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Research article

Bipolar soft functions

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Abstract: In this paper, we introduce and study bipolar soft functions. Later, the inverse image of bipolar soft sets is developed and some of its properties are discussed. The relationships between bipolar soft image and inverse image of bipolar soft sets are investigated.

Keywords: bipolar soft sets; bipolar soft functions; image of bipolar soft set; inverse image of bipolar soft set
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1. Introduction

Many crucial fields in our lives such as computer sciences, economics, engineering, decision making and medical sciences are vital of disciplines involving uncertainties where traditional methods cannot deal with them successfully. Taking this into account, many theories have been established to solve these problems. Molodtsov [1] presented the notion of soft set theory as a novel concept dealing with uncertainties. After that, some operations, properties and applications of soft set theory were investigated by some researchers [2–4]. Later, soft set theory was extended to many variants [5–8].

Functions are important mathematical tools that have a lot of applications in many fields of our lives. Therefore, many definitions of soft functions were introduced. Majumdar and Samanta [9] introduced the notions of soft mapping, image and pre-image of crisp sets under soft mapping, and image of soft sets under soft mapping associated with some properties and examples. They also presented an application of soft mapping in medical diagnosis. Kharal and Ahmad [10] introduced the concepts of soft mapping on soft classes and the inverse image of soft set along with some of their properties. They also supported their work by an application in medical diagnosis. Followed by Zorlutuna et al. [11], who extended the definition of soft mappings defined by Kharal and Ahmad [10], constructed some properties of inverse image of soft sets under soft mapping and introduced the notion of soft continuous function along with some equivalent notions to it. Furthermore, many concepts related to soft functions such as continuity of soft functions, soft open functions, soft closed functions and soft homeomorphism functions were discussed [12–14] . Besides that, soft composite functions and soft projection of soft functions were investigated [15]. Another two definitions of soft mapping were introduced by Babitha and Sunil [16], and Wardowski [17].

The notion of bipolar soft set was first introduced by Shabir and Naz [18] as a combination between bipolarity [19] and soft set theory [1]. According to Dubois and Prade [19], our decision making depending on two sides positive and negative, and we choose according to which one is stronger. Therefore, bipolar soft set is a combination of two soft sets one of them represents the positive side where the other represents the negative. In [18], some notions, properties, operations and an application of bipolar soft set in decision making problems were investigated. Later, Karaaslan and Karatas [20] gave another definition of bipolar soft set using an extended definition of the not set of parameters which was used by Shabir and Naz [18]. Besides that, they investigated some operations and application of the new definition. Followed by a discussion on some algebraic structure of bipolar soft sets [21]. In [22], another definition of bipolar soft sets was introduced as a generalization of the previous two definitions of bipolar soft sets and some algebraic structure on it were presented. They also defined the notion of image and pre-image of bipolar soft set, which transfers a bipolar soft set to another bipolar soft set defined on the same universal set depending only on a function defined on their parameters sets. Moreover, the topological structure of bipolar soft sets was studied in [23–26].

The main aim of this paper is to continue investigating bipolar soft sets. Motivated by the interest of mathematicians in soft functions which appears clearly in defining many versions of soft functions, finding many applications on them and studying the behaviour of some topological structure of them, we define bipolar soft function and discuss some of its properties. We also develop the notion of inverse image of bipolar soft set along with some properties of it. The relationships between bipolar soft image and inverse image of bipolar soft sets are also demonstrated. In addition, examples are provided to support our works. We are hopeful that these results will help the researchers to enhance and promote research on bipolar soft sets.

2. Preliminaries

In this section, we recall few definitions, operations and properties regarding bipolar soft sets.

Throughout this paper, T and L represent the universal sets; $P(T)$ and $P(L)$ stand for the power sets of T and L, respectively; $\mathcal{P}$ and $\mathcal{P}'$ refer to the sets of parameters; and I, J, K, H, and M are non-empty sets of parameters where $I, J, K \subseteq \mathcal{P}$ and $H, M \subseteq \mathcal{P}'$.

Definition 2.1. [1] Let T be a universal set and $\mathcal{P}$ be a set of parameters. The pair $(F, \mathcal{P})$ is called a soft set (over T) if F is a function from $\mathcal{P}$ to the power set of T.

Definition 2.2. [2] Let $\mathcal{P} = \{\kappa_i : i = 1, 2, .., n\}$ be a set of parameters. The not set of $\mathcal{P}$ is the set $\neg\mathcal{P} = \{\neg\kappa_i : i = 1, 2, .., n\}$, where $\neg\kappa_i =$ not κ_i for all i.

The following definition of bipolar soft sets is based on Shabir and Naz [18].

Definition 2.3. [18] Let T be a universal set and $\mathcal{P}$ be a set of parameters. We define a bipolar soft set on T with a non-empty set of parameters $I \subseteq \mathcal{P}$ as a triple (W^+, W^-, I) where $W^+ : I \to P(T)$ and $W^- : \neg I \to P(T)$ are two functions, which satisfy the condition $W^+(\kappa) \cap W^-(\neg\kappa) = \emptyset$ for all $\kappa \in I$.

The collection of all bipolar soft sets on T, which the set of parameters of each one of them is a non-empty subset of $\mathcal{P}$ is denoted by $BS(T_{\mathcal{P}})$ and we shall write the bipolar soft set (W^+, W^-, I) as

$$(W^+, W^-, I) = \{(\kappa, W^+(\kappa), W^-(\neg\kappa)) : \kappa \in I, \neg\kappa \in \neg I\}.$$

In our paper we will use Shabir and Naz definition of bipolar soft set.

Definition 2.4. [18] Let $(W^+, W^-, I), (Z^+, Z^-, J) \in BS(T_{\mathcal{P}})$. Then,

1. (W^+, W^-, I) is a bipolar soft subset of (Z^+, Z^-, J) denoted by $(W^+, W^-, I)\tilde{\subseteq}(Z^+, Z^-, J)$, if $I \subseteq J$, $W^+(\kappa) \subseteq Z^+(\kappa)$ and $Z^-(\neg\kappa) \subseteq W^-(\neg\kappa)$ for all $\kappa \in I$.
2. (W^+, W^-, I) and (Z^+, Z^-, J) are equal if $(W^+, W^-, I)\tilde{\subseteq}(Z^+, Z^-, J)$ and $(Z^+, Z^-, J)\tilde{\subseteq}(W^+, W^-, I)$.
3. If $W^+(\kappa) = T$, for all $\kappa \in I$ and $W^-(\neg\kappa) = \emptyset$ for all $\neg\kappa \in \neg I$, (W^+, W^-, I) is called a relative absolute bipolar soft set, denoted by $(\tilde{T}, \Phi, I)$.
4. If $W^+(\kappa) = \emptyset$, for all $\kappa \in I$ and $W^-(\neg\kappa) = T$ for all $\neg\kappa \in \neg I$, (W^+, W^-, I) is called a relative null bipolar soft set, denoted by $(\Phi, \tilde{T}, I)$.
5. The complement of (W^+, W^-, I) is the bipolar soft set $(W^+, W^-, I)^c = ((W^+)^c, (W^-)^c, I)$ where $(W^+)^c(\kappa) = W^-(\neg\kappa)$ and $(W^-)^c(\neg\kappa) = W^+(\kappa)$ for all $\kappa \in I$.
6. The intersection of (W^+, W^-, I) and (Z^+, Z^-, J) denoted by $(W^+, W^-, I)\tilde{\cap}(Z^+, Z^-, J)$ is the bipolar soft set (S^+, S^-, K) on T where $K = I \cap J$ is a non-empty set, and is defined as $S^+(\kappa) = W^+(\kappa) \cap Z^+(\kappa)$ and $S^-(\neg\kappa) = W^-(\neg\kappa) \cup Z^-(\neg\kappa)$ for all $\kappa \in K$.
   Whenever, we talk about the intersection of two bipolar soft sets, we assume that the intersection of their sets of parameters is a non-empty set.
7. The union of (W^+, W^-, I) and (Z^+, Z^-, J) denoted by $(W^+, W^-, I)\tilde{\cup}(Z^+, Z^-, J)$ is the bipolar soft set (S^+, S^-, K) on T where $K = I \cup J$ and

$$S^+(\kappa) = \begin{cases} W^+(\kappa), & \text{if } \kappa \in I \setminus J \\ Z^+(\kappa), & \text{if } \kappa \in J \setminus I \\ W^+(\kappa) \cup Z^+(\kappa), & \text{if } \kappa \in I \cap J \end{cases}$$

$$S^-(\neg\kappa) = \begin{cases} W^-(\neg\kappa), & \text{if } \neg\kappa \in (\neg I) \setminus (\neg J) \\ Z^-(\neg\kappa), & \text{if } \neg\kappa \in (\neg J) \setminus (\neg I) \\ W^-(\neg\kappa) \cap Z^-(\neg\kappa), & \text{if } \neg\kappa \in (\neg I) \cap (\neg J). \end{cases}$$

The intersection (union) is reflexive and associative on $BS(T_{\mathcal{P}})$.

Definition 2.5. [25] Let (W^+, W^-, I), $(Z^+, Z^-, J) \in BS(T_{\mathcal{P}})$ with $I \cap J \neq \emptyset$. The difference between (W^+, W^-, I) and (Z^+, Z^-, J) is defined as $(W^+, W^-, I) \setminus (Z^+, Z^-, J) = (W^+, W^-, I)\tilde{\cap}(Z^+, Z^-, J)^c$.

