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Research article

Putnam-Fuglede type theorem for class $\mathcal{A}_k$ operators

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Abstract: We will call $U \in B(X)$ as an operator of class $\mathcal{A}_k$ if for some integer k, the following inequality is satisfied:

$$|U^{k+1}|^{\frac{2}{k+1}} \geq |U|^2.$$

In the present article, some basic spectral properties of this class are given, also the asymmetric Putnam-Fuglede theorem and the range kernel orthogonality for class $\mathcal{A}_k$ operators are proved.

Keywords: Putnam-Fuglede theorem; hyponormal operator; class $\mathcal{A}_k$ operator
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1. Introduction

Spectral theory has a key important role in the modern functional analysis and its applications in various fields [4, 15]. Basically, it is incorporated with specific inverse operators, their common properties and their dealings with the original operators. Such inverse operators play a major role in solving systems of linear algebraic equations, differential and Sylvester equations.

Everywhere in this paper, a complex Hilbert space of infinite dimension with the inner product $\langle \cdot, \cdot \rangle$ will be denoted by X and $B(X)$ indicates the algebra of all linear bounded operators which act on X. Spectrum, approximate spectrum, residual spectrum, and point spectrum of an operator U will be denoted by $\sigma(U)$, $\sigma_a(U)$, $\sigma_r(U)$, and $\sigma_p(U)$, respectively. The kernel of an operator U will be denoted by $\ker(U)$ and the range by $ran\,(U)$.

For each operator $U \in B(X)$, we set, as usual $|U| = (U^*U)^{1/2}$, and review the following standard (familiar) definitions:

U is normal if $U^*U = UU^*$, and

U is hyponormal if $|U^*|^2 \leq |U|^2$,

(i.e. equivalently, if $\|U^*x\| \leq \|Ux\|$ for every $x \in X$).

An operator $U \in B(X)$ is said to be of class $\mathcal{A}$ if and only if $|U^2| \geq |U|^2$.

The class of hyponormal operator has been studied by many authors. In recent years this class has been generalized, in some sense, to the larger sets of so called class $p-$hyponormal, log $-hyponormal$ [21], w-hyponormal [2] and class $\mathcal{A}$ operators [19].

Definition 1. An operator $U \in B(X)$ is said to be class $\mathcal{A}_k$ operator if

$$|U^{k+1}|^{\frac{2}{k+1}} \geq |U|^2,$$

holds for some integer k.

The class $\mathcal{A}$ coincides with class $\mathcal{A}_k$ when $k = 1$.

Example 2. If $U \in B(X)$ is a bilateral shift operator with weights $\{\alpha_n\}$, $\alpha_n \neq 0$, then U is class $\mathcal{A}_k$ if and only if

$$|\alpha_{n+1}| \cdots |\alpha_{n+k}| \geq |\alpha_n|^k.$$

Our first goal is to prove that the class $\mathcal{A}$ shares many properties with that of hyponormal operators.

The following inclusions give the relationships between these operators

$$\begin{aligned} \text{hyponormal} &\subset p\text{-hyponormal} \\ &\subset \log\text{-}hyponormal \\ &\subset w\text{-hyponormal} \\ &\subset \text{class } \mathcal{A} \\ &\subset \text{class } \mathcal{A}_k. \end{aligned}$$

The generalized derivation $\delta_{U,T} : B(X) \to B(X)$ for $U, T \in B(X)$ is defined by $\delta_{U,T}(H) = UH - HT$ for $H \in B(X)$, and we note $\delta_{U,U} = \delta_U$. If the following inequality

$$\|T - (UH - HU)\| \geq \|T\|,$$

holds for all $T \in \ker \delta_U$ and for all $H \in B(X)$, then we remark that the range of δ_U is orthogonal to the kernel of δ_U.

The familiar Putnam-Fuglede's theorem affirms that if both $U \in B(X)$ and $T \in B(X)$ are normal operators and $UH = HT$ for some $H \in B(X)$, then $U^*H = HT^*$ (see [17]). This theorem attracted attention of many researchers and they extended it for several nonnormal classes of operators (see [2–4, 10, 12–15, 18, 19, 21–23]).

In this artcle, our second goal is extend this theorem to class $\mathcal{A}_k$ operators and prove the range kernel orthogonality for class $\mathcal{A}_k$ operators.

