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Research article

Well defined extinction time of solutions for a class of weak-viscoelastic parabolic equation with positive initial energy

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Abstract: In the present paper, we consider an important problem from the point of view of application in sciences and mechanic, namely, a class of $p(x)$-Laplacian type parabolic equation with weak-viscoelasticity. Here, we are concerned with global in time non-existence under suitable conditions on the exponents $q(x)$ and $p(x)$ with positive initial energy. We show that the weak-memory term is unable to stabilize problem (1.2) under conditions (1.5) and (1.7). Our main interest in this paper arose in the first place in consequence of a query to blow-up phenomenon.

Keywords: heat equation; blow up; weak-memory; Sobolev spaces, variable exponents
Mathematics Subject Classification: 35B44, 35K45

1. Literature overview and new contributions

Let $1 < p < +\infty$ and Ω be an open bounded domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary $\partial\Omega$. The p-Laplacian Δ_p is a nonlinear differential operator of order 2 defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \text{ for all } \quad u \in W^{1,p}(\Omega).$$

This differential operator intervenes in various domains among which we can mention: the modeling of the mechanical phenomena, the image processing and some physical problems. To model the movement of some non-Newtonian fluids, O. Ladyzhenskaya proposed in [17] the following system

$$\begin{cases} u = (u_1, u_2, \ldots, u_n), u_i \in L^\infty(0, T, L^1(\Omega)) \cap L^p(0, T, W^{1,p}(\Omega)), i = 1, \ldots, n, \\ \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\, \nabla u) = 0 \text{ of distributions sense in } \Omega \times (0, T). \end{cases} \tag{1.1}$$

Let $u(x, t)$ be the concentration of a particle component (or the density of heat) in Ω. The equation

$$\partial_t u - \operatorname{div}(a(x, t, u, \nabla u)\nabla u) = b(x, t, \nabla u) + c(x, t, u), \quad in \quad \Omega \times (0, T),$$

describes the evolution of the concentration during the propagation of particles in Ω. Here $c(x,t,u)$ describes a source if it is positive or a bowl if it is negative, the diffusion coefficient $a(x,t,u,\nabla u)$ reflects the intrinsic ability of diffusion in particles in the medium. Needful to say, it has a numerous generalizations, we can also do with $p(x)$-Laplacian denoted by $\Delta_{p(x)}$, which has an exponent variable property.

The fact that $p(x)$-Laplacian is not homogeneous, makes non-linearities more complicated than the operator p-Laplacian. Studies of various mathematical systems with variable exponent growth conditions have received considerable attention in recent years, which is justified by their various physical applications. However, few papers have treated evolutionary equations of non-local $p(x)$-Laplacian type (Please see [1,5,12,13,24,25]). Viscoelastic materials demonstrate properties between those of elastic materials and viscous fluid. As a consequence of the widespread use of polymers and other modern materials which exhibit stress relaxation, the theory of visco-elasticity has provided important applications in materials science and engineering (Please see [7,8,10,11,19,21,22]).

The viscoelastic materials show a behavior which is something between that of elastic solids and Newtonian fluids. Indeed, the stresses in these media depend on the entire history of their deformation, not only on their current state of deformation or their current state of motion. This is the reason why they are called materials with memory. The viscoelastic equations with fading memory in a bounded space has been deeply studied by several authors, in view of its wide applicability.

The lack of stability of solutions of viscoelastic partial differential equations is a huge restriction for qualitative studies. In the present paper, we consider

$$\partial_t w - div\Big(|\nabla_x w|^{q(x)-2}\nabla_x w\Big) + \sigma(t)\int_0^t \mu\,(t-s)\,\Delta_x w\,ds = |w|^{p(x)-2}w, \tag{1.2}$$

for $x\in\Omega,\ 0<t<\infty$ with initial and boundary conditions

$$w\,(x,0) = w_0\,(x) \in W_0^{1,q(\cdot)}(\Omega), \tag{1.3}$$

$$w = 0 \text{ on } \partial\Omega\times(0,T), \tag{1.4}$$

where $q(\cdot)$ and $p(\cdot)$ are two continuous functions on $\overline{\Omega}$ such that

$$2 < q_- \le q\,(x) \le q_+ < p_- \le p\,(x) \le p_+ < q_*\,(x)\,, \tag{1.5}$$

with

$$q_*\,(x) = \begin{cases} \frac{nq(x)}{n-q(x)} & \text{if } n > q_+ \\ +\infty & \text{if } n \le q_+. \end{cases}$$

The viscoelastic term is represented as $\int_0^t \mu\,(t-s)\,\Delta_x w\,ds$, it is called "weak-viscoelastic" when it comes with the time weighted function $\sigma(t)$, which is considered as a dissipative term and causes stability of systems. The nonlinear term $|w|^{p(x)-2}w$ is known as the source of instability. The importance of our study lies in the study of the interaction between the exponents of source term and the Laplacian with the presence of weak-viscoelastic term. We take the exponents as a variable functions, with their difficulties in the mathematical point of view, to obtain a very large applications. These contributions extend the early results in literature.