Proposition 2.6. *[18, 23] Let $(W^+, W^-, I), (Z^+, Z^-, I) \in BS(T_{\mathcal{P}})$. Then,*

1. $\left((W^+, W^-, I)^c\right)^c = (W^+, W^-, I)$.
2. $(W^+, W^-, I)\tilde{\subseteq}(Z^+, Z^-, I) \Rightarrow (Z^+, Z^-, I)^c\tilde{\subseteq}(W^+, W^-, I)^c$.
3. $(\Phi, \tilde{T}, I)\tilde{\subseteq}(W^+, W^-, I)\tilde{\cap}(W^+, W^-, I)^c\tilde{\subseteq}(W^+, W^-, I)\tilde{\cup}(W^+, W^-, I)^c\tilde{\subseteq}(\tilde{T}, \Phi, I)$.
4. $\left((W^+, W^-, I)\tilde{\cup}(Z^+, Z^-, I)\right)^c = (W^+, W^-, I)^c\tilde{\cap}(Z^+, Z^-, I)^c$.

Now, we present the notions of bipolar soft set according to Karaaslan and Karatas [20], and Karaaslan et al. [22]. The notions of image and pre-image of bipolar soft set will also be provided.

Definition 2.7. [20] Let T be a universal set, $\mathcal{P}$ be a set of parameters, I and J be two non-empty disjoint subsets of $\mathcal{P}$ where $I \cup J = \mathcal{P}$ and $g : I \to J$ be an injective function. If $W : I \to P(T)$ and $Q : J \to P(T)$ are two functions, which satisfy the condition $W(\kappa) \cap Q(g(\kappa)) = \emptyset$ for all $\kappa \in I$. Then the triple $(W, Q, \mathcal{P})$ is called a bipolar soft set on T.

The bipolar soft set $(W, Q, \mathcal{P})$ is presented as

$$(W, Q, \mathcal{P}) = \Big\{\Big\langle(\kappa, W(\kappa)), (g(\kappa), Q(g(\kappa)))\Big\rangle : \kappa \in I \text{ and } W(\kappa) \cap Q(g(\kappa)) = \emptyset\Big\}.$$

If for some $\kappa \in I$, $W(\kappa) = Q(g(\kappa)) = \emptyset$, then $\langle(\kappa, \emptyset), (g(\kappa), \emptyset)\rangle$ will not be written in the bipolar soft set $(W, Q, \mathcal{P})$.

It can be seen that instead of the notion of the not set which was used by Shabir and Naz [18], the authors defined an injective function between two subsets of the parameters set $\mathcal{P}$ as one of them is the not set of the other.

Definition 2.8. [22] Let $\mathcal{P}$ be a set of parameters, I be a non-empty subset of $\mathcal{P}$ and $g : I \to \mathcal{P}$ be an injective function. Then the set $I \cup g(I)$ denoted by $\Im_I$ is said to be the extended set of parameters of I.

If the set of parameters is language expression, then $g : I \to \mathcal{P}$ is the function $g(\kappa) = \text{not } \kappa$, for all $\kappa \in I$

Definition 2.9. [22] Let T be a universal set, $\mathcal{P}$ be a set of parameters, I be a non-empty subsets of $\mathcal{P}$ and $\Im_I = I \cup g(I)$ where $g : I \to \mathcal{P}$ is an injective function. Then the triple $(W, Q, \mathcal{P})$ is called a bipolar soft set where $W : I \to P(T)$ and $Q : g(I) \to P(T)$ are two functions, which satisfy the condition $W(\kappa) \cap Q(g(\kappa)) = \emptyset$.

We can represent the bipolar soft set $(W, Q, \mathcal{P})$ as

$$g_I = (W, Q, \mathcal{P}) = \{(\kappa, W(\kappa), Q(g(\kappa))) : \kappa \in \mathcal{P} \text{ and } W(\kappa) \cap Q(g(\kappa)) = \emptyset\},$$

where $W(\kappa) = \emptyset$ and $Q(g(\kappa)) = T$ if $\kappa \in \mathcal{P} \setminus I$ and $g(\kappa) \in \mathcal{P} \setminus \Im_I$. If for some $\kappa \in \mathcal{P}$, $W(\kappa) = \emptyset$ and $Q(g(\kappa)) = T$, then $(\kappa, \emptyset, T)$ will not be written in the bipolar soft set $(W, Q, \mathcal{P})$.

For simplicity, the set $W(\kappa)$ will be denoted as $g_I^+(\kappa)$, $Q(g(\kappa))$ as $g_I^-(\kappa)$ and the bipolar soft set $(W, Q, \mathcal{P})$ as $(g_I^+, g_I^-, \mathcal{P})$. The image of the parameter κ will be written as $g_I(\kappa) = (g_I^+(\kappa), g_I^-(\kappa))$.

Definition 2.10. [22] Let T be a universal set, $\mathcal{P}$ be a set of parameters, $\emptyset \neq I, J \subset \mathcal{P}$, and g_I and g_J be two bipolar soft sets on T. If Ψ is a function from I to J, then the bipolar soft image of g_I under Ψ denoted by $\Psi(g_I)$ is defined as

$$\Psi(g_I)(\kappa') = \begin{cases} (\cup g_I^+(\kappa), \cap g_I^-(\kappa)), & \text{if } \kappa \in I, \Psi(\kappa) = \kappa' \\ (\emptyset, T), & \text{otherwise} \end{cases}$$

for all $\kappa' \in J$. The bipolar soft pre-image of g_J under Ψ denoted by $\Psi^{-1}(g_J)$ is defined as $\Psi^{-1}(g_J)(\kappa) = g_J(\Psi(\kappa))$ for all $\kappa \in I$.

In Definition 2.10, we can only find the bipolar soft image and the bipolar soft pre-image for the bipolar soft sets on T whose set of parameters is a subset from $\mathcal{P}$. Also, we only have a function defined on a subsets from $\mathcal{P}$ without any mention for a function on T or a subset from it.

3. Bipolar soft functions

In this section, we introduce the notion of bipolar soft function associated with some of its properties and supported by some illustrative examples.

Definition 3.1. Let $\gamma : T \to L$ be an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$. Then, a bipolar soft function $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ is defined as: for any bipolar soft set $(W^+, W^-, I) \in BS(T_{\mathcal{P}})$, the image of (W^+, W^-, I) under $\varphi_{\gamma\rho\vartheta}$, $\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big) = (\varphi_{\gamma\rho\vartheta}(W^+), \varphi_{\gamma\rho\vartheta}(W^-), \mathcal{P}')$ is a bipolar soft set in $BS(L_{\mathcal{P}'})$ given as, for all $\sigma \in \mathcal{P}'$

$$\begin{aligned}
\varphi_{\gamma\rho\vartheta}(W^+)(\sigma) &= \begin{cases} \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa)\Big), & \text{if } \rho^{-1}(\sigma) \cap I \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases} \\
\varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma) &= \begin{cases} \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg I} W^-(\neg\kappa)\Big), & \text{if } \vartheta^{-1}(\neg\sigma) \cap \neg I \neq \emptyset \\ L, & \text{otherwise.} \end{cases}
\end{aligned}$$

Throughout our work, the bipolar soft function $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ is associated with the three functions $\gamma : T \to L$, $\rho : \mathcal{P} \to \mathcal{P}'$, and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ where γ is an injective function and $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$.

Example 3.2. Let $T = \{t_1, t_2, t_3\}$ and $L = \{l_1, l_2, l_3, l_4\}$ be two sets, $\mathcal{P} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and $\mathcal{P}' = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be two sets of parameters, $\gamma : T \to L$ be a function defined as $\gamma(t_i) = l_i$, $i = 1, 2, 3$, the function $\rho : \mathcal{P} \to \mathcal{P}'$ be defined as $\rho(\kappa_1) = \rho(\kappa_2) = \sigma_1$, $\rho(\kappa_3) = \sigma_3$, $\rho(\kappa_4) = \sigma_4$, the function $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be defined as $\vartheta(\neg\kappa_i) = \neg\rho(\kappa_i)$, $i = 1, 2, 3, 4$ and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ be a bipolar soft function. We will find the image of $(W^+, W^-, I) = \{(\kappa_1, \{t_1\}, \{t_2\}), (\kappa_2, \{t_3\}, \{t_1, t_2\}), (\kappa_3, \{t_3\}, \{t_1\})\}$.
First, $\rho(I) = \rho(\{\kappa_1, \kappa_2, \kappa_3\}) = \{\sigma_1, \sigma_3\}$.
For σ_1: $\rho^{-1}(\sigma_1) \cap I = \{\kappa_1, \kappa_2\} \cap \{\kappa_1, \kappa_2, \kappa_3\} = \{\kappa_1, \kappa_2\}$. We get

$$\begin{aligned}
\varphi_{\gamma\rho\vartheta}(W^+)(\sigma_1) &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma_1)\cap I} W^+(\kappa)\Big) \\
&= \gamma\Big(W^+(\kappa_1) \cup W^+(\kappa_2)\Big) \\
&= \gamma\Big(\{t_1\} \cup \{t_3\}\Big) \\
&= \{l_1, l_3\}.
\end{aligned}$$