Let $U \in B(X)$ and let $\{e_n\}$ be an orthonormal basis of a Hilbert space X. The Hilbert-Schmidt norm is given by

$$\|U\|_2 = \left(\sum_{n=1}^{\infty} \|Ue_n\|^2 \right)^{\frac{1}{2}}.$$

An operator U is called to be a Hilbert-Schmidt operator if $\|U\|_2 < \infty$ (see [8] for details). $C_2(X)$ denotes a set of all Hilbert-Schmidt operators. For $T, U \in B(X)$, the operator $\Gamma_{T,U}$ defined as $\Gamma_{T,U} : C_2(X) \ni H \to THU \in C_2(X)$ has been studied in [6]. It is known that $\|\Gamma\| \leq \|T\|\|U\|$ and $(\Gamma_{T,U})^*H = T^*HU^* = \Gamma_{T^*,U^*}H$. If $U \geq 0$ and $T \geq 0$, then $\Gamma_{U,T} \geq 0$. For more information see [6].

We organise our paper as follows: Section 2 deals with some properties for class $\mathcal{A}_k$ operators which will be needed to prove our main results. We present our main theorems, like the asymmetric Putnam-Fuglede's theorem for some $\mathcal{A}_k$ class operators and also some orthogonality results in section 3.

2. Materials and method

Properties of class $\mathcal{A}_k$ operators

Theorem 3. *[11] If $U \in B(X)$ is a p-hyponormal or a* log*-hyponormal operator, then U is class $\mathcal{A}_k$ operator, for each positive integer k.*

Corollary 4. *Every hyponormal operator is a class $\mathcal{A}_k$ operator.*

Theorem 5. *[11] If $U \in B(X)$ is an invertible class $\mathcal{A}$, then U is class $\mathcal{A}_k$ operator for every k.*

A number $\lambda \in \mathbb{C}$ is said to be in the joint spectrum of operator U if there exist a joint eigenvector v corresponding to U and U^* such that $Uv = \lambda v$ and $U^*v = \bar{\lambda}v$, where $\bar{\lambda}$ is the complex conjugate of λ. We will denote the joint point spectrum and the point spectrum of operator U by $\sigma_{jp}(U)$ and $\sigma_p(U)$, respectively.

Theorem 6. *Let $U \in B(X)$ be a class $\mathcal{A}_k$ operator. Then the following hold*

(i) If $Uv = \lambda v,\ \lambda \neq 0$, then $U^*v = \bar{\lambda}v$,
(ii) $\sigma_{jp}(U) - \{0\} = \sigma_p(U) - \{0\}$,
(iii) Let $Uv = \lambda v$ and $Uw = \mu w$ with $\lambda \neq \mu$. Then $v \perp w$.

Proof. (i) We have that the following

$$\begin{aligned}
|\lambda|^2\|v\|^2 &= \|Uv\|^2 \\
&= \langle |U|^2 v, v\rangle \\
&\leq \langle |U^{k+1}|^{\frac{2}{k+1}} v, v\rangle \\
&\leq \langle |U^{k+1}| v, v\rangle^{\frac{2}{k+1}} \|v\|^{\frac{2}{k+1}} \\
&\leq \||U^{k+1}x|\|^{\frac{2}{k+1}} \|v\|^{\frac{2}{k+1}} \\
&= \left(|\lambda|^{2(k+1)}\|v\|^2\right)^{\frac{1}{k+1}} \|v\|^{\frac{2}{k+1}} \\
&= |\lambda|^2\|v\|^2
\end{aligned}$$

follow from using Holder-McCarthy and Schwarz's inequalities.

Hence

$$|\lambda|^2\langle v, v\rangle = \langle U^*Uv, v\rangle = \langle |U^{k+1}|^{\frac{2}{k+1}} v, v\rangle.$$

Since $|U^{k+1}|^{\frac{2}{k+1}} v$ and v are linearly independent [16], we get

$$|U^{k+1}|^{\frac{2}{k+1}} v = |\lambda|^2 v.$$

Also,

$$\|(|U^{k+1}|^{\frac{2}{k+1}} - U^*U)^{\frac{1}{2}}v\|^2 = \langle(|U^{k+1}|^{\frac{2}{k+1}} - U^*U)v, v\rangle = 0.$$

Therefore

$$U^*Uv = |U^{k+1}|^{\frac{2}{k+1}}v = |\lambda|^2 v,$$

and so

$$(U-\lambda)^*v = 0.$$

(ii) We can easily see that (ii) follows from the definition of the joint point spectrum and (i).