We assume that $q(x)$ satisfies the Zhikov-Fan condition, i.e. for all $x, y \in \Omega$,

$$|q(x) - q(y)| \leq \frac{K}{-\log|x-y|} \text{ with } |x-y| < \kappa, \tag{1.6}$$

with $K > 0,\ 0 < \kappa < 1$ and

$$\operatorname*{ess\,inf}_{x\in\Omega} (q_*(x) - p(x)) > 0.$$

Since the relaxation function μ lies with the stability of solutions, we state assumptions on μ and σ as: $\mu, \sigma \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$\sigma'(t) < 0,\ \mu'(t) < -\mu(t) < 0,\ \int_0^{+\infty} \mu(s)\, ds < \left(\frac{1}{q_+} - \frac{1}{p_-}\right) q_- \|\sigma\|_\infty^{-1}. \tag{1.7}$$

For positive constant C depending only on Ω determined by Lemma 2.1, we set for some constant $\lambda > 0$ (will be specified later)

$$\alpha = \left(\frac{q_-}{Cp_+}\right)^{\frac{q_-}{p_+-q_-}}, \quad E_1 = \rho\alpha, \quad \rho = \frac{p_+ - q_-}{q_- p_+}. \tag{1.8}$$

Problem (1.2)–(1.4) is related to the parabolic problem and when the exponents $q(x) = q,\ p(x) = p$, the existence/non-existence results have been extensively studied (please see [2–4, 14, 15, 23]).

Extinction phenomenon for parabolic equation with nonlinearities in divergence form are investigated in [18], under nonlinear boundary flux in bounded star-shaped region. The authors assumed conditions on weight function to guarantee that the solution exists globally or blows up at finite time. Moreover, using the modified differential inequality, upper and lower bounds for the blow-up time of solutions were derived in higher dimensional spaces.

In the case where $q(\cdot)$ and $p(\cdot)$ are constants, the existence of local solutions of initial-boundary value problem

$$\partial_t u - \Delta_q u = |u|^{p-2} u + f(x,t) \text{ in } \Omega \times (0, T),$$

is proved by Akagi in [2] for initial data $u_0 \in L^r(\Omega)$ and $2 \leq q < p < +\infty$ and Ω is an open bounded domain in $\mathbb{R}^n$, under $r > n(p-q)/q$. For the case where $q(\cdot)$ and $p(\cdot)$ are two measurable functions, it is well known that some additional techniques must be used to study the existence/nonexistence of solutions for (1.2)–(1.4) and of course the classical methods may be failed unless some developments are made.

For the case of nonlocal $p(x)$-Laplacian equations and in the absence of the memory term ($\mu \equiv 0$), problem (1.2)–(1.4) has been studied by Ôtani [20]. The author treated the question of existence and qualitative studies of solutions of (1.2)–(1.4) and showed that the difficulties come from the use of non-monotone perturbation theory. To complete these studies, the questions of blow-up of solutions for the same problem are discussed later by many authors.

In this paper we shall establish a blow-up result of solutions for problem (1.2)–(1.4) in the Lebesgue and Sobolev spaces with variable exponents and positive initial energy.

2. Preliminary and well known results

We try to list here some useful mathematical tools.

First, let $p : \Omega \longrightarrow (1, \infty)$ be a measurable function. Denoting by

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \text{ and } p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

We define the $p(\cdot)$ modular of a measurable function $w : \Omega \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ as

$$Q_{p(\cdot)}(w) = \int_{\Omega\setminus\Omega_\infty} |w|^{p(x)}\, dx + \operatorname*{ess\,sup}_{x\neq\Omega_\infty} |w(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega;\ p(x) = \infty\}.$$

The special Orlicz Musielak space $L^{p(\cdot)}(\Omega)$ is a Lebesgue space with variable-exponent and it consists of all the measurable function w defined on Ω for which

$$Q_{p(\cdot)}(\lambda w) < \infty, \text{ for some } \lambda > 0.$$

Let

$$\|w\|_{p(\cdot)} = \inf\left\{\lambda > 0; Q_{p(\cdot)}\left(\frac{w}{\lambda}\right) \leq 1\right\},$$

be the Luxembourg norm on this space (see [16]).

