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Research article

Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with $(p, q)-$ Lucas polynomials

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Abstract: By means of $(p, q)-$ Lucas polynomials, a class of Bazilevič functions of order $\vartheta + i\delta$ in the open unit disk $\mathbb{U}$ of analytic and bi-univalent functions is introduced. Further, we estimate coefficients bounds and Fekete-Szegö inequalities for functions belonging to this class. Several corollaries and consequences of the main results are also obtained.

Keywords: Bazilevič functions; Lucas polynomial; analytic functions; univalent functions; bi-univalent functions
Mathematics Subject Classification: 30C45

1. Introduction and preliminaries

Let $\mathcal{A}$ indicate an analytic functions family, which is normalized under the condition $f(0) = f'(0) - 1 = 0$ in $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$\omega(0) = 0,\ |\omega(z)| < 1,\ (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)).$$

We denote this subordination by

$$f \prec g \text{ or } f(z) \prec g(z).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0),\ f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Koebe-One Quarter Theorem [11] asserts that image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disc of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f), r_0(f) > \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent functions in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exists a function $g \in \mathcal{S}$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathbb{U}$. Let Λ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Λ (see details in [20]). However, the familiar Koebe function is not bi-univalent. Lewin [17] investigated the class of *bi-univalent* functions Λ and obtained a bound $|a_2| \leq 1.51$. Motivated by the work of Lewin [17], Brannan and Clunie [9] conjectured that $|a_2| \leq \sqrt{2}$. The coefficient estimate problem for $|a_n|$ $(n \in \mathbb{N},\ n \geq 3)$ is still open ([20]). Brannan and Taha [10] also worked on certain subclasses of the bi-univalent function class Λ and obtained estimates for their initial coefficients. Various classes of bi-univalent functions were introduced and studied in recent times, the study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava et al. [20]. Motivated by this, many researchers [1], [4–8], [13–15], [20], [21], and [27–29], also the references cited there in) recently investigated several interesting subclasses of the class Λ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients. Recently, many researchers have been exploring bi-univalent functions, few to mention Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Pell polynomials, Lucas–Lehmer polynomials, orthogonal polynomials and the other special polynomials and their generalizations are of great importance in a variety of branches such as physics, engineering, architecture, nature, art, number theory, combinatorics and numerical analysis. These polynomials have been studied in several papers from a theoretical point of view (see, for example, [23–30] also see references therein).

We recall the following results relevant for our study as stated in [3].

Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p, q)-$Lucas polynomials $\mathcal{L}_{p,q,n}(x)$ are defined by the recurrence relation

$$\mathcal{L}_{p,q,n}(x) = p(x)\mathcal{L}_{p,q,n-1}(x) + q(x)\mathcal{L}_{p,q,n-2}(x) \quad (n \geq 2),$$

from which the first few Lucas polynomials can be found as

$$\begin{aligned}
\mathcal{L}_{p,q,0}(x) &= 2, \\
\mathcal{L}_{p,q,1}(x) &= p(x), \\
\mathcal{L}_{p,q,2}(x) &= p^2(x) + 2q(x), \\
\mathcal{L}_{p,q,3}(x) &= p^3(x) + 3p(x)q(x), ...
\end{aligned} \tag{1.3}$$

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given $\mathcal{L}_{x,1,n}(x) \equiv \mathcal{L}_n(x)$ Lucas polynomials, $\mathcal{L}_{2x,1,n}(x) \equiv \mathcal{D}_n(x)$ Pell–Lucas polynomials, $\mathcal{L}_{1,2x,n}(x) \equiv j_n(x)$ Jacobsthal–Lucas polynomials, $\mathcal{L}_{3x,-2,n}(x) \equiv F_n(x)$ Fermat–Lucas polynomials, $\mathcal{L}_{2x,-1,n}(x) \equiv T_n(x)$ Chebyshev polynomials first kind.