Next, $\vartheta(\neg I) = \{\neg\sigma_1, \neg\sigma_3\}$. For $\neg\sigma_1$: $\vartheta^{-1}(\neg\sigma_1) \cap \neg I = \{\neg\kappa_1, \neg\kappa_2\}$. We have

$$\begin{aligned}
\varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma_1) &= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma_1)\cap\neg I} W^-(\neg\kappa)\Big) \\
&= \gamma\Big(W^-(\neg\kappa_1) \cap W^-(\neg\kappa_2)\Big) \\
&= \gamma\Big(\{t_2\} \cap \{t_1, t_2\}\Big) \\
&= \gamma\Big(\{t_2\}\Big) \\
&= \{l_2\}.
\end{aligned}$$

We can write $\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)(\sigma_1) = (\sigma_1, \{l_1, l_3\}, \{l_2\})$.
For σ_3: $\rho^{-1}(\sigma_3) \cap I = \{\kappa_3\} \cap \{\kappa_1, \kappa_2, \kappa_3\} = \{\kappa_3\}$. Thus,

$$\begin{aligned}\varphi_{\gamma\rho\vartheta}(W^+)(\sigma_3) &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma_3)\cap I} W^+(\kappa)\Big)\\ &= \gamma\Big(W^+(\kappa_3)\Big)\\ &= \gamma\Big(\{t_3\}\Big)\\ &= \{l_3\}.\end{aligned}$$

For $\neg\sigma_3$: $\vartheta^{-1}(\neg\sigma_3) \cap \neg I = \{\neg\kappa_3\}$. Therefore,

$$\begin{aligned}\varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma_3) &= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma_3)\cap\neg I} W^-(\neg\kappa)\Big)\\ &= \gamma\Big(W^-(\neg\kappa_3)\Big)\\ &= \gamma\Big(\{t_1\}\Big)\\ &= \{l_1\}.\end{aligned}$$

Therefore, $\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big) = \{(\sigma_1, \{l_1, l_3\}, \{l_2\}), (\sigma_2, \emptyset, L), (\sigma_3, \{l_3\}, \{l_1\}), (\sigma_4, \emptyset, L)\}$.

In the following example, we will explain why the function $\gamma : T \to L$ should be injective.

Example 3.3. Let $\varphi_{\gamma\rho\vartheta}$ be the bipolar soft functions in Example 3.2 but change $\gamma(t_2)$ to be equal to l_1 instead of l_2. Then, $\varphi_{\gamma\rho\vartheta}(W^+)(\sigma_1) = \{l_1, l_3\}$ and $\varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma_1) = \{l_1\}$. It can be seen that $\varphi_{\gamma\rho\vartheta}(W^+)(\sigma_1) \cap \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma_1) \neq \emptyset$. Which contradicts the definition of bipolar soft set. Therefore, the condition that the function γ is injective is not redundant.

Definition 3.4. Let $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ be a bipolar soft function and $(W^+, W^-, I), (Z^+, Z^-, J) \in BS(T_{\mathcal{P}})$. Then, for $\sigma \in \mathcal{P}'$, the bipolar soft union image of (W^+, W^-, I) and (Z^+, Z^-, J) in $BS(L_{\mathcal{P}'})$ under $\varphi_{\gamma\rho\vartheta}$ is defined as

$$\Big(\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\tilde{\cup}\varphi_{\gamma\rho\vartheta}\Big((Z^+, Z^-, J)\Big)\Big)(\sigma) = \Big(\sigma, \varphi_{\gamma\rho\vartheta}(W^+)(\sigma) \cup \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma), \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma) \cap \varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma)\Big) \quad (3.1)$$

and the bipolar soft intersection image is defined as

$$\Big(\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\tilde{\cap}\varphi_{\gamma\rho\vartheta}\Big((Z^+, Z^-, J)\Big)\Big)(\sigma) = \Big(\sigma, \varphi_{\gamma\rho\vartheta}(W^+)(\sigma) \cap \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma), \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma) \cup \varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma)\Big). \quad (3.2)$$

Definition 3.5. Let $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ be a bipolar soft function, where $\gamma : T \to L$ is an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ are two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$. Then, $\varphi_{\gamma\rho\vartheta}$ is called

1. a bipolar soft surjective function if γ and ρ are surjective functions.

2. a bipolar soft injective function if γ and ρ are injective functions.
3. a bipolar soft bijective function if γ and ρ are bijective functions.

Remark 3.6. The definition of bipolar soft surjective function in the previous definition is equivalent to the classical definition of surjective function, that is, $\varphi_{\gamma\rho\vartheta}$ is a bipolar soft surjective function if and only if $\varphi_{\gamma\rho\vartheta}\big((\tilde{T},\Phi,\mathcal{P})\big) = (\tilde{L},\Phi,\mathcal{P}')$.

Remark 3.7. The definition of bipolar soft injective function in Definition 3.5 is not equivalent to the classical definition of injective function, that is, $\varphi_{\gamma\rho\vartheta}$ can a be a bipolar soft injective function and $\varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\big) = \varphi_{\gamma\rho\vartheta}\big((Z^+,Z^-,J)\big)$ but $(W^+,W^-,I) \neq (Z^+,Z^-,J)$. However, the equivalent happens when the two bipolar soft sets have the same set of parameters. In other words, $\varphi_{\gamma\rho\vartheta}$ is a bipolar soft injective function if and only if for all $(W^+,W^-,I),(Z^+,Z^-,I) \in BS(T_{\mathcal{P}})$ when $\varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\big) = \varphi_{\gamma\rho\vartheta}\big((Z^+,Z^-,I)\big)$ we get $(W^+,W^-,I) = (Z^+,Z^-,I)$.

Theorem 3.8. *Let $\gamma : T \to L$ be an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$ and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$. If $(W^+,W^-,I),(Z^+,Z^-,J) \in BS(T_{\mathcal{P}})$, then*

1. *$\varphi_{\gamma\rho\vartheta}\big((\Phi,\tilde{T},\mathcal{P})\big) \supseteq (\Phi,\tilde{L},\mathcal{P}')$. The equality holds when γ is surjective.*
2. *$\varphi_{\gamma\rho\vartheta}\big((\tilde{T},\Phi,\mathcal{P})\big) \subseteq (\tilde{L},\Phi,\mathcal{P}')$.*
3. *$(W^+,W^-,I)\tilde{\subseteq}(Z^+,Z^-,J) \Rightarrow \varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\big)\tilde{\subseteq}\varphi_{\gamma\rho\vartheta}\big((Z^+,Z^-,J)\big)$.*
4. *$\varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\tilde{\cup}(Z^+,Z^-,J)\big) = \varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\big)\tilde{\cup}\varphi_{\gamma\rho\vartheta}\big((Z^+,Z^-,J)\big)$.*
5. *$\varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I) \tilde{\cap} (Z^+,Z^-,J)\big)\tilde{\subseteq}\varphi_{\gamma\rho\vartheta}\big((W^+,W^-,I)\big) \tilde{\cap} \varphi_{\gamma\rho\vartheta}\big((Z^+,Z^-,J)\big)$. The equality holds if $\varphi_{\gamma\rho\vartheta}$ is a bipolar soft injective function.*

Proof. 1. and 2. are trivial.

3. We need to show that for all $\sigma \in \mathcal{P}'$, $\varphi_{\gamma\rho\vartheta}(W^+)(\sigma) \subseteq \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma)$ and for all $\neg\sigma \in \neg\mathcal{P}'$, $\varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma) \subseteq \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma)$. Let $\sigma \in \rho(I) \subseteq \rho(J) \subseteq \mathcal{P}'$ $\big($if $\sigma \notin \rho(I)$, then $\varphi_{\gamma\rho\vartheta}(W^+)(\sigma) = \emptyset \subseteq \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma)\big)$, then

$$\begin{aligned}\varphi_{\gamma\rho\vartheta}(W^+)(\sigma) &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa)\Big)\\ &\subseteq \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap J} Z^+(\kappa)\Big), \text{ since } W^+(\kappa) \subseteq Z^+(\kappa), \text{ for all } \kappa \in I\\ &= \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma).\end{aligned}$$

Now, for $\neg\sigma \in \vartheta(\neg I) \subseteq \vartheta(\neg J) \subseteq \neg\mathcal{P}'$ $\big($if $\neg\sigma \notin \vartheta(\neg I)$, then $\varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma) = L \supseteq \varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma)\big)$,

$$\begin{aligned}\varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma) &= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg J} Z^-(\neg\kappa)\Big)\\ &\subseteq \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg I} Z^-(\neg\kappa)\Big)\\ &\subseteq \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg I} W^-(\neg\kappa)\Big), \text{ since } Z^-(\neg\kappa) \subseteq W^-(\neg\kappa), \text{ for all } \neg\kappa \in \neg I\end{aligned}$$

$$= \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma).$$

Therefore, $\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\tilde{\subseteq}\varphi_{\gamma\rho\vartheta}\Big((Z^+, Z^-, J)\Big)$.