The Sobolev space $W^{1,q(\cdot)}(\Omega)$ consists of functions $w \in L^{q(\cdot)}(\Omega)$ whose distributional gradient $\nabla_x w$ exists and satisfies $|\nabla_x w| \in L^{q(\cdot)}(\Omega)$. This space is a Banach with respect to the norm

$$\|w\|_{1,q(\cdot)} = \|w\|_{q(\cdot)} + \|\nabla_x w\|_{q(\cdot)}.$$

Lemma 2.1 (Corollaries 8.2.5 and 8.3.2 in [9])**.** (1) *If* (1.6) *holds with $q(x)$, then*

$$\|w\|_{q(\cdot)} \leq C \|\nabla_x w\|_{q(\cdot)}, \forall w \in W_0^{1,q(\cdot)}(\Omega),$$

where Ω is a bounded domain and $C > 0$ is a constant. The norm of space $W_0^{1,q(\cdot)}(\Omega)$ is given by

$$\|w\|_{1,q(\cdot)} = \|\nabla_x w\|_{q(\cdot)}, \forall w \in W_0^{1,q(\cdot)}(\Omega).$$

(2) *If*

$$q \in C\left(\bar{\Omega}\right),\ p : \Omega \longrightarrow [1, \infty),$$

is a measurable function and

$$\operatorname*{ess\,inf}_{x\in\Omega} (q^*(x) - p(x)) > 0,$$

with

$$q^* = \frac{nq(x)}{(n - q(x))_+},$$

then

$$W_0^{1,q(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega),$$

with continuous and compact embedding and

$$\|w\|_{p(\cdot)} \leq C_S \|\nabla w\|_{q(\cdot)},$$

where $C_S > 0$ is an embedding constant.

Proposition 2.2 (Section 1 and Lemma 3.2.20 in [9]). *Let $1 < p_- \leq p_+ < +\infty$. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ \forall x \in \Omega.$$

For $w \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\left|\int_\Omega w(x)v(x)\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{(p')_-}\Big)\|w\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

Lemma 2.3 (Lemma 3.2.4 in [9]). *If $p(.) \geq 1$ is a measurable function on Ω and $w \in L^{p(\cdot)}(\Omega)$, then $\|w\|_{p(\cdot)} \leq 1$ and $Q_{p(\cdot)}(w) \leq 1$ are equivalent. For $w \in L^{p(\cdot)}(\Omega)$, we have*
(1) $\|w\|_{p(\cdot)} \leq 1$ *implies* $Q_{p(\cdot)}(w) \leq \|w\|_{p(\cdot)}$.
(2) $\|w\|_{p(\cdot)} > 1$ *implies* $Q_{p(\cdot)}(w) \geq \|w\|_{p(\cdot)}$.

Lemma 2.4 (Section 2 in [6], Lemma 3.2.5 in [9]). *If $p(.) \in [1, \infty)$ is a measurable function on Ω, then*

$$\min\{\|w\|^{p_-}_{p(\cdot)}, \|w\|^{p_+}_{p(\cdot)}\} \leq Q_{p(\cdot)}(w) \leq \max\{\|w\|^{p_-}_{p(\cdot)}, \|w\|^{p_+}_{p(\cdot)}\},$$

for all $w \in L^{p(\cdot)}(\Omega)$.

Lemma 2.5 (Lemma 3.2.20 in [9]). *If $p_1(.) \geq p_2(.) \geq 1$ a.e. in Ω, there is a continuous inclusion $L^{p_1(\cdot)}(\Omega) \subset L^{p_2(\cdot)}(\Omega)$ and for all $w \in L^{p_1(\cdot)}(\Omega)$,*

$$\|w\|_{p_2(\cdot)} \leq 2\|1\|_{r(\cdot)}\|w\|_{p_1(\cdot)},$$

where

$$\frac{1}{r(x)} \equiv \frac{1}{p_2(x)} - \frac{1}{p_1(x)}.$$

The following notation will be used throughout this paper

$$(\mu \circ v)(t) = \int_0^t \mu(t-s)\,\|v(t) - v(s)\|_2^2 \, ds,$$

for $v \in L^2(\Omega)$ and $t \geq 0$. We have the following technical Lemma.