Lemma 1.1. *[16] Let $G\{\mathcal{L}(x)\}(z)$be the generating function of the $(p,q)-$Lucas polynomial sequence $\mathcal{L}_{p,q,n}(x)$. Then,*

$$G\{\mathcal{L}(x)\}(z) = \sum_{n=0}^{\infty} \mathcal{L}_{p,q,n}(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}$$

and

$$\mathcal{G}_{\{\mathcal{L}(x)\}}(z) = G\{\mathcal{L}(x)\}(z) - 1 = 1 + \sum_{n=1}^{\infty} \mathcal{L}_{p,q,n}(x)z^n = \frac{1 + q(x)z^2}{1 - p(x)z - q(x)z^2}.$$

Definition 1.2. [22] For $\vartheta \geq 0$, $\delta \in \mathbb{R}$, $\vartheta + i\delta \neq 0$ and $f \in \mathcal{A}$, let $\mathcal{B}(\vartheta, \delta)$ denote the class of Bazilevič function if and only if

$$Re\left[\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\vartheta+i\delta}\right] > 0 \ .$$

Several authors have researched different subfamilies of the well-known Bazilevič functions of type ϑ from various viewpoints (see [3] and [19]). For Bazilevič functions of order $\vartheta + i\delta$, there is no much work associated with Lucas polynomials in the literature. Initiating an exploration of properties of Lucas polynomials associated with Bazilevič functions of order $\vartheta + i\delta$ is the main goal of this paper. To do so, we take into account the following definitions. In this paper motivated by the very recent work of Altinkaya and Yalcin [3] (also see [18]) we define a new class $\mathcal{B}(\vartheta, \delta)$, bi-Bazilevič function of Λ based on $(p,q)-$ Lucas polynomials as below:

Definition 1.3. For $f \in \Lambda$, $\vartheta \geq 0$, $\delta \in \mathbb{R}$, $\vartheta + i\delta \neq 0$ and let $\mathcal{B}(\vartheta, \delta)$ denote the class of Bi-Bazilevič functions of order $\vartheta + i\delta$ if only if

$$\left[\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\vartheta+i\delta}\right] \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(z) \quad (z \in \mathbb{U}) \tag{1.4}$$

and

$$\left[\left(\frac{zg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^{\vartheta+i\delta}\right] \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(w) \quad (w \in \mathbb{U}), \tag{1.5}$$

where $\mathcal{G}_{\mathcal{L}_{p,q,n}}(z) \in \Phi$ and the function g is described as $g(w) = f^{-1}(w)$.

Remark 1.4. We note that for $\delta = 0$ the class $R(\vartheta, 0) = R(\vartheta)$ is defined by Altinkaya and Yalcin [2].

The class $\mathcal{B}(0,0) = \mathcal{S}^*_\Lambda$ is defined as follows:

Definition 1.5. A function $f \in \Lambda$ is said to be in the class $\mathcal{S}^*_\Lambda$, if the following subordinations hold

$$\frac{zf'(z)}{f(z)} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(z)(z \in \mathbb{U})$$

and

$$\frac{wg'(w)}{g(w)} \prec \mathcal{G}_{\{\mathcal{L}(x)\}}(w)(w \in \mathbb{U})$$

where $g(w) = f^{-1}(w)$.

We begin this section by finding the estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}(\vartheta, \delta)$.