4. To make our proof easy, let

$$\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\tilde{\cup}\varphi_{\gamma\rho\vartheta}\Big((Z^+, Z^-, J)\Big) = (B^+, B^-, \mathcal{P}')$$

$$\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\tilde{\cup}(Z^+, Z^-, J)\Big) = \varphi_{\gamma\rho\vartheta}\Big((P^+, P^-, I \cup J)\Big) = (R^+, R^-, \mathcal{P}').$$

We need to show that $R^+(\sigma) = B^+(\sigma)$ for all $\sigma \in \mathcal{P}'$ and $R^-(\neg\sigma) = B^-(\neg\sigma)$ for all $\neg\sigma \in \mathcal{P}'$. Let $\sigma \in \rho(I \cup J) = \rho(I) \cup \rho(J) = H \cup M$ (when $\sigma \notin H \cup M$, the case is trivial), then

$$R^+(\sigma) = \varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) = \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\cup J)} P^+(\kappa)\Big).$$

We have three cases for the position of σ:

- case 1: $\sigma \in H \setminus M$, in this case $\rho^{-1}(\sigma) \cap J = \emptyset$ (otherwise, there exists $\kappa \in \rho^{-1}(\sigma) \cap J$ such that $\rho(\kappa) = \sigma \in \rho(J) = M$ which is a contradiction since $\sigma \in M \cap M^c$), therefore $R^+(\sigma) = \varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) = \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\setminus J)} W^+(\kappa)\Big)$.
- case 2: $\sigma \in M\setminus H$, in this case $\rho^{-1}(\sigma)\cap I = \emptyset$ (otherwise, we get a contradiction since $\sigma \in H\cap H^c$). Thus, $R^+(\sigma) = \varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) = \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(J\setminus I)} Z^+(\kappa)\Big)$.
- case 3: $\sigma \in M \cap H$, in this case $\rho^{-1}(\sigma) \in I \cup J$. Thus,

$$\begin{aligned} R^+(\sigma) &= \varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) \\ &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa) \cup \bigcup_{\kappa\in\rho^{-1}(\sigma)\cap J} Z^+(\kappa)\Big) \\ &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa)\Big) \cup \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap J} Z^+(\kappa)\Big). \end{aligned}$$

We have,

$$\begin{aligned} R^+(\sigma) &= \begin{cases} \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\setminus J)} W^+(\kappa)\Big), & \text{if } \sigma \in H \setminus M \\ \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(J\setminus I)} Z^+(\kappa)\Big), & \text{if } \sigma \in M \setminus H \\ \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa)\Big) \cup \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap J} Z^+(\kappa)\Big), & \text{if } \sigma \in H \cap M \end{cases} \\ &= \begin{cases} \varphi_{\gamma\rho\vartheta}\Big(W^+\Big)(\sigma), & \text{if } \sigma \in H \setminus M \\ \varphi_{\gamma\rho\vartheta}\Big(Z^+\Big)(\sigma), & \text{if } \sigma \in M \setminus H \\ \varphi_{\gamma\rho\vartheta}\Big(W^+\Big)(\sigma) \cup \varphi_{\gamma\rho\vartheta}\Big(Z^+\Big)(\sigma), & \text{if } \sigma \in H \cap M \end{cases} \end{aligned}$$

Since $\varphi_{\gamma\rho\vartheta}\Big(Z^+\Big)(\sigma) = \emptyset$ for $\sigma \in H \setminus M$ and $\varphi_{\gamma\rho\vartheta}\Big(W^+\Big)(\sigma) = \emptyset$ for $\sigma \in M \setminus H$, we get that for all $\sigma \in \mathcal{P}'$,

$$\begin{aligned} R^+(\sigma) &= \varphi_{\gamma\rho\vartheta}\Big(W^+\Big)(\sigma) \cup \varphi_{\gamma\rho\vartheta}\Big(Z^+\Big)(\sigma) \\ &= B^+(\sigma), \qquad \text{using Eq (3.1).} \end{aligned}$$

We get $R^+(\sigma) = B^+(\sigma)$ for all $\sigma \in \mathcal{P}'$.
For $\neg\sigma \in \neg(H \cup M) = \neg H \cup \neg M$ (when $\neg\sigma \notin \neg(H \cup M)$ the case is trivial), we have

$$R^-(\neg\sigma) = \varphi_{\gamma\rho\vartheta}\Big(P^-\Big)(\neg\sigma) = \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\cup\neg J)} P^-(\neg\kappa)\Big).$$

By the same strategy as σ, we can find that there are also three cases for the position of $\neg\sigma$ and since γ is injective, we get

$$R^-(\neg\sigma) = \begin{cases} \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\setminus\neg J)} W^-(\neg\kappa)\Big), & \text{if } \neg\sigma \in \neg H \setminus \neg M \\ \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg J\setminus\neg I)} Z^-(\neg\kappa)\Big), & \text{if } \neg\sigma \in \neg M \setminus \neg H \\ \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg I} W^-(\neg\kappa)\Big) \cap \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg J} Z^-(\neg\kappa)\Big), & \text{if } \neg\sigma \in \neg H \cap \neg M. \end{cases}$$

$$= \begin{cases} \varphi_{\gamma\rho\vartheta}\Big(W^-\Big)(\neg\sigma), & \text{if } \neg\sigma \in \neg H \setminus \neg M \\ \varphi_{\gamma\rho\vartheta}\Big(Z^-\Big)(\neg\sigma), & \text{if } \neg\sigma \in \neg M \setminus \neg H \\ \varphi_{\gamma\rho\vartheta}\Big(W^-\Big)(\neg\sigma) \cap \varphi_{\gamma\rho\vartheta}\Big(Z^-\Big)(\neg\sigma), & \text{if } \neg\sigma \in \neg H \cap \neg M. \end{cases}$$

Since $\varphi_{\gamma\rho\vartheta}\Big(Z^-\Big)(\neg\sigma) = L$ for $\neg\sigma \in \neg H \setminus \neg M$ and $\varphi_{\gamma\rho\vartheta}\Big(W^-\Big)(\neg\sigma) = L$ for $\neg\sigma \in \neg M \setminus \neg H$, we get that for all $\neg\sigma \in \neg\mathcal{P}'$

$$\begin{aligned} R^-(\neg\sigma) &= \varphi_{\gamma\rho\vartheta}\Big(W^-\Big)(\neg\sigma) \cap \varphi_{\gamma\rho\vartheta}\Big(Z^-\Big)(\neg\sigma) \\ &= B^-(\neg\sigma), \qquad \text{using Eq (3.1).} \end{aligned}$$

We have $R^-(\neg\sigma) = B^-(\neg\sigma)$ for all $\neg\sigma \in \neg\mathcal{P}'$. This completes our proof.

5. To make our proof easy, let

$$\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\tilde{\cap}(Z^+, Z^-, J)\Big) = \varphi_{\gamma\rho\vartheta}(P^+, P^-, I \cap J)$$

$$\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\tilde{\cap}\varphi_{\gamma\rho\vartheta}\Big((Z^+, Z^-, J)\Big) = (B^+, B^-, \mathcal{P}').$$

We want to show that for all $\sigma \in \mathcal{P}'$, $\varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) \subseteq B^+(\sigma)$ and for all $\neg\sigma \in \neg\mathcal{P}'$, $B^-(\neg\sigma) \subseteq \varphi_{\gamma\rho\vartheta}\Big(P^-\Big)(\neg\sigma)$. For $\sigma \in \rho(I \cap J) \subseteq \mathcal{P}'$ (otherwise, we have trivial case)

$$\begin{aligned} \varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\cap J)} P^+(\kappa)\Big) \\ &= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\cap J)} W^+(\kappa) \cap Z^+(\kappa)\Big) \\ &\subseteq \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\cap J)} W^+(\kappa)\Big) \cap \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap(I\cap J)} Z^+(\kappa)\Big) \\ &\subseteq \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap I} W^+(\kappa)\Big) \cap \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap J} Z^+(\kappa)\Big) \\ &= \varphi_{\gamma\rho\vartheta}(W^+)(\sigma) \cap \varphi_{\gamma\rho\vartheta}(Z^+)(\sigma) \\ &= B^+(\sigma), \qquad \text{using Eq (3.2).} \end{aligned}$$

Next, for $\neg\sigma \in \vartheta(\neg(I \cap J)) = \vartheta(\neg I \cap \neg J) \subseteq \neg\mathcal{P}'$ (otherwise, we have trivial case)

$$\begin{aligned}
\varphi_{\gamma\rho\vartheta}\Big(P^-\Big)(\neg\sigma) &= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\cap\neg J)} P^-(\neg\kappa)\Big)\\
&= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\cap\neg J)} W^-(\neg\kappa)\cup Z^-(\neg\kappa)\Big)\\
&\supseteq \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\cap\neg J)} W^-(\neg\kappa)\Big)\cup\gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap(\neg I\cap\neg J)} Z^-(\neg\kappa)\Big)\\
&\supseteq \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg I} W^-(\neg\kappa)\Big)\cup\gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg J} Z^-(\neg\kappa)\Big)\\
&= \varphi_{\gamma\rho\vartheta}(W^-)(\neg\sigma)\cup\varphi_{\gamma\rho\vartheta}(Z^-)(\neg\sigma)\\
&= B^-(\neg\sigma), \qquad \text{using Eq (3.2).}
\end{aligned}$$

Since for all $\sigma \in \mathcal{P}'$, $\varphi_{\gamma\rho\vartheta}\Big(P^+\Big)(\sigma) \subseteq B^+(\sigma)$ and $B^-(\neg\sigma) \subseteq \varphi_{\gamma\rho\vartheta}\Big(P^-\Big)(\neg\sigma)$ for all $\neg\sigma \in \neg\mathcal{P}'$, our proof is completed. □

Next, we show that the equality in Theorem 3.8 (5) does not hold.