Lemma 2.6. *Let $\kappa \in \mathbb{N}$. For any $\Delta_x^p v \in C^1\left(0, T, H_0^1(\Omega)\right)$ with $p = 0, 1, \ldots, \kappa - 1$, we have*

$$\int_\Omega \sigma(t) \int_0^t \mu(t-s)\Delta_x^\kappa v(s)\partial_t v(t)\,ds\,dx$$

$$
\begin{aligned}
&= \frac{(-1)^{\kappa+1}}{2}\frac{d}{dt}\left[\sigma(t)\left(\mu\circ\nabla_x^{\kappa}v\right)(t)\right]\\
&\quad+\frac{(-1)^{\kappa}}{2}\frac{d}{dt}\left[\sigma(t)\int_0^t\mu(s)\,ds\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx\right]\\
&\quad+\frac{(-1)^{\kappa}}{2}\sigma(t)\left(\partial_t\mu\circ\nabla_x^{\kappa}v\right)(t)+\frac{(-1)^{\kappa+1}}{2}\sigma(t)\mu(t)\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx\\
&\quad+\frac{(-1)^{\kappa}}{2}\partial_t\sigma(t)\left(\mu\circ\nabla_x^{\kappa}v\right)(t)+\frac{(-1)^{\kappa+1}}{2}\partial_t\sigma(t)\int_0^t\mu(s)\,ds\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx.
\end{aligned}
$$

Proof. Since

$$\int_\Omega\Delta_x^{\kappa}vw\,dx=(-1)^{\kappa}\int_\Omega\nabla_x^{\kappa}v\nabla_x^{\kappa}w\,dx,$$

holds, we have

$$
\begin{aligned}
&\int_\Omega\sigma(t)\int_0^t\mu(t-s)\Delta_x^{\kappa}v(s)\partial_tv(t)\,ds\,dx\\
&= (-1)^{\kappa}\sigma(t)\int_0^t\mu(t-s)\int_\Omega\nabla_x^{\kappa}\partial_tv(t)\left[\nabla_x^{\kappa}v(s)-\nabla_x^{\kappa}v(t)+\nabla_x^{\kappa}v(t)\right]\,dx\,ds\\
&= (-1)^{\kappa}\sigma(t)\int_0^t\mu(t-s)\int_\Omega\nabla_x^{\kappa}\partial_tv(t)\left[\nabla_x^{\kappa}v(s)-\nabla_x^{\kappa}v(t)\right]\,dx\,ds\\
&\quad+(-1)^{\kappa}\sigma(t)\int_0^t\mu(s)\,ds\int_\Omega\nabla_x^{\kappa}\partial_tv(t)\nabla_x^{\kappa}v(t)\,dx.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\int_\Omega\sigma(t)\int_0^t\mu(t-s)\Delta_x^{\kappa}v(s)\partial_tv(t)\,ds\,dx\\
&= \frac{(-1)^{\kappa+1}}{2}\sigma(t)\int_0^t\mu(t-s)\frac{d}{dt}\int_\Omega\left|\nabla_x^{\kappa}v(s)-\nabla_x^{\kappa}v(t)\right|^2\,dx\,ds\\
&\quad+\frac{(-1)^{\kappa}}{2}\sigma(t)\int_0^t\mu(s)\,ds\frac{d}{dt}\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_\Omega\sigma(t)\int_0^t\mu(t-s)\Delta_x^{\kappa}v(s)\partial_tv(t)\,ds\,dx\\
&= \frac{(-1)^{\kappa+1}}{2}\frac{d}{dt}\left[\sigma(t)\int_0^t\mu(t-s)\int_\Omega\left|\nabla_x^{\kappa}v(s)-\nabla_x^{\kappa}v(t)\right|^2\,dx\,ds\right]\\
&\quad+\frac{(-1)^{\kappa}}{2}\frac{d}{dt}\left[\sigma(t)\int_0^t\mu(s)\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx\,ds\right]\\
&\quad+\frac{(-1)^{\kappa}}{2}\sigma(t)\int_0^t\partial_t\mu(t-s)\int_\Omega\left|\nabla_x^{\kappa}v(s)-\nabla_x^{\kappa}v(t)\right|^2\,dx\,ds\\
&\quad+\frac{(-1)^{\kappa+1}}{2}\sigma(t)\mu(t)\int_\Omega\left|\nabla_x^{\kappa}v(t)\right|^2\,dx
\end{aligned}
$$

$$+\frac{(-1)^{\kappa}}{2}\partial_t\sigma(t)\int_0^t\mu(t-s)\int_\Omega\left|\nabla_x^\kappa v(s)-\nabla_x^\kappa v(t)\right|^2\,dx\,ds$$
$$+\frac{(-1)^{\kappa+1}}{2}\partial_t\sigma(t)\int_0^t\mu(s)\,ds\int_\Omega\left|\nabla_x^\kappa v(t)\right|^2\,dx.$$