(iii) Let $Uv = \lambda v$ and $Uw = \mu w$, then

$$\begin{aligned}\langle Uv, w\rangle &= \langle \lambda v, w\rangle \\ &= \lambda\langle v, w\rangle \\ &= \langle v, U^*w\rangle \\ &= \langle v, \bar{\mu}w\rangle \\ &= \mu\langle v, w\rangle.\end{aligned}$$

Since $\lambda \neq \mu$, then $\langle v, w\rangle = 0$, i.e., $v \perp w$.

□

Definition 7. We say that $U \in B(\mathcal{H})$ is finite if the distance $\operatorname{dist}(I, \operatorname{ran}(\delta_U)) \geq 1$ from the identity to the range of δ_U.

Definition 8. If $U \in B(\mathcal{H})$, we denote by $\sigma_{ar}(U)$ the reduisant approximate spectrum, the set of scalars λ for which there is a normalized sequence $\{x_n\} \subset \mathcal{H}$ verifying

$$(U-\lambda)x_n \longrightarrow 0 \ \ and \ \ (U-\lambda)^*x_n \longrightarrow 0$$

Proposition 9. *[1] Let $U \in B(\mathcal{H})$, if σ_{ra} is not empty, then U is finite.*

Proposition 10. *(Berberian Technique) [5]*
*Let $\mathcal{H}$ be a complex Hilbert space, then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\varphi : B(\mathcal{H}) \to B(\mathcal{K})$ $(U \mapsto \tilde{U})$ satisfying: φ is an *-isomorphism preserving the order such that:*

(i) $\varphi(U^*) = \varphi(U)^*, \qquad \varphi(I) = \tilde{I};$
(ii) $\varphi(\alpha U + \beta V) = \alpha\varphi(U) + \beta\varphi(V), \qquad \varphi(UV) = \varphi(U)\varphi(V);$
(iii) $\|\varphi(U)\| = \|U\|$
(iv) $\varphi(U) \leq \varphi(V)$ *if* $U \leq V$*, for all* $U, V \in B(\mathcal{H}), \quad \alpha, \beta \in \mathbb{C};$
(v) $\sigma(U) = \sigma(\tilde{U}), \quad \sigma_a(U) = \sigma_a(\tilde{U}) = \sigma_p(\tilde{U}).$

Proposition 11. *If $U \in B(\mathcal{H})$ is a class $\mathcal{A}_k$, then $\varphi(U)$ is a class $\mathcal{A}_k$.*

Proof. By using Berberian technique, we prove easily that

$$\begin{aligned}|\varphi(U)^{k+1}|^{\frac{2}{k+1}} &= |\varphi(U^{k+1})|^{\frac{2}{k+1}} \\ &= \varphi(|U^{k+1}|^{\frac{2}{k+1}}) \\ &\geq \varphi(|U|^2)\end{aligned}$$

$$= \quad |\varphi(U)|^2,$$

this means that $\varphi(U)$ is a class $\mathcal{A}_k$. □

Proposition 12. *If $U \in B(\mathcal{H})$ is a class $\mathcal{A}_k$, then U is finite.*

Proof. From Proposition 11 $\varphi(U)$ is a class $\mathcal{A}_k$, with $\sigma_a(U) = \sigma_a(\tilde{U}) = \sigma_p(\tilde{U})$ using Berberian technique, since $\sigma_a(U)$ is never empty and $\sigma_{jp}(U) - \{0\} = \sigma_p(U) - \{0\}$, so by Theorem 6, it follows that $\sigma_{ra}(U) \neq \emptyset$ implying U is finite. □

Proposition 13. *If $U \in \mathcal{A}_k$, then $U^* \notin \mathrm{ran}(\delta_U)$.*

Proof. Let $\lambda \in \sigma_{ra} - \{0\} \neq \emptyset$, then there is a normalized sequence $\{x_n\}$ such that

$$(U - \lambda)x_n \longrightarrow 0 \;\; and \;\; (U - \lambda)^* x_n \longrightarrow 0$$

and let $X \in B(\mathcal{H})$, then

$$\begin{aligned} \|UX - XU - U^*\| &= \|(U-\lambda)X - X(U-\lambda) - (U^* - \overline{\lambda}) - \overline{\lambda}| \\ &\geq \|(\langle(U-\lambda)Xx_n, x_n\rangle - \langle X(U-\lambda)x_n, x_n\rangle - \langle(U^* - \overline{\lambda}\rangle - \overline{\lambda}\| \end{aligned}$$

letting $n \to \infty$, we get $\|UX - XU - U^*\| \geq |\lambda\|$ implying $U^* \notin \mathrm{ran}(\delta_U)$. □

