$$
$$= \prod_{j=1}^m \int_a^y (y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j}) \Big[\frac{1}{\Gamma(\rho)} \int_a^u (u-t)^{\rho-1} h(t) dt \Big] du. \tag{3.10}$$

Interchanging the order of integration and using Eq (1.17), we get

$$\Big\{ \mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} [I_{a^+}^{\rho} h] \Big\}(y)$$
$$= \prod_{j=1}^m \int_a^y \Big[\frac{1}{\Gamma(\rho)} \int_t^y (y-u)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(y-u)^{\alpha_j}) (u-t)^{\rho-1} du \Big] h(t) dt. \tag{3.11}$$

Setting $y - u = x$ and using theorem (2.2) then

$$\Big\{ \mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} [I_{a^+}^{\rho} h] \Big\}(y)$$
$$= \prod_{j=1}^m \int_a^y \Big[\frac{1}{\Gamma(\rho)} \int_{y-t}^0 (x)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(x)^{\alpha_j}) (y-x-t)^{\rho-1} (-dx) \Big] h(t) dt$$
$$= \prod_{j=1}^m \int_a^y \Big[\frac{1}{\Gamma(\rho)} \int_0^{y-t} (x)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(x)^{\alpha_j}) (y-t-x)^{\rho-1} dx \Big] h(t) dt$$
$$= \prod_{j=1}^m \int_a^y I_{0^+}^{\rho} \Big[(x)^{\beta_j} J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(x)^{\alpha_j}) \Big](y-t) \times h(t) dt$$
$$= \prod_{j=1}^m \int_a^y (y-t)^{\beta_j+\rho} J_{\gamma,k}^{(\alpha_j,\beta_j+\rho)_m}(r(y-t)^{\alpha_j}) h(t) dt$$

$$= \left\{\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j+\rho)_m}\right\}(y). \tag{3.12}$$

Thus, we obtain the desired results by combining Eqs (3.9) and (3.12). □

Corollary 3.2. *Composition of right-sided (FIO) with right-sided Riemann-Liouville fractional integral for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$, $\Re(\rho) > 0$ *then*

$$\left\{I^{\rho}_{b}[\mathcal{I}^{r;\gamma,k}_{b;(\alpha_j,\beta_j)_m} h]\right\}(y) = \left\{\mathcal{I}^{r;\gamma,k}_{b;(\alpha_j,\beta_j+\rho)_m}\right\}(y) = \left\{\mathcal{I}^{r;\gamma,k}_{b;(\alpha_j,\beta_j)_m}[I^{\rho}_{b} h]\right\}(y). \tag{3.13}$$

Corollary 3.3. *Consider the composition of Riemann-Liouville fractional differential operator with (FIO) for* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$, $\Re(\rho) > 0$ *then*

$$\left\{D^{\rho}_{a^+}[\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h]\right\}(y) = \left\{\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j-\rho)_m}\right\}(y) = \left\{\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m}[D^{\rho}_{a^+} h]\right\}(y). \tag{3.14}$$

Theorem 3.3. *If* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j = 1, 2 \cdots m)$, $\Re(\alpha_j) > 0$, $\Re(\beta_j) > -1$, $\sum_{j=1}^{m} \Re(\alpha)_j > \max\{0; \Re(k) - 1\}$, $k > 0$ *then*

$$||\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h||_1 \leq A||h||_1, \tag{3.15}$$

where

$$A = \prod_{j=1}^{m} \sum_{n=0}^{\infty} \frac{|(y-a)^{\Re(\beta_j)+1}||(\gamma)_{kn}||(-r(y-a)^{\Re(\alpha_j)})^n|}{n!|\prod_{j=1}^{m}\Gamma(\alpha_j n + \beta_j + 1)|\Re(\alpha_j)n + \Re(\beta_j) + 1}. \tag{3.16}$$

Proof. Let V_n be the nth-terms of (3.16); we have

$$\begin{aligned}
\left|\frac{V_{n+1}}{V_n}\right| &= \prod_{j=1}^{m} \left|\frac{(\gamma)_{kn+k}}{(\gamma)_{kn}}\right| \frac{n!}{(n+1)!} \left|\frac{\prod_{j=1}^{m}\Gamma(\alpha_j n + \beta_j + 1)}{\prod_{j=1}^{m}\Gamma(\alpha_j n + \alpha_j + \beta_j + 1)}\right| \\
&\times \frac{\Re(\alpha_j)n + \Re(\beta_j) + 1}{\Re(\alpha_j)n + \Re(\alpha_j) + \Re(\beta_j) + 1} \left|\frac{(-1)^{n+1}}{(-1)^n}\right| \left|r(y-a)^{\Re(\alpha_j)}\right| \\
&\approx \prod_{j=1}^{m} \frac{(kn)^k \left| - r(y-a)^{\Re(\alpha_j)}\right|}{(n+1)\left|\prod_{j=1}^{m}|(\alpha_j)|n^{(\alpha_j)}\right|}, \quad as \quad n \to \infty
\end{aligned} \tag{3.17}$$