This completes the proof. □

3. Main results

Here, we present without proof, the first known result concerning local existence (in time) for problem (1.2)–(1.4). (see [22])

Theorem 3.1. *Assume that* (1.5), (1.6) *and* (1.7) *hold. Then problem* (1.2)–(1.4) *has a unique local solution w satisfying*

$$w\in C\left([0,T_0];W_0^{1,q(\cdot)}(\Omega)\right),\ \partial_t w\in C\left([0,T_0],L^2(\Omega)\right)\cap L^2\left([0,T_0];H_0^1(\Omega)\right),$$

for $T_0>0$ depending on $\|w_0\|_{1,q(\cdot)}$.

Now, to prove the blow up result, we should define the energy functional $E(t)$, associated with our problem by

$$\begin{aligned}E(t) &= \int_\Omega\frac{|\nabla_x w(x,t)|^{q(x)}}{q(x)}\,dx-\int_\Omega\frac{|w(x,t)|^{p(x)}}{p(x)}\,dx+\frac{1}{2}\sigma(t)(\mu\circ\nabla_x w)(t)\\ &\quad -\frac{1}{2}\sigma(t)\int_0^t\mu(s)\,ds\int_\Omega|\nabla_x w(x,t)|^2\,dx.\end{aligned}$$

Lemma 3.2. *Let w be a solution of* (1.2)–(1.4) *with* (1.5)–(1.7). *Then the energy functional satisfies*

$$\begin{aligned}2\partial_t E(t) &= -2\|\partial_t w\|_2^2+\sigma(t)(\partial_t\mu\circ\nabla_x w)-\sigma(t)\mu(t)\|\nabla_x w\|_2^2\\ &\quad +\partial_t\sigma(t)(\mu\circ\nabla_x w)-\partial_t\sigma(t)\int_0^t\mu(s)\,ds\,\|\nabla_x w\|_2^2\\ &\le 0.\end{aligned}$$

Proof. Multiplying (1.2) by $\partial_t w$, integrating by parts over Ω, using (1.7) and Lemma 2.6, we get the desired result. □

By using conditions (1.6), (1.8) and thanks to

$$\left(\int_\Omega|w|^2dx\right)^{1/2}\le\left(\int_\Omega|w|^r dx\right)^{1/r},r\ge 2,$$

there exists a constant

$$\lambda=\frac{1}{q_+}-\frac{1}{q_-}\|\sigma\|_\infty\int_0^\infty\mu(s)\,ds,$$

by (1.5) and (1.7), we have

$$\frac{1}{p_-} < \lambda < \frac{1}{q_+} < \frac{1}{2}, \tag{3.1}$$

such that

$$E(t) \geq \lambda \int_\Omega |\nabla_x w(x,t)|^{q(x)}\, dx - \int_\Omega \frac{|w(x,t)|^{p(x)}}{p(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t). \tag{3.2}$$

Let f be a function defined by

$$\begin{array}{rccc} f: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & \psi & \longmapsto & \psi - C\psi^{\frac{p_+}{q_-}}, \end{array}$$

Then f is increasing in $(0,\alpha)$, decreasing for $\psi > \alpha$, $f(\psi) \longrightarrow -\infty$ as $\psi \longrightarrow +\infty$ and

$$f(\alpha) = \frac{p_+ - q_-}{q_- p_+}\alpha \equiv E_1, \tag{3.3}$$

for $\alpha = \left(\frac{q_-}{Cp_+}\right)^{\frac{q_-}{p_+ - q_-}}$.

Lemma 3.3. *Let w be strong solution of* (1.2)–(1.4) *with* (1.5)–(1.7) *and initial condition satisfying*

$$0 < E(0) < E_1 \text{ and } \int_\Omega |\nabla_x w_0|^{q(x)}\, dx > \alpha. \tag{3.4}$$

Then there exists a constant $\beta > \alpha$ such that

$$\lambda \int_\Omega |\nabla_x w(t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \geq \beta, \tag{3.5}$$

and

$$\int_\Omega |w|^{p(x)}\, dx \geq C\beta^{p_+}, \tag{3.6}$$

for all $t \in [0, T_0)$.