Example 3.9. Let $T = \{1, 2, 3, ..., 9\}$, $L = \{1, 2, 3, ..., 90\}$, $\mathcal{P} = \{\kappa_i, i = 2, 3, 4, 5, 6\}$ and $\mathcal{P}' = \{\sigma_0, \sigma_5\}$. Define $\gamma : T \to L$ as $\gamma(t) = t^2$, $\rho : \mathcal{P} \to \mathcal{P}'$ as

$$\rho(\kappa_i) = \begin{cases}\sigma_0, & \text{if } i < 5\\ \sigma_5, & \text{if } i \geq 5,\end{cases}$$

$\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ as $\vartheta(\neg\kappa_i) = \neg\rho(\kappa_i)$, $i = 2, 3, 4, 5, 6$ and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$. Suppose

$$\begin{aligned}
(K^+, K^-, I) &= \{(\kappa_2, \{3\}, \{4\}), (\kappa_4, \{5, 7\}, \{6\}), (\kappa_6, \{1, 7, 9\}, \{2, 8\})\},\\
(A^+, A^-, J) &= \{(\kappa_2, \{3\}, \{4, 6, 8\}), (\kappa_3, \{5\}, \{3, 6, 9\}), (\kappa_5, \{7\}, \{5\})\}.
\end{aligned}$$

It yields $(K^+, K^-, I)\tilde{\cap}(A^+, A^-, J) = \{(\kappa_2, \{3\}, \{4, 6, 8\})\}$. Therefore, its image under $\varphi_{\gamma\rho\vartheta}$ is $\varphi_{\gamma\rho\vartheta}\Big((K^+, K^-, I)\tilde{\cap}(A^+, A^-, J)\Big) = \{(\sigma_0, \{9\}, \{16, 36, 64\}), (\sigma_5, \emptyset, L)\}$. Now,

$$\begin{aligned}
\varphi_{\gamma\rho\vartheta}\Big((K^+, K^-, I)\Big) &= \{(\sigma_0, \{9, 25, 49\}, \emptyset), (\sigma_5, \{1, 49, 81\}, \{4, 64\})\},\\
\varphi_{\gamma\rho\vartheta}\Big((A^+, A^-, J)\Big) &= \{(\sigma_0, \{9, 25\}, \{36\}), (\sigma_5, \{49\}, \{25\})\}.
\end{aligned}$$

This yields that

$$\varphi_{\gamma\rho\vartheta}\Big((K^+, K^-, I)\Big)\tilde{\cap}\varphi_{\gamma\rho\vartheta}\Big((A^+, A^-, J)\Big) = \{(\sigma_0, \{9, 25\}, \{36\}), (\sigma_5, \{49\}, \{4, 25, 64\})\}.$$

It is clear that $\varphi_{\gamma\rho\vartheta}\Big((K^+, K^-, I)\tilde{\cap}(A^+, A^-, J)\Big) \neq \varphi_{\gamma\rho\vartheta}\Big((K^+, K^-, I)\Big)\tilde{\cap}\varphi_{\gamma\rho\vartheta}\Big((A^+, A^-, J)\Big)$.

4. Inverse image of bipolar soft sets

In this section, we first introduce the concept of inverse image of bipolar soft sets and discuss some of its properties. Then, we investigate the relationships between image and the inverse image of bipolar soft sets. Next, some examples are established to explain our work.

Definition 4.1. Let $\gamma : T \to L$ be an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$, and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ be a bipolar soft function. The inverse image of the bipolar soft set $(Q^+, Q^-, H) \in BS(L_{\mathcal{P}'})$ under $\varphi_{\gamma\rho\vartheta}$, $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big) = (\varphi^{-1}_{\gamma\rho\vartheta}(Q^+), \varphi^{-1}_{\gamma\rho\vartheta}(Q^-), \mathcal{P})$ is a bipolar soft set in $BS(T_{\mathcal{P}})$ given as, for all $\kappa \in \mathcal{P}$,

$$\varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) = \begin{cases} \gamma^{-1}\Big(Q^+(\rho(\kappa))\Big), & \text{if } \rho(\kappa) \in H \\ \emptyset, & \text{if } \rho(\kappa) \notin H. \end{cases}$$

$$\varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa) = \begin{cases} \gamma^{-1}\Big(Q^-(\vartheta(\neg\kappa))\Big), & \text{if } \vartheta(\neg\kappa) \in \neg H \\ T, & \text{if } \vartheta(\neg\kappa) \notin \neg H. \end{cases}$$

The following example illustrates the previous definition.

Example 4.2. Consider the bipolar soft functions defined in Example 3.2. We will find the inverse image of the bipolar soft set $(Q^+, Q^-, H) = \{(\sigma_1, \{l_1, l_3\}, \{l_2\}), (\sigma_2, L, \emptyset)\}$.
First, $\rho^{-1}(H) = \rho^{-1}(\{\sigma_1, \sigma_2\}) = \{\kappa_1, \kappa_2\}$. Since $\rho(\kappa_1) = \sigma_1 \in H$, then

$$\begin{aligned} \varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa_1) &= \gamma^{-1}\Big(Q^+(\rho(\kappa_1))\Big) \\ &= \gamma^{-1}\Big(Q^+(\sigma_1)\Big) \\ &= \gamma^{-1}\Big(\{l_1, l_3\}\Big) \\ &= \{t_1, t_3\}. \end{aligned}$$

Now, $\vartheta^{-1}(\neg H) = \vartheta^{-1}(\{\neg\sigma_1, \neg\sigma_2\}) = \{\neg\kappa_1, \neg\kappa_2\}$. Since $\vartheta(\neg\kappa_1) = \neg\sigma_1 \in \neg H$, then

$$\begin{aligned} \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa_1) &= \gamma^{-1}\Big(Q^-(\vartheta(\neg\kappa_1))\Big) \\ &= \gamma^{-1}\Big(Q^-(\neg\sigma_1)\Big) \\ &= \gamma^{-1}\Big(\{l_2\}\Big) \\ &= \{t_2\}. \end{aligned}$$

We can write $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)(\kappa_1) = (\kappa_1, \{t_1, t_3\}, \{t_2\})$.
Since $\rho(\kappa_2) = \sigma_1$ and $\vartheta(\neg\kappa_2) = \neg\sigma_1$, we get $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)(\kappa_2) = (\kappa_2, \{t_1, t_3\}, \{t_2\})$. Therefore, $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big) = \{(\kappa_1, \{t_1, t_3\}, \{t_2\}), (\kappa_2, \{t_1, t_3\}, \{t_2\}), (\kappa_3, \emptyset, T), (\kappa_4, \emptyset, T)\}$.

Definition 4.3. Let $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$ be a bipolar soft function and $(Q^+, Q^-, H), (S^+, S^-, M) \in BS(L_{\mathcal{P}'})$. Then, for $\kappa \in \mathcal{P}$, the bipolar soft union of the inverse image of (Q^+, Q^-, H) and (S^+, S^-, M) in $BS(T_{\mathcal{P}})$ under $\varphi_{\gamma\rho\vartheta}$ is defined as

$$\begin{aligned} &\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big) \tilde{\cup} \varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)\Big)(\kappa) = \\ &\qquad \Big(\kappa, \varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) \cup \varphi^{-1}_{\gamma\rho\vartheta}(S^+)(\kappa), \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa) \cap \varphi^{-1}_{\gamma\rho\vartheta}(S^-)(\neg\kappa)\Big) \end{aligned} \quad (4.1)$$

and the bipolar soft intersection of the inverse image is defined as

$$\begin{aligned} &\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big) \tilde{\cap} \varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)\Big)(\kappa) = \\ &\qquad \Big(\kappa, \varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) \cap \varphi^{-1}_{\gamma\rho\vartheta}(S^+)(\kappa), \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa) \cup \varphi^{-1}_{\gamma\rho\vartheta}(S^-)(\neg\kappa)\Big). \end{aligned} \quad (4.2)$$

Theorem 4.4. *Let $\gamma : T \to L$ be an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$, and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$. If $(Q^+, Q^-, H), (S^+, S^-, M) \in BS(L_{\mathcal{P}'})$, then*

1. $\varphi^{-1}_{\gamma\rho\vartheta}\Big((\tilde{L}, \Phi, \mathcal{P}')\Big) = (\tilde{T}, \Phi, \mathcal{P})$.
2. $\varphi^{-1}_{\gamma\rho\vartheta}\Big((\Phi, \tilde{L}, \mathcal{P}')\Big) = (\Phi, \tilde{T}, \mathcal{P})$.
3. $(Q^+, Q^-, H)\tilde{\subseteq}(S^+, S^-, M) \Rightarrow \varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)\tilde{\subseteq}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)$.
4. $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\tilde{\cup}(S^+, S^-, M)\Big) = \varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)\tilde{\cup}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)$.
5. $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\tilde{\cap}(S^+, S^-, M)\Big) = \varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)\tilde{\cap}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)$.
6. $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, \mathcal{P}')^c\Big) = \Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, \mathcal{P}')\Big)\Big)^c$.

Proof. 1. and 2. are clear.