2. Coefficient bounds for the function class $\mathcal{B}(\vartheta, \delta)$

Theorem 2.1. *Let the function $f(z)$ given by 1.1 be in the class $\mathcal{B}(\vartheta, \delta)$. Then*

$$|a_2| \leq \frac{p(x)\sqrt{2p(x)}}{\sqrt{|\{\left((\vartheta + i\delta)^2 + 3(\vartheta + i\delta) + 2\right) - 2(\vartheta + i\delta + 1)^2\}p^2(x) - 4q(x)(\vartheta + i\delta + 1)^2|}}.$$

and

$$|a_3| \leq \frac{p^2(x)}{(\vartheta + 1)^2 + \delta^2} + \frac{p(x)}{\sqrt{(\vartheta + 2)^2 + \delta^2}}.$$

Proof. Let $f \in \mathcal{B}(\vartheta, \delta)$ there exist two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = 0 = v(0)$, such that $|u(z)| < 1$, $|v(w)| < 1$, we can write from (1.4) and (1.5), we have

$$\left[\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\vartheta + i\delta}\right] = \mathcal{G}_{\{\mathcal{L}(x)\}}(z) \quad (z \in \mathbb{U}), \tag{2.1}$$

and

$$\left[\left(\frac{zg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^{\vartheta + i\delta}\right] = \mathcal{G}_{\{\mathcal{L}(x)\}}(w) \quad (w \in \mathbb{U}), \tag{2.2}$$

It is fairly well known that if

$$|u(z)| = |u_1 z + u_2 z^2 + \cdots| < 1,$$

and

$$|v(w)| = |v_1 w + v_2 w^2 + \cdots| < 1,$$

then

$$|u_k| \leq 1 \qquad \text{and} \qquad |v_k| \leq 1 \quad (k \in \mathbb{N}),$$

so we have,

$$\begin{aligned}\mathcal{G}_{\{\mathcal{L}(x)\}}(u(z)) &= 1+\mathcal{L}_{p,q,1}(x)u(z)+\mathcal{L}_{p,q,2}(x)u^2(z)+\dots \\ &= 1+\mathcal{L}_{p,q,1}(x)u_1z+[\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2]z^2+\dots \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathcal{G}_{\{\mathcal{L}(x)\}}(v(w)) &= 1+\mathcal{L}_{p,q,1}(x)v(w)+\mathcal{L}_{p,q,2}(x)v^2(w)+\dots \\ &= 1+\mathcal{L}_{p,q,1}(x)v_1w+[\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2]w^2+\dots \end{aligned} \tag{2.4}$$

From the equalities (2.1) and (2.2), we obtain that

$$\left[\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\vartheta+i\delta}\right]=1+\mathcal{L}_{p,q,1}(x)u_1z+[\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2]z^2+\dots, \tag{2.5}$$

and

$$\left[\left(\frac{zg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^{\vartheta+i\delta}\right]=1+\mathcal{L}_{p,q,1}(x)v_1w+[\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2]w^2+\dots, \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$(\vartheta+i\delta+1)\,a_2=\mathcal{L}_{p,q,1}(x)u_1, \tag{2.7}$$

$$\frac{(\vartheta+i\delta-1)\,(\vartheta+i\delta+2)}{2}a_2^2-(\vartheta+i\delta+2)\,a_3=\mathcal{L}_{p,q,1}(x)u_2+\mathcal{L}_{p,q,2}(x)u_1^2, \tag{2.8}$$

and

$$-\,(\vartheta+i\delta+1)\,a_2=\mathcal{L}_{p,q,1}(x)v_1, \tag{2.9}$$

$$\frac{(\vartheta+i\delta+2)\,(\vartheta+i\delta+3)}{2}a_2^2+(\vartheta+i\delta+2)\,a_3=\mathcal{L}_{p,q,1}(x)v_2+\mathcal{L}_{p,q,2}(x)v_1^2. \tag{2.10}$$