Proposition 14. *If U is a class $\mathcal{A}_k$ and N is a normal opeartor such that $UN = NU$, then for every $\lambda \in \sigma_p(N)$*

$$|\lambda| \leq \mathrm{dist}(N, \mathrm{ran}(\delta_U))$$

Proof. Let $\lambda \in \sigma_p(N)$ and $\mathcal{M}_\lambda$ be the eigenspace associated to λ. Since $NU = UN$, then $U^*N = NU^*$ by Putnam-Fuglede Theorem. Hence $\mathcal{M}_\lambda$ reduces orthogonaly U and N. Let $T \in B(\mathcal{H})$, we can write U, N and T according to the decmpositiom of $\mathcal{H} = \mathcal{M}_\lambda \oplus \mathcal{M}_\lambda^\perp$ as follows:

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \;\; U = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \;\; \text{and} \;\; U = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

We have

$$\begin{aligned} \|N + UT - TU\| &= \left\| \begin{bmatrix} \lambda + U_1T_1 - T_1U_1 & * \\ * & * \end{bmatrix} \right\| \\ &\geq \|\lambda + U_1T_1 - T_1U_1\| \\ &\geq |\lambda\| \left\| I + U_1\Big(\frac{T_1}{\lambda}\Big) - \Big(\frac{T_1}{\lambda}\Big) \right\| \\ &\leq |\lambda|. \end{aligned}$$

□

Proposition 15. *If U is a class $\mathcal{A}_k$, then for every normal operator N such that $UN = NU$, we have $\|N\| \leq \mathrm{dist}(N, \mathrm{ran}(\delta_U))$.*

Proof. Let $\lambda \in \sigma(N) = \sigma_a(N)$ [1], from proposition 10, $\tilde{N}$ is normal and $\tilde{U}$ is a class $\mathcal{A}_k$, $\tilde{N}U = \tilde{N}\tilde{U} = \tilde{U}\tilde{N}$, also $\lambda \in \sigma_p(\tilde{N}$. Applying proposition (14), we get for every $T \in B(\mathcal{H})$

$$|\lambda| \leq ||\tilde{N} + \tilde{U}\tilde{T} - \tilde{T}\tilde{U}|vert = ||N + UT - TU||$$

Therefore

$$\sup_{\lambda\in\sigma(\tilde{N}} |\lambda| = ||\tilde{N}|| = ||N|| \leq ||N + UT - TU||.$$

□

We will denote by $U \otimes T$, the tensor product of some non-zero operators $U, T \in B(X)$, on the product space $X \oplus X$. We can see the importance the tensor product operation $U \oplus T$ as it preserves many properties of $U, T \in B(X)$. It can be checked that the tensor product of operators U and T i.e. $U \oplus T$ is hyponormal if and only if U and T are hyponormal [9].

We will obtain an analogous result for class $\mathcal{A}_k$ operators in this section. Before stating our main theorems, we need some preliminary results.

Lemma 16. *[20] Let $U_1, U_2 \in B(X), T_1, T_2 \in B(X)$ be non-negative operators. If U_1 and T_1 are non-zero, then the following assertions are equivalent*

1. *$U_1 \oplus T_1 \leq U_2 \oplus T_2$*
2. *There exists $c > 0$ for which $U_1 \leq U_2$ and $T_1 \leq c^{-1}T_2$.*

Lemma 17. *If $U, T \in B(X)$ are class $\mathcal{A}_k$ operators, then $U \oplus T$ is class $\mathcal{A}_k$ operator.*

Proof. Since U and T are class $\mathcal{A}_k$ operators, then

$$\begin{aligned} |(U \oplus T)^{k+1}|^{\frac{2}{k+1}} &= |U^{k+1}|^{\frac{2}{k+1}} \oplus |T^{k+1}|^{\frac{2}{k+1}} \\ &\geq |U|^2 \oplus |T|^2 \\ &= |U \oplus T|. \end{aligned}$$

Hence $U \oplus T$ is a class $\mathcal{A}_k$ operator. □

Theorem 18. *[11] If U is a class $\mathcal{A}_k$ operator and $\mathcal{M}$ is an invariant subspace of U, the restriction $U |_{\mathcal{M}}$ is also a class $\mathcal{A}_k$.*

3. Main results

In the following, we prove that if H is a Hilbert-Schmidt operator, U is a class $\mathcal{A}_k$ operator and T^* is an invertible class $\mathcal{A}$ following the relation $UH = HT$, then $U^*H = HT^*$.