3. We need to show that for all $\kappa \in \mathcal{P}$, $\varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) \subseteq \varphi^{-1}_{\gamma\rho\vartheta}(S^+)(\kappa)$ and for all $\neg\kappa \in \neg\mathcal{P}$, $\varphi^{-1}_{\gamma\rho\vartheta}(S^-)(\neg\kappa) \subseteq \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa)$. Let $\kappa \in \mathcal{P}$ where $\rho(\kappa) \in H \subseteq M$ $\Big($for $\rho(\kappa) \notin H$, $\varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) = \emptyset \subseteq \varphi^{-1}_{\gamma\rho\vartheta}(S^+)(\kappa)$ $\Big)$, then

$$\begin{aligned}
\varphi^{-1}_{\gamma\rho\vartheta}(Q^+)(\kappa) &= \gamma^{-1}\Big(Q^+(\rho(\kappa))\Big)\\
&\subseteq \gamma^{-1}\Big(S^+(\rho(\kappa))\Big), \text{ since } Q^+(\sigma) \subseteq S^+(\sigma), \text{ for all } \sigma \in H\\
&= \varphi^{-1}_{\gamma\rho\vartheta}(S^+)(\kappa).
\end{aligned}$$

Now, for $\neg\kappa \in \neg\mathcal{P}$ where $\vartheta(\neg\kappa) \in \neg H \subseteq \neg M$ $\Big($for $\vartheta(\neg\kappa) \notin \neg H$, $\varphi^{-1}_{\gamma\rho\vartheta}(S^-)(\neg\kappa) \subseteq T = \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa)\Big)$,

$$\begin{aligned}
\varphi^{-1}_{\gamma\rho\vartheta}(S^-)(\neg\kappa) &= \gamma^{-1}\Big(S^-(\vartheta(\neg\kappa))\Big)\\
&\subseteq \gamma^{-1}\Big(Q^-(\vartheta(\neg\kappa))\Big), \text{ since } S^-(\neg\sigma) \subseteq Q^-(\neg\sigma), \text{ for all } \neg\sigma \in \neg H\\
&= \varphi^{-1}_{\gamma\rho\vartheta}(Q^-)(\neg\kappa).
\end{aligned}$$

Thus, $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)\tilde{\subseteq}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big)$.

4. For the sake of simplicity, we shall assume that

$$\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\tilde{\cup}(S^+, S^-, M)\Big) = \varphi^{-1}_{\gamma\rho\vartheta}\Big((Z^+, Z^-, H \cup M)\Big) = (A^+, A^-, \mathcal{P})$$

$$\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, H)\Big)\tilde{\cup}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^+, S^-, M)\Big) = (B^+, B^-, \mathcal{P}).$$

We need to show that $A^+(\kappa) = B^+(\kappa)$ for all $\kappa \in \mathcal{P}$ and $A^-(\neg\kappa) = B^-(\neg\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$. Let $\kappa \in \mathcal{P}$ where $\rho(\kappa) \in H \cup M$ $\Big($since for $\rho(\kappa) \notin (H \cup M)$, $A^+(\kappa) = B^+(\kappa) = \emptyset\Big)$, then

$$\begin{aligned}
A^+(\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(Z^+)(\kappa)\\
&= \gamma^{-1}(Z^+(\rho(\kappa)))\\
&= \begin{cases} \gamma^{-1}(Q^+(\rho(\kappa))), & \text{if } \rho(\kappa) \in H \setminus M\\ \gamma^{-1}(S^+(\rho(\kappa))), & \text{if } \rho(\kappa) \in M \setminus H\\ \gamma^{-1}(Q^+(\rho(\kappa)) \cup S^+(\rho(\kappa))), & \text{if } \rho(\kappa) \in H \cap M \end{cases}
\end{aligned}$$

$$= \begin{cases} \gamma^{-1}(Q^{+}(\rho(\kappa))), & \text{if } \rho(\kappa) \in H \setminus M \\ \gamma^{-1}(S^{+}(\rho(\kappa))), & \text{if } \rho(\kappa) \in M \setminus H \\ \gamma^{-1}(Q^{+}(\rho(\kappa))) \cup \gamma^{-1}(S^{+}(\rho(\kappa))), & \text{if } \rho(\kappa) \in H \cap M \end{cases}$$

$$= \begin{cases} \varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa), & \text{if } \rho(\kappa) \in H \setminus M \\ \varphi^{-1}_{\gamma\rho\vartheta}(S^{+})(\kappa), & \text{if } \rho(\kappa) \in M \setminus H \\ \varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa) \cup \varphi^{-1}_{\gamma\rho\vartheta}(S^{+})(\kappa), & \text{if } \rho(\kappa) \in H \cap M. \end{cases}$$

Since $\varphi^{-1}_{\gamma\rho\vartheta}(S^{+})(\kappa) = \emptyset$ if $\rho(\kappa) \in H \setminus M$ and $\varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa) = \emptyset$ if $\rho(\kappa) \in M \setminus H$, we get

$$\begin{aligned} A^{+}(\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa) \cup \varphi^{-1}_{\gamma\rho\vartheta}(S^{+})(\kappa) \\ &= B^{+}(\kappa), \qquad \text{using Eq (4.1).} \end{aligned}$$

Therefore, $A^{+}(\kappa) = B^{+}(\kappa)$ for all $\kappa \in \mathcal{P}$.
Now, for $\neg\kappa \in \neg\mathcal{P}$ where $\vartheta(\neg\kappa) \in \neg(H \cup M) = \neg H \cup \neg M$ $\Big($since, for $\vartheta(\neg\kappa) \notin \neg(H \cup M)$, $A^{-}(\neg\kappa) = B^{-}(\neg\kappa) = T\Big)$, then

$$\begin{aligned} A^{-}(\neg\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(Z^{-})(\neg\kappa) \\ &= \gamma^{-1}(Z^{-}(\vartheta(\neg\kappa))) \\ &= \begin{cases} \gamma^{-1}(Q^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg H \setminus \neg M \\ \gamma^{-1}(S^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg M \setminus \neg H \\ \gamma^{-1}(Q^{-}(\vartheta(\neg\kappa)) \cap S^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg H \cap \neg M \end{cases} \\ &= \begin{cases} \gamma^{-1}(Q^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg H \setminus \neg M \\ \gamma^{-1}(S^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg M \setminus \neg H \\ \gamma^{-1}(Q^{-}(\vartheta(\neg\kappa))) \cap \gamma^{-1}(S^{-}(\vartheta(\neg\kappa))), & \text{if } \vartheta(\neg\kappa) \in \neg H \cap \neg M \end{cases} \\ &= \begin{cases} \varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa), & \text{if } \vartheta(\neg\kappa) \in \neg H \setminus \neg M \\ \varphi^{-1}_{\gamma\rho\vartheta}(S^{-})(\neg\kappa), & \text{if } \vartheta(\neg\kappa) \in \neg M \setminus \neg H \\ \varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa) \cap \varphi^{-1}_{\gamma\rho\vartheta}(S^{-})(\neg\kappa), & \text{if } \vartheta(\neg\kappa) \in \neg H \cap \neg M \end{cases} \end{aligned}$$

Since $\varphi^{-1}_{\gamma\rho\vartheta}(S^{-})(\neg\kappa) = T$ if $\vartheta(\neg\kappa) \in \neg H \setminus \neg M$ and $\varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa) = T$ if $\vartheta(\neg\kappa) \in \neg M \setminus \neg H$, we get

$$\begin{aligned} A^{-}(\neg\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa) \cap \varphi^{-1}_{\gamma\rho\vartheta}(S^{-})(\neg\kappa) \\ &= B^{-}(\neg\kappa) \qquad \text{using Eq (4.1).} \end{aligned}$$

This implies that $A^{-}(\neg\kappa) = B^{-}(\neg\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$ and completes our proof.

5. Assume that

$$\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, H)\tilde{\cap}(S^{+}, S^{-}, M)\Big) = \varphi^{-1}_{\gamma\rho\vartheta}\Big((Z^{+}, Z^{-}, H \cap M)\Big) = (A^{+}, A^{-}, \mathcal{P})$$

$$\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, H)\Big)\tilde{\cap}\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^{+}, S^{-}, M)\Big) = (B^{+}, B^{-}, \mathcal{P}).$$

We have to show that $A^{+}(\kappa) = B^{+}(\kappa)$ for all $\kappa \in \mathcal{P}$ and $A^{-}(\neg\kappa) = B^{-}(\neg\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$. Let $\kappa \in \rho^{-1}(H \cap M) = \rho^{-1}(H) \cap \rho^{-1}(M)$ $\Big($for $\kappa \notin \rho^{-1}(H \cap M)$, $A^{+}(\kappa) = B^{+}(\kappa) = \emptyset$ $\Big)$, then

$$\begin{aligned} A^{+}(\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(Z^{+})(\kappa) \\ &= \gamma^{-1}\Big(Q^{+}(\rho(\kappa)) \cap S^{+}(\rho(\kappa))\Big) \\ &= \gamma^{-1}\Big(Q^{+}(\rho(\kappa))\Big) \cap \gamma^{-1}\Big(S^{+}(\rho(\kappa))\Big) \\ &= \varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa) \cap \varphi^{-1}_{\gamma\rho\vartheta}(S^{+})(\kappa) \\ &= B^{+}(\kappa), \qquad \text{using Eq (4.2).} \end{aligned}$$

Next, let $\neg\kappa \in \vartheta^{-1}(\neg(H \cap M)) = \vartheta^{-1}(\neg H \cap \neg M) = \vartheta^{-1}(\neg H) \cap \vartheta^{-1}(\neg M)$ $\Big($for $\neg\kappa \notin \vartheta^{-1}(\neg(H \cap M))$, $A^{-}(\neg\kappa) = B^{-}(\neg\kappa) = T\Big)$, then

$$\begin{aligned} A^{-}(\neg\kappa) &= \varphi_{\gamma\rho\vartheta}(Z^{-})(\neg\kappa) \\ &= \gamma^{-1}\Big(Q^{-}(\vartheta(\neg\kappa)) \cup S^{-}(\vartheta(\neg\kappa))\Big) \\ &= \gamma^{-1}\Big(Q^{-}(\vartheta(\neg\kappa))\Big) \cup \gamma^{-1}\Big(S^{-}(\vartheta(\neg\kappa))\Big) \\ &= \varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa) \cup \varphi^{-1}_{\gamma\rho\vartheta}(S^{-})(\neg\kappa) \\ &= B^{-}(\neg\kappa), \qquad \text{using Eq (4.2).} \end{aligned}$$

This completes our proof.