From (2.7) and (2.9)

$$u_1=-v_1 \tag{2.11}$$

and

$$2\,(\vartheta+i\delta+1)^2\,a_2^2=\mathcal{L}_{p,q,1}^2(x)(u_1^2+v_1^2), \tag{2.12}$$

by adding (2.8) to (2.10) , we get

$$\Big((\vartheta+i\delta)^2+3\,(\vartheta+i\delta)+2\Big)\,a_2^2=\mathcal{L}_{p,q,1}(x)(u_2+v_2)+\mathcal{L}_{p,q,2}(x)(u_1^2+v_1^2), \tag{2.13}$$

by using (2.12) in equality (2.13), we have

$$\left[\Big((\vartheta+i\delta)^2+3\,(\vartheta+i\delta)+2\Big)-\frac{2\mathcal{L}_{p,q,2}(x)\,(\vartheta+i\delta+1)^2}{\mathcal{L}_{p,q,1}^2(x)}\right]a_2^2 \;=\; \mathcal{L}_{p,q,1}(x)(u_2+v_2),$$

$$a_2^2 \;=\; \frac{\mathcal{L}_{p,q,1}^3(x)(u_2+v_2)}{\left[\Big((\vartheta+i\delta)^2+3\,(\vartheta+i\delta)+2\Big)\mathcal{L}_{p,q,1}^2(x)-2\mathcal{L}_{p,q,2}(x)\,(\vartheta+i\delta+1)^2\right]}. \tag{2.14}$$

Thus, from (1.3) and (2.14) we get

$$|a_2| \leq \frac{p(x)\sqrt{2p(x)}}{\sqrt{\left|\left\{\left((\vartheta+i\delta)^2+3(\vartheta+i\delta)+2\right)-2(\vartheta+i\delta+1)^2\right\}p^2(x)-4q(x)(\vartheta+i\delta+1)^2\right|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we obtain

$$\begin{aligned} 2(\vartheta+i\delta+2)a_3-2(\vartheta+i\delta+2)a_2^2 &= \mathcal{L}_{p,q,1}(x)(u_2-v_2)+\mathcal{L}_{p,q,2}(x)(u_1^2-v_1^2) \\ 2(\vartheta+i\delta+2)a_3 &= \mathcal{L}_{p,q,1}(x)(u_2-v_2)+2(\vartheta+i\delta+2)a_2^2 \\ a_3 &= \frac{\mathcal{L}_{p,q,1}(x)(u_2-v_2)}{2(\vartheta+i\delta+2)}+a_2^2 \end{aligned} \tag{2.15}$$

Then, in view of (2.11) and (2.12), we have from (2.15)

$$a_3 = \frac{\mathcal{L}_{p,q,1}^2(x)}{2(\vartheta+i\delta+2)^2}(u_1^2+v_1^2)+\frac{\mathcal{L}_{p,q,1}(x)}{2(\vartheta+i\delta+2)}(u_2-v_2).$$

$$\begin{aligned} |a_3| &\leq \frac{p^2(x)}{|\vartheta+i\delta+1|^2}+\frac{p(x)}{|\vartheta+i\delta+2|} \\ &= \frac{p^2(x)}{(\vartheta+1)^2+\delta^2}+\frac{p(x)}{\sqrt{(\vartheta+2)^2+\delta^2}} \end{aligned}$$

This completes the proof. □

Taking $\delta=0$, in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}(\vartheta)$. Then*

$$|a_2| \leq \frac{p(x)\sqrt{2p(x)}}{\sqrt{\left|\left\{(\vartheta^2+3\vartheta+2)-2(\vartheta+1)^2\right\}p^2(x)-4q(x)(\vartheta+1)^2\right|}}$$

and

$$|a_3| \leq \frac{p^2(x)}{(\vartheta+2)^2}+\frac{p(x)}{\vartheta+2}$$

Also, taking $\vartheta=0$ and $\delta=0$, in Theorem 2.1, we get the results given in [18].

3. Fekete-Szegö inequality for the class $\mathcal{B}(\vartheta,\delta)$

Fekete-Szegö inequality is one of the famous problems related to coefficients of univalent analytic functions. It was first given by [12], the classical Fekete-Szegö inequality for the coefficients of $f \in \mathcal{S}$ is

$$\left|a_3-\mu a_2^2\right| \leq 1+2\exp\left(-2\mu/(1-\mu)\right) \text{ for } \mu \in [0,1).$$

As $\mu \to 1^-$, we have the elementary inequality $\left|a_3 - a_2^2\right| \le 1$. Moreover, the coefficient functional

$$\varsigma_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the unit disk $\mathbb{U}$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\varsigma\mu(f)$ is called the Fekete-Szegö problem.