Proof. We define the sets

$$\Omega_- = \{x \in \Omega / |w| < 1\},$$

and

$$\Omega_+ = \{x \in \Omega / |w| \geq 1\},$$

By (3.2) and the Sobolev embedding, we get

$$\begin{aligned} E(t) &\geq \lambda \int_\Omega |\nabla_x w(x,t)|^{q(x)}\, dx - \int_\Omega \frac{|w(x,t)|^{p(x)}}{p(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\ &\geq \lambda \int_\Omega |\nabla_x w(x,t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\ &- \frac{1}{p_-}\Big[\int_{\Omega_-} |w(x,t)|^{p(x)}\, dx + \int_{\Omega_+} |w(x,t)|^{p(x)}\, dx\Big] \end{aligned}$$

$$
\begin{aligned}
&\geq \lambda \int_{\Omega} |\nabla_x w(x,t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\
&- \frac{1}{p_-}\Big[\int_{\Omega_-} |w(x,t)|^{p_-}\, dx + \int_{\Omega_+} |w(x,t)|^{p_+}\, dx\Big] \\
&\geq \lambda \int_{\Omega} |\nabla_x w(x,t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\
&- \frac{2}{p_-}\int_{\Omega} |w(x,t)|^{p_+}\, dx \\
&\geq \lambda \int_{\Omega} |\nabla_x w(x,t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\
&- \frac{2C^{p_+}}{p_-}\lambda^{p_+}\Big(\int_{\Omega} |\nabla_x w(x,t)|^{2}\, dx\Big)^{p_+/2} \\
&\geq \lambda \int_{\Omega} |\nabla_x w(x,t)|^{q(x)}\, dx + \frac{1}{2}\sigma(t)(\mu \circ \nabla_x w)(t) \\
&- \frac{2C^{p_+}}{p_-}\Big[\lambda \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t)\Big]^{p_+/q_+}. \qquad (3.7)
\end{aligned}
$$

Since $E(0) < E_1$, there exists $\beta > \alpha$ such that $f(\beta) = E(0)$, by using (3.7) we have $f\Big(\int_{\Omega} |\nabla_x w(x,0)|^{p(x)}\, dx\Big) \leq E(0)$, which implies that $\int_{\Omega} |\nabla_x w(x,0)|^{p(x)}\, dx \geq \beta$.
Now, to establish (3.5), we suppose by contradiction that

$$
\Big[\Big(\frac{1}{q_+} - \frac{1}{q_-}\sigma(t_0) \int_0^{t_0} \mu(\tau)d\tau\Big) \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t_0)\Big]^{1/q_+} < \beta,
$$

for some $t_0 \geq 0$ and by continuity of

$$
\lambda \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t_0),
$$

we have to choose t_0 such that

$$
\Big[\Big(\frac{1}{q_+} - \frac{1}{q_-}\sigma(t_0) \int_0^{t_0} \mu(\tau)d\tau\Big) \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t_0)\Big]^{1/q_+} < \alpha.
$$

Again the use of (3.7) leads to

$$
\begin{aligned}
E(t_0) &\geq f\Big(\Big[\Big(\frac{1}{q_+} - \frac{1}{q_-}\sigma(t_0) \int_0^{t_0} \mu(\tau)d\tau\Big) \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t_0)\Big]^{1/q_+}\Big) \\
&> f(\beta) = E(0).
\end{aligned}
$$

which is not possible, since $E(t) \leq E(0), \forall t \in [0,T)$. Then (3.5) is proved.
We use (3.2) and (3.7) to prove (3.6)

$$
\lambda \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t) \leq E(0) + \int_{\Omega} \frac{|w(x,t)|^{p(x)}\, dx}{p(x)},
$$

this imply

$$
\int_{\Omega} \frac{|w(x,t)|^{p(x)}\, dx}{p(x)} \quad \geq \quad \lambda \int_{\Omega} |\nabla_x w(x,t)|^{p(x)}\, dx + \sigma(t)(\mu \circ \nabla_x w)(t) - E(0)
$$

$$\geq \beta^{q_+} - E(0)$$
$$= \beta^{q_+} - f(\beta).$$

Therefore the desired results are proved. □

Combining with the estimates obtained in the above Lemmas, we state the main result concerned with finite time blow-up.

Theorem 3.4. *Assume that* (1.5)–(1.7) *hold. Given* $w_0 \in W_0^{1,q(\cdot)}(\Omega)$ *satisfying* (3.4)*, any solution of* (1.2) *with* (1.3) *and* (1.4) *blows up in finite time* $t^* < \infty$.