6. Suppose $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, \mathcal{P}')\Big) = (R^{+}, R^{-}, \mathcal{P})$. Let $\kappa \in \mathcal{P}$

$$\begin{aligned} \Big(\varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa)\Big)^{c} &= \bigg(\gamma^{-1}\Big(Q^{+}(\rho(\kappa))\Big)\bigg)^{c} \\ &= \Big(R^{+}(\kappa)\Big)^{c} \\ &= R^{-}(\neg\kappa). \end{aligned}$$

Now, $(Q^{+}, Q^{-}, \mathcal{P}')^{c} = \Big((Q^{+})^{c}, (Q^{-})^{c}, \mathcal{P}'\Big)$. Therefore,

$$\begin{aligned} \varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+})^{c}\Big)(\kappa) &= \gamma^{-1}\Big((Q^{+})^{c}(\rho(\kappa))\Big) \\ &= \gamma^{-1}\Big(Q^{-}(\neg\rho(\kappa))\Big) \\ &= \gamma^{-1}\Big(Q^{-}(\vartheta(\neg\kappa))\Big) \\ &= R^{-}(\neg\kappa). \end{aligned}$$

This implies that

$$\Big(\varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})(\kappa)\Big)^{c} = \varphi^{-1}_{\gamma\rho\vartheta}(Q^{+})^{c}(\kappa)$$

for all $\kappa \in \mathcal{P}$. By the same strategy, we can prove that

$$\Big(\varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})(\neg\kappa)\Big)^{c} = \varphi^{-1}_{\gamma\rho\vartheta}(Q^{-})^{c}(\neg\kappa)$$

for all $\neg\kappa \in \neg\mathcal{P}$. Therefore, $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, \mathcal{P}')^{c}\Big) = \Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, \mathcal{P}')\Big)\Big)^{c}$. □

In this present segment, we discuss the relationships between image and inverse image of bipolar soft sets.

Theorem 4.5. *Let $\gamma : T \to L$ be an injective function, $\rho : \mathcal{P} \to \mathcal{P}'$ and $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be two functions where $\vartheta(\neg\kappa) = \neg\rho(\kappa)$ for all $\neg\kappa \in \neg\mathcal{P}$, and $\varphi_{\gamma\rho\vartheta} : BS(T_{\mathcal{P}}) \to BS(L_{\mathcal{P}'})$. If $(W^+, W^-, I) \in BS(T_{\mathcal{P}})$ and $(Q^+, Q^-, \mathcal{P}') \in BS(L_{\mathcal{P}'})$, then*

1. *$(W^+, W^-, I)\tilde{\subseteq}\varphi^{-1}_{\gamma\rho\vartheta}\Big(\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\Big)$. The equality holds if $I = \mathcal{P}$ and $\varphi_{\gamma\rho\vartheta}$ is a bipolar soft injective function.*
2. *If γ is a surjective function, then $\varphi_{\gamma\rho\vartheta}\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, \mathcal{P}')\Big)\Big)\tilde{\subseteq}(Q^+, Q^-, \mathcal{P}')$. The equality holds if $\varphi_{\gamma\rho\vartheta}$ is a bipolar soft surjective function.*

Proof. 1. Let $\varphi^{-1}_{\gamma\rho\vartheta}\Big(\varphi_{\gamma\rho\vartheta}\Big((W^+, W^-, I)\Big)\Big) = \varphi^{-1}_{\gamma\rho\vartheta}\Big((R^+, R^-, \mathcal{P}')\Big) = (Z^+, Z^-, \mathcal{P})$. It is clear that $I \subseteq \mathcal{P}$. Therefore, we need to show that $W^+(\kappa) \subseteq Z^+(\kappa)$ for all $\kappa \in I$ and for all $\neg\kappa \in \neg I$, $Z^-(\neg\kappa) \subseteq W^-(\neg\kappa)$. Let $\kappa \in I$, then

$$\begin{aligned} Z^+(\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(R^+)(\kappa) \\ &= \gamma^{-1}(R^+(\rho(\kappa))) \\ &= \gamma^{-1}\Big(\gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\rho(\kappa))\cap I} W^+(\kappa)\Big)\Big) \\ &= \bigcup_{\kappa\in\rho^{-1}(\rho(\kappa))\cap I} \gamma^{-1}\Big(\gamma\Big(W^+(\kappa)\Big)\Big) \\ &= \bigcup_{\kappa\in\rho^{-1}(\rho(\kappa))\cap I} W^+(\kappa), \qquad \text{since } \gamma \text{ is injective} \\ &\supseteq W^+(\kappa). \end{aligned}$$

Now, for $\neg\kappa \in \neg I$

$$\begin{aligned} Z^-(\neg\kappa) &= \varphi^{-1}_{\gamma\rho\vartheta}(R^-)(\neg\kappa) \\ &= \gamma^{-1}(R^-(\vartheta(\neg\kappa))) \\ &= \gamma^{-1}\Big(\gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\vartheta(\neg\kappa))\cap\neg I} W^-(\neg\kappa)\Big)\Big) \\ &= \bigcap_{\neg\kappa\in\vartheta^{-1}(\vartheta(\neg\kappa))\cap\neg I} \gamma^{-1}\Big(\gamma\Big(W^-(\neg\kappa)\Big)\Big), \text{ since } \gamma \text{ is injective} \\ &= \bigcap_{\neg\kappa\in\vartheta^{-1}(\vartheta(\neg\kappa))\cap\neg I} W^-(\neg\kappa), \text{ since } \gamma \text{ is injective} \\ &\subseteq W^-(\neg\kappa). \end{aligned}$$

Our proof is completed.

2. Assume that $\varphi_{\gamma\rho\vartheta}\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^+, Q^-, \mathcal{P}')\Big)\Big) = \varphi_{\gamma\rho\vartheta}\Big((V^+, V^-, \mathcal{P})\Big) = (S^+, S^-, \mathcal{P}')$. Since $\mathcal{P}' = \mathcal{P}'$, we need to show that for all $\sigma \in \mathcal{P}'$, $S^+(\sigma) \subseteq Q^+(\sigma)$ and for all $\neg\sigma \in \neg\mathcal{P}'$, $Q^-(\neg\sigma) \subseteq S^-(\neg\sigma)$. Let $\sigma \in \rho(\rho^{-1}(\mathcal{P}')) = \rho(\mathcal{P}) \subseteq \mathcal{P}'$ $\Big($for $\sigma \in \mathcal{P}' \setminus \rho(\rho^{-1}(\mathcal{P}'))$, $S^+(\sigma) = \emptyset \subseteq Q^+(\sigma)\Big)$, then

$$S^+(\sigma) = \varphi_{\gamma\rho\vartheta}(V^+)(\sigma)$$

$$
\begin{aligned}
&= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)\cap\mathcal{P}} V^{+}(\kappa)\Big) \\
&= \gamma\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)} \gamma^{-1}\Big(Q^{+}(\rho(\kappa))\Big)\Big), \text{ where } \rho(\kappa)=\sigma \\
&= \gamma\Bigg(\gamma^{-1}\Big(\bigcup_{\kappa\in\rho^{-1}(\sigma)} Q^{+}(\rho(\kappa))\Big)\Bigg) \\
&= \gamma\Bigg(\gamma^{-1}\Big(Q^{+}(\sigma)\Big)\Bigg), \text{ since } Q^{+}(\rho(\kappa)) = Q^{+}(\sigma), \text{ for all } \kappa\in\rho^{-1}(\sigma) \\
&= Q^{+}(\sigma), \quad \text{since } \gamma \text{ is surjective function.}
\end{aligned}
$$

Now, for $\neg\sigma \in \vartheta(\vartheta^{-1}(\neg\mathcal{P}')) = \vartheta(\neg\mathcal{P}) \subseteq \neg\mathcal{P}'$ $\Big($for $\neg\sigma \in \neg\mathcal{P}' \setminus \vartheta(\vartheta^{-1}(\neg\mathcal{P}')), S^{-}(\neg\sigma) = L \supseteq Q^{-}(\neg\sigma)\Big)$, then

$$
\begin{aligned}
S^{-}(\neg\sigma) &= \varphi_{\gamma\rho\vartheta}(V^{-})(\neg\sigma) \\
&= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)\cap\neg\mathcal{P}} V^{-}(\neg\kappa)\Big) \\
&= \gamma\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)} \gamma^{-1}\Big(Q^{-}(\vartheta(\neg\kappa))\Big)\Big), \text{ where } \vartheta(\neg\kappa)=\neg\sigma \\
&= \gamma\Bigg(\gamma^{-1}\Big(\bigcap_{\neg\kappa\in\vartheta^{-1}(\neg\sigma)} Q^{-}(\vartheta(\neg\kappa))\Big)\Bigg) \\
&= \gamma\Big(\gamma^{-1}\Big(Q^{-}(\neg\sigma)\Big)\Big), \text{ since } Q^{-}(\vartheta(\neg\kappa)) = Q^{-}(\neg\sigma), \text{ for all } \neg\kappa\in\vartheta^{-1}(\neg\sigma) \\
&= Q^{-}(\neg\sigma), \quad \text{since } \gamma \text{ is surjective function.}
\end{aligned}
$$

This completes our proof. □

Example 4.6. Consider the bipolar soft function $\varphi_{\gamma\rho\vartheta}$ in Example 3.2.