In this section, we are ready to find the sharp bounds of Fekete-Szegö functional $\varsigma_\mu(f)$ defined for $f \in \mathcal{B}(\vartheta, \delta)$ given by (1.1).

Theorem 3.1. *Let f given by (1.1) be in the class $\mathcal{B}(\vartheta, \delta)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{p(x)}{\sqrt{(\vartheta+2)^2+\delta^2}}, & 0 \le |h(\mu)| \le \frac{1}{2\sqrt{(\vartheta+2)^2+\delta^2}} \\ 2p(x)\,|h(\mu)|, & |h(\mu)| \ge \frac{1}{2\sqrt{(\vartheta+2)^2+\delta^2}} \end{cases}$$

where

$$h(\mu) = \frac{\mathcal{L}_{p,q,1}^2(x)(1-\mu)}{\left((\vartheta + i\delta)^2 + 3\,(\vartheta + i\delta) + 2\right)\mathcal{L}_{p,q,1}^2(x) - 2\mathcal{L}_{p,q,2}(x)\,(\vartheta + i\delta + 1)^2}.$$

Proof. From (2.14) and (2.15), we conclude that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{\mathcal{L}_{p,q,1}^3(x)(u_2+v_2)}{\left[\left((\vartheta + i\delta)^2 + 3\,(\vartheta + i\delta) + 2\right)\mathcal{L}_{p,q,1}^2(x) - 2\mathcal{L}_{p,q,2}(x)\,(\vartheta + i\delta + 1)^2\right]} \\ &+ \frac{\mathcal{L}_{p,q,1}(x)}{2\,(\vartheta + i\delta + 2)}(u_2 - v_2) \end{aligned}$$

$$= \mathcal{L}_{p,q,1}(x)\left[\left(h(\mu) + \frac{1}{2(\vartheta + i\delta + 2)}\right)u_2 + \left(h(\mu) - \frac{1}{2(\vartheta + i\delta + 2)}\right)v_2\right]$$

where

$$h(\mu) = \frac{\mathcal{L}_{p,q,1}^2(x)(1-\mu)}{\left((\vartheta + i\delta)^2 + 3\,(\vartheta + i\delta) + 2\right)\mathcal{L}_{p,q,1}^2(x) - 2\mathcal{L}_{p,q,2}(x)\,(\vartheta + i\delta + 1)^2}.$$

Then, in view of (1.3), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{p(x)}{\sqrt{(\vartheta+2)^2+\delta^2}}, & 0 \le |h(\mu)| \le \frac{1}{2\sqrt{(\vartheta+2)^2+\delta^2}} \\ 2p(x)\,|h(\mu)|, & |h(\mu)| \ge \frac{1}{2\sqrt{(\vartheta+2)^2+\delta^2}} \end{cases}$$

□

We end this section with some corollaries.

Taking $\mu = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.2. *If $f \in \mathcal{B}(\vartheta, \delta)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{p(x)}{\sqrt{(\vartheta+2)^2+\delta^2}}.$$

Taking $\delta = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{B}(\vartheta, 0)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{p(x)}{\vartheta+2}, & 0 \leq |h(\mu)| \leq \frac{1}{2(\vartheta+2)} \\ 2p(x)\,|h(\mu)|, & |h(\mu)| \geq \frac{1}{2(\vartheta+2)} \end{cases}$$

Also, taking $\vartheta = 0$, $\delta = 0$ and $\mu = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.4. *Let f given by (1.1) be in the class $\mathcal{B}$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{p(x)}{2}.$$

Conflict of interest

All authors declare no conflicts of interest in this paper.

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