Proof. Set $H(t) \leq E_1 - E(t)$. We define

$$L(t) = \frac{1}{2}\int_\Omega |w(t)|^2\,dx.$$

By differentiating L, we get

$$\begin{aligned}
\partial_t L(t) &= \textstyle\int_\Omega w\partial_t w\,dx \\
&= \textstyle\int_\Omega w\Big[\operatorname{div}\Big(|\nabla_x w|^{q(x)-2}\,\nabla_x w\Big) - \sigma(t)\int_0^t \mu(t-s)\,\Delta_x w(s)\,ds + |w|^{p(x)-2}\,w\Big]\,dx \\
&= -\textstyle\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx + \int_\Omega |w(t)|^{p(x)}\,dx \\
&\quad + \textstyle\int_\Omega \sigma(t)\int_0^t \mu(t-s)\,\nabla_x w(s)\,\nabla_x w(t)\,ds\,dx.
\end{aligned}$$

Using Cauchy Schwarz's inequality and Lemma 2.6 to obtain

$$\begin{aligned}
&\int_\Omega \nabla_x w(s)\,\nabla_x w(t)\,dx \\
&= \int_\Omega |\nabla_x w(t)|^2\,dx - \int_\Omega \nabla_x w(t)\,(\nabla_x w(t) - \nabla_x w(s))\,dx \\
&\geq \int_\Omega |\nabla_x w(t)|^2\,dx - \int_\Omega \frac{1}{C_0}\,|\nabla_x w(t)| \times C_0\,|\nabla_x w(t) - \nabla_x w(s)|\,dx \\
&\geq \frac{2C_0^2 - 1}{2C_0^2}\,\|\nabla_x w(t)\|_2^2 - \frac{C_0^2}{2}\,\|\nabla_x w(t) - \nabla_x w(s)\|_2^2,
\end{aligned}$$

for some positive constant $C_0 > 0$ (to be determined later). Then, we have

$$\begin{aligned}
\partial_t L(t) \geq\; & -\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx + \int_\Omega |w(t)|^{p(x)}\,dx + \frac{2C_0^2-1}{2C_0^2}\sigma(t)\,\|\nabla_x w(t)\|_2^2\int_0^t \mu(s)\,ds \\
& -\frac{C_0^2}{2}\sigma(t)\,(\mu \circ \nabla_x w)(t).
\end{aligned}$$

By using Young's inequality, (1.8) and Lemma 2.6 for some constant $c > 0$, we obtain

$$\partial_t L(t) \geq -c\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx + \int_\Omega |w(t)|^{p(x)}\,dx - \sigma(t)\,(\mu \circ \nabla_x w)(t). \tag{3.8}$$

We then substitute for $\sigma(t)\,(\mu\circ\nabla_x w)\,(t)$ from (3.2), hence (3.8) becomes

$$
\begin{aligned}
\partial_t L(t) &\geq -c\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx+\int_\Omega |w(t)|^{p(x)}\,dx \\
&- \Big[2E(t)-\lambda\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx+\frac{2}{p_+}\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx\Big] \\
&\geq \Big(1-\frac{2}{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx-(c-\lambda)\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx+2H(t)-2E_1,
\end{aligned}
\tag{3.9}
$$

where $0\leq H(t)\leq E_1-E(t)$. By using (1.8) and (3.6), the estimate (3.9) takes the form

$$
\begin{aligned}
\partial_t L(t) &\geq 2H(t)+\Big(1-\frac{2}{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx-(c-\lambda)\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx-2E_1 \\
&\geq \Big(1-\frac{2}{p_+}-\rho\frac{2p_+}{p_-}\Big(\frac{\alpha}{\beta}\Big)^{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx-(c-\lambda)\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx.
\end{aligned}
$$

This implies that, we can choose $c>C_0+\lambda$ to get

$$
\partial_t L(t) \geq \Big(-\rho\frac{2p_+}{p_-}\Big(\frac{\alpha}{\beta}\Big)^{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx+C_0\Big(\int_\Omega |w(t)|^{p(x)}\,dx-\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx\Big),
$$

where $C_0=1-\frac{2}{p_-}$. Then

$$
\begin{aligned}
\partial_t L(t) &\geq C_0\Big[\frac{1}{\lambda}\Big(\lambda-\frac{2}{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx+\frac{1}{\lambda p_+}\int_\Omega |w(t)|^{p(x)}\,dx-\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx\Big] \\
&- \Big(\rho\frac{2p_+}{p_-}\Big(\frac{\alpha}{\beta}\Big)^{p_+}\Big)\int_\Omega |w(t)|^{p(x)}\,dx \\
&\geq \Big[C_0\frac{1}{\lambda}\Big(\lambda-\frac{2}{p_+}\Big)-\rho\frac{2p_+}{p_-}\Big(\frac{\alpha}{\beta}\Big)^{p_+}\Big]\int_\Omega |w(t)|^{p(x)}\,dx.
\end{aligned}
$$