hence, $\left|\frac{V_{n+1}}{V_n}\right| \to 0$ as $n \to \infty$, and $k < \Re(\alpha_j)$ which means that right hand side of (3.16) is convergent and finite under the given condition. The condition of boundedness of the integral operator $(\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h)(y)$ is discussed in the space of Lebesgue measure $L(a, y)$ of a continuous function, where $y > a$. Consider the Lebesgue measurable space (1.22) and FIO (1.15), we have

$$\begin{aligned}
||\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h||_1 &= \int_a^y \left|\mathcal{I}^{r;\gamma,k}_{a^+;(\alpha_j,\beta_j)_m} h\right| du \\
&= \int_a^y \left|\prod_{j=1}^{m} \int_a^u (u-\tau)^{\beta_j} J^{(\alpha_j,\beta_j)_m}_{\gamma,k}(r(u-\tau)^{\alpha_j}) h(\tau) d\tau\right| du
\end{aligned}$$

$$\leq \int_a^y \Big[\prod_{j=1}^m \int_\tau^y (u-\tau)^{\beta_j} \Big| J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(u-\tau)^{\alpha_j})\Big| du\Big] |h(\tau)| d\tau. \tag{3.18}$$

Putting $u-\tau=\lambda, u=y \Rightarrow \lambda=y-\tau; u=\tau \Rightarrow \lambda=0, du=d\lambda$ in Eq (3.18), we get

$$\begin{aligned} \|\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} h\|_1 &\leq \int_a^y \Big[\prod_{j=1}^m \int_0^{y-\tau} \lambda^{\Re(\beta_j)} \Big| J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(\lambda)^{\alpha_j})\Big| d\lambda\Big] |h(\tau)| d\tau \\ &\leq \int_a^y \Big[\prod_{j=1}^m \int_0^{y-a} \lambda^{\Re(\beta_j)} \Big| J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(\lambda)^{\alpha_j})\Big| d\lambda\Big] |h(\tau)| d\tau, \end{aligned}$$

where

$$\begin{aligned} &\prod_{j=1}^m \int_0^{y-a} \lambda^{\Re(\beta_j)} \Big| J_{\gamma,k}^{(\alpha_j,\beta_j)_m}(r(\lambda)^{\alpha_j})\Big| d\lambda \\ &\leq \prod_{j=1}^m \int_0^{y-a} \lambda^{\Re(\beta_j)} \Big| \sum_{n=0}^\infty \frac{(\gamma)_{kn} |(-r)^n (\lambda)^{\alpha_j b}|}{n! \prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+1)} \Big| d\lambda \\ &\leq \sum_{n=0}^\infty \frac{|(\gamma)_{kn}||(-r)^n|}{n! |\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+1)|} \prod_{j=1}^m \int_0^{y-a} \lambda^{\Re(\alpha_j)n+\Re(\beta_j)} d\lambda \\ &\leq \prod_{j=1}^m \sum_{n=0}^\infty \frac{|(y-a)^{\Re(\beta_j)+1}||(\gamma)_{kn}||(-r(y-a)^{\Re(\alpha_j)})^n|}{n! |\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+1)| \Re(\alpha_j)n+\Re(\beta_j)+1} \qquad (3.19) \\ &= A. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} h\|_1 &\leq \int_a^y A|h(\tau)| d\tau \leq A\|h\|_1 \\ &\Rightarrow \|\mathcal{I}_{a^+;(\alpha_j,\beta_j)_m}^{r;\gamma,k} h\|_1 \leq A\|h\|_1 \end{aligned}$$

□

Corollary 3.4. *If* $r, \alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j=1,2\cdots m)$, $\Re(\alpha_j)>0$, $\Re(\beta_j)>-1$, $\sum_{j=1}^m \Re(\alpha)_j > \max\{0; \Re(k)-1\}$, $k>0$ *then*

$$\|\mathcal{I}_{b;(\alpha_j,\beta_j)_m}^{r;\gamma,k} h\|_1 \leq A\|h\|_1, \tag{3.20}$$

where

$$A=\prod_{j=1}^m (b-y)^{\Re(\beta_j)+1} \sum_{n=0}^\infty \frac{|(\gamma)_{kn}||(-r(b-y)^{\Re(\alpha_j)})^n|}{n! |\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+1)| \Re(\alpha_j)n+\Re(\beta_j)+1}. \tag{3.21}$$

4. Conclusions

The results we discussed in this paper is creating a chain of fractional operators with kernels, having convergence and boundedness, continuity, symmetric properties and composition with Riemann-Liouville operators [9, 12, 20–22, 25, 30, 31] in fractional calculus. We constructed the fractional operator with generalized multi-index Bessel function as a kernel, and discussed its properties, continuity, and check the behaviour with Riemann-Liouville fractional operators. We analyzed the generalized multi-index Bessel function *nth*-derivative and integral in the field of fractional calculus.

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Conflict of interest

The authors declare that they have no competing interests.

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