Theorem 19. *Let U and $T \in B(X)$. Then $\Gamma_{U,T}$ is a class $\mathcal{A}_k$ operator on $C_2(X)$ if and only if U and T^* belong to $\mathcal{A}_k$ operators.*

Proof. The unitary operator

$$\mathcal{U} : C_2(X) \to X \oplus X$$

defined by

$$(v \oplus w)^* = v \oplus w$$

induces the $*$-isomorphism

$$\psi : B(C_2(X)) \to B(X \oplus X)$$

by a map

$$H \mapsto \mathcal{U} H \mathcal{U}^*.$$

Then we can obtain

$$\psi(\Gamma_{U,T}) = U \oplus T^*,$$

see [7] for details. This completes the proof by Lemma 17. □

Theorem 20. *Let U be a class $\mathcal{A}_k$ operator and T^* an invertible class $\mathcal{A}$ operator. If $UH = HT$ for some $H \in C_2(X)$, then $U^*H = HT^*$.*

Proof. Let Γ be defined on $C_2(X)$ by

$$\Gamma(V) = UVT^{-1}.$$

The operator T is an invertible class $\mathcal{A}$, then T is a class $\mathcal{A}_k$ by Theorem 5.

Since U and $(T^{-1})^* = (T^*)^{-1}$ are $\mathcal{A}_k$ operators, we have by Theorem 19, we can say that Γ is also an $\mathcal{A}_k$ operator. Moreover,

$$\Gamma(H) = UHT^{-1} = H$$

because of $UH = HT$. Hence, H is an eigenvector of Γ. By Theorem 6, we have

$$\Gamma^*(H) = U^*H(T^{-1})^* = H,$$

that is,

$$U^*H = HT^*$$

as desired. □

Corollary 21. *Let $U \in B(X)$ be a class $\mathcal{A}$ and T^* be an invertible class $\mathcal{A}$ such that $UH = HT$ for some $H \in C_2(X)$. Then, $U^*H = HT^*$.*

Corollary 22. *Let $U \in B(X)$ be hyponormal and T^* be an invertible class $\mathcal{A}$ such that $UH = HT$ for some $H \in C_2(X)$. Then, $U^*H = HT^*$.*

Corollary 23. *Let $U \in B(X)$ be a class $\mathcal{A}_k$ and T^* be an invertible hyponormal such that $UH = HT$ for some $H \in C_2(X)$. Then, $U^*H = HT^*$.*

Corollary 24. *Let $U \in B(X)$ be a class $\mathcal{A}$ and T^* be an invertible hyponormal such that $UH = HT$ for some $H \in C_2(X)$. Then, $U^*H = HT^*$.*

Now, we are ready to extend the orthogonality results to some class $\mathcal{A}_k$ operators.

Theorem 25. *Let $U, T \in B(X)$ and $V \in C_2(X)$. Then*

$$\|\delta_{U,T}(H) + V\|_2^2 = \|\delta_{U,T}(H)\|_2^2 + \|V\|_2^2, \tag{3.1}$$

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}(H)\|_2^2 + \|V\|_2^2, \tag{3.2}$$

if and only if $\delta_{U,T}(V) = 0 = \delta_{U^*,T^*}(V)$ for all $V \in C_2(X)$.

Proof. It is known that the Hilbert-Schmidt class $C_2(X)$ is a Hilbert space. Note that

$$\begin{aligned}\|\delta_{U,T}(H) + V\|_2^2 &= \|\delta_{U,T}\|_2^2 + \|V\|_2^2 + Re\langle \delta_{U,T}(H), V\rangle \\ &= \|\delta_{U,T}\|_2^2 + \|V\|_2^2 + Re\langle H, \delta^*_{U,T}(V)\rangle,\end{aligned}$$

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}\|_2^2 + \|V\|_2^2 + Re\langle H, \delta^*_{U,T}(V)\rangle. \tag{3.3}$$

Hence by the equality $\delta_{U,T}(V) = 0 = \delta_{U^*,T^*}(V)$, we obtain (3.1) and (3.2). So, this completes the proof as our claim is verified. □

Corollary 26. *Let U, T