Since

$$
\frac{1}{\lambda p_+}\int_\Omega |w(t)|^{p(x)}\,dx-\int_\Omega |\nabla_x w(t)|^{q(x)}\,dx.
$$

Then

$$
\begin{aligned}
\partial_t L(t) &\geq C_1\Big(\int_{\Omega_-} |w(t)|^{p(x)}\,dx+\int_{\Omega_+} |w(t)|^{p(x)}\,dx\Big) \\
&\geq C_2\Big(\Big(\int_{\Omega_-} |w(t)|^2\,dx\Big)^{p_+/2}+\Big(\int_{\Omega_+} |w(t)|^2\,dx\Big)^{p_-/2}\Big).
\end{aligned}
$$

This implies that

$$
\partial_t L(t) \geq C_2\Big(\int_{\Omega_+} |w(t)|^2\,dx\Big)^{p_-/2},
$$

and

$$
\partial_t L(t) \geq C_2\Big(\int_{\Omega_-} |w(t)|^2\,dx\Big)^{p_+/2}.
$$

Then

$$(\partial_t L(t))^{2/p_-} \;\geq\; C_2^{2/p_-} \int_{\Omega_+} |w(t)|^2 \, dx,$$

and

$$(\partial_t L(t))^{2/p_+} \;\geq\; C_2^{2/p_-} \int_{\Omega_-} |w(t)|^2 \, dx.$$

By addition, it leads to

$$(\partial_t L(t))^{2/p_-} + (\partial_t L(t))^{2/p_+} \;\geq\; C_3 L(t), \forall t \geq 0, \tag{3.10}$$

where $C_3 = \min\{C_2^{2/p_-}, C_2^{2/p_+}\}$. Or

$$(\partial_t L(t))^{2/p_-}\Big[1 + (\partial_t L(t))^{2(1/p_+ - 1/p_-)}\Big] \geq C_3 L(t). \tag{3.11}$$

Using (3.10) and since $0 < L(0) \leq L(T)$, we have for any $t > 0$ either

$$(\partial_t L(t))^{2/p_-} \geq \frac{C_3}{2} L(t) \geq \frac{C_3}{2} L(0). \tag{3.12}$$

Or

$$(\partial_t L(t))^{2/p_+} \geq \frac{C_3}{2} L(t) \geq \frac{C_3}{2} L(0), \tag{3.13}$$

which gives, in turn

$$\partial_t L(t) \geq \Big(\frac{C_3}{2} L(0)\Big)^{p_-/2}. \tag{3.14}$$

Or

$$\partial_t L(t) \geq \Big(\frac{C_3}{2} L(0)\Big)^{p_+/2}. \tag{3.15}$$

Therefore $\partial_t L(t) \geq \gamma$, where $\gamma = \min\Big\{\Big(\frac{C_3}{2}\Big)^{p_-/2}, \Big(\frac{C_3}{2}\Big)^{p_+/2}\Big\}$, since $\frac{1}{p_+} - \frac{1}{p_-} \leq 0$ and (1.6), (3.11) yields

$$\partial_t L(t) > \delta\, (L(t))^{\frac{q_-}{2}} \;\text{ for all } t > 0, \tag{3.16}$$

where

$$\delta = \Big(C_0^2 \lambda - 1\Big)\left(\frac{1}{2(1 + C_P^2)} \|1\|^{-2}_{\frac{2q}{q-2}}\right)^{\frac{q_-}{2}} > 0.$$

A direct integration of (3.16) over $[0, t]$ yields

$$(L(t))^{1-\frac{q_-}{2}} < \Big(1 - \frac{q_-}{2}\Big)\delta t + (L(0))^{1-\frac{q_-}{2}},$$

which implies that

$$L(t) > \frac{1}{\Big((L(0))^{1-\frac{q_-}{2}} - \Big(\frac{q_-}{2} - 1\Big)\delta t\Big)^{\frac{2}{q_- - 2}}},$$

along with $1 - \frac{q_-}{2} < 0$. Finally, we have

$$L(t) \longrightarrow +\infty \;\text{ when }\; t \longrightarrow t^{*-}, \;\text{ where }\; t^* = \frac{L(0)^{1-\frac{q_-}{2}}}{\Big(\frac{q_-}{2} - 1\Big)\delta}.$$

□

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Conflict of interest

The author agree with the contents of the manuscript, and there is no conflict of interest among the author.

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