1. To show that the equality does not hold in Theorem 4.5 (1), let

$$(S^{+}, S^{-}, J) = \{(\kappa_1, \{t_1, t_2\}, \emptyset), (\kappa_4, T, \emptyset)\}.$$

Then,

$$\varphi_{\gamma\rho\vartheta}\Big((S^{+}, S^{-}, J)\Big) = \{(\sigma_1, \{l_1, l_2\}, \emptyset), (\sigma_2, \emptyset, L), (\sigma_3, \emptyset, L), (\sigma_4, \{l_1, l_2, l_3\}, \emptyset)\}.$$

It yields

$$
\begin{aligned}
\varphi^{-1}_{\gamma\rho\vartheta}\Big(\varphi_{\gamma\rho\vartheta}\Big((S^{+}, S^{-}, J)\Big)\Big) &= \{(\kappa_1, \{t_1, t_2\}, \emptyset), (\kappa_2, \{t_1, t_2\}, \emptyset), (\kappa_3, \emptyset, T), (\kappa_4, T, \emptyset)\} \\
&\neq (S^{+}, S^{-}, J).
\end{aligned}
$$

2. To show that the subset relation does not hold in Theorem 4.5 (2) if γ is not surjective, let

$$(Q^{+}, Q^{-}, \mathcal{P}') = \{(\sigma_1, \{l_4\}, \{l_3\}), (\sigma_2, \{l_1\}, \{l_2\}), (\sigma_3, \{l_1\}, \{l_3, l_4\}), (\sigma_4, \emptyset, L)\}.$$

Then, $\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, \mathcal{P}')\Big) = \{(\kappa_1, \emptyset, \{t_3\}), (\kappa_2, \emptyset, \{t_3\}), (\kappa_3, \{t_1\}, \{t_3\}), (\kappa_4, \emptyset, T)\}$. Therefore,

$$\begin{aligned}\varphi_{\gamma\rho\vartheta}\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((Q^{+}, Q^{-}, \mathcal{P}')\Big)\Big) &= \{(\sigma_1, \emptyset, \{l_3\}), (\sigma_2, \emptyset, L), (\sigma_3, \{l_1\}, \{l_3\}), (\sigma_4, \emptyset, \{l_1, l_2, l_3\})\} \\ &\not\subseteq (Q^{+}, Q^{-}, \mathcal{P}').\end{aligned}$$

In the following example we will show that the equality does not hold in Theorem 4.5 (2) if $\varphi_{\gamma\rho\vartheta}$ is not a bipolar soft surjective function.

Example 4.7. Let $\mathcal{P} = \{\kappa_1, \kappa_2\}$ and $\mathcal{P}' = \{\sigma_1, \sigma_2\}$ be two sets of parameters, $\gamma : \mathbb{R} \to \mathbb{R}$ be a function on the set of real numbers defined as $\gamma(x) = 2x + 1$ for all $x \in \mathbb{R}$, $\rho : \mathcal{P} \to \mathcal{P}'$ be defined as $\rho(\kappa_1) = \rho(\kappa_2) = \sigma_1$, $\vartheta : \neg\mathcal{P} \to \neg\mathcal{P}'$ be defined as $\vartheta(\neg\kappa_i) = \neg\rho(\kappa_i)$, $i = 1, 2$ and $\varphi_{\gamma\rho\vartheta} : BS(\mathbb{R}_{\mathcal{P}}) \to BS(\mathbb{R}_{\mathcal{P}'})$.
It is clear that γ is a homeomorphism function but $\varphi_{\gamma\rho\vartheta}$ is not a bipolar soft surjective function since ρ is not surjective. Let $(S^{+}, S^{-}, \mathcal{P}') = \{(\sigma_1, \{3, 9\}, \{5\}), (\sigma_2, \{1, 6\}, \{7\})\}$. Then,

$$\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^{+}, S^{-}, \mathcal{P}')\Big) = \{(\kappa_1, \{1, 4\}, \{2\}), (\kappa_2, \{1, 4\}, \{2\})\}.$$

Therefore,

$$\varphi_{\gamma\rho\vartheta}\Big(\varphi^{-1}_{\gamma\rho\vartheta}\Big((S^{+}, S^{-}, \mathcal{P}')\Big)\Big) = \{(\sigma_1, \{3, 9\}, \{5\}), (\sigma_2, \emptyset, \mathbb{R})\} \neq (S^{+}, S^{-}, \mathcal{P}').$$

5. Conclusion

Functions are fundamental mathematical concept used in many fundamental areas of science and have numerous applications. Throughout this paper, we presented bipolar soft function along with some notions and properties related to it. This is followed by the definition of inverse image of bipolar soft set associated with some of its results. The relationships between image and inverse image of bipolar soft sets are also discussed. We found that, the main difference between classical functions, soft functions and bipolar soft functions is related to the relationship between the image and the inverse image of a set. We know that for any classical function, the image of the inverse image of a set is subset from this set. The same goes for soft functions [11]. In bipolar soft functions, this is not true and a condition should be added in order to satisfy this property. In future work, we will focus on adopting these results to some real life applications from the domains of decision making and medical diagnosis, and to investigate the behaviour of some topological and algebraic structures of bipolar soft functions. We will also extend our work to define fuzzy bipolar soft function and discuss its applications.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. D. A. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31.
2. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562.
3. M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553.
4. N. Çağman, S. Enginoğlu, Soft set theory and uni-int decision making, *Eur. J. Oper. Res.*, **207** (2010), 848–855.
5. K. Hayat, M. I. Ali, B. Y. Cao, X. P. Yang, A new type-2 soft set: Type-2 soft graphs and their applications, *Adv. Fuzzy Syst.*, **2017** (2017), 6162753.
6. K. Hayat, B. Y. Cao, M. I. Ali, F. Karaaslan, Z. Qin, Characterizations of certain types of type 2 soft graphs, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 8535703.
7. M. F. Aslam, M. I. Ali, T. Mahmood, M. M. U. Rehman, N. Sarfraz, Study of fuzzy soft sets with some order on set of parameters, *Int. J. Algebra Stat.*, **8** (2019), 50–65.
8. K. Hayat, M. I. Ali, F. Karaaslan, B. Y. Cao, M. H. Shah, Design concept evaluation using soft sets based on acceptable and satisfactory levels: an integrated TOPSIS and Shannon entropy, *Soft Comput.*, **24** (2020), 2229–2263.
9. P. Majumdar, S. K. Samanta, On soft mappings, *Comput. Math. Appl.*, **60** (2010), 2666–2672.
10. A. Kharal, B. Ahmad, Mappings on soft classes, *New Mathematics and Natural Computation*, **7** (2011), 471–481.
11. I. Zorlutuna, M. Akdag, W. K. Min, S. Atmaca, Remarks on soft topological spaces, *Ann. Fuzzy Math. Inform.*, **3** (2012), 171–185.
12. D. N. Georgiou, A. C. Megaritis, Soft set theory and topology, *Appl. Gen. Topol.*, **15** (2014), 93–109.
13. I. Zorlutuna, H. Çakir, On continuity of soft mappings, *Appl. Math. Inf. Sci.*, **9** (2015), 403–409.
14. C. G. Aras, A. Sonmez, H. Çakallı, An approach to soft functions, *J. Math. Anal.*, **8** (2017), 129–138.
15. A. Aygünoğlu, H. Aygün, Some notes on soft topological spaces, *Neural Comput. Appl.*, **21** (2012), 113–119.
16. K. V. Babitha, J. J. Sunil, Soft set relations and functions, *Comput. Math. Appl.*, **60** (2010), 1840–1849.
17. D. Wardowski, On a soft mapping and its fixed points, *Fixed Point Theory A.*, **2013** (2013), 1–11.
18. M. Shabir, M. Naz, On bipolar soft sets, *arXiv preprint arXiv:1303.1344*, 2013.
19. D. Dubois, H. Prade, An introduction to bipolar representations of information and preference, *Int. J. Intell. Syst.*, **8** (2008), 866–877.
20. F. Karaaslan, S. Karataş, A new approach to bipolar soft sets and its applications, *Discrete Math. Algorithm. Appl.*, **7** (2015), 1550054.

21. K. Hayat, T. Mahmood, Some applications of bipolar soft set: characterizations of two isomorphic hemi-rings via BSI-h-ideals, *Journal of Advances in Mathematics and Computer Science,* **13** (2015), 1–21.
22. F. Karaaslan, I. Ahmad, A. Ullah, Bipolar soft groups, *J. Intell. Fuzzy Syst.,* **31** (2016), 651–662.
23. M. Shabir, A. Bakhtawar, Bipolar soft connected, bipolar soft disconnected and bipolar soft compact spaces, *Songklanakari J. Sci. Technol.,* **39** (2017), 359–371.
24. T.Y. Öztürk, On bipolar soft topological spaces, *J. New Theory,* **20** (2018), 64–75.
25. A. Fadel, N. Hassan, Separation axioms of bipolar soft topological space, *J. Phys. Conf. Ser.*, **1212** (2019), 012017.
26. A. Fadel, S. C. Dzul-Kifli, Bipolar soft topological spaces, *European Journal of Pure and Applied Mathematics*, **13** (2020), 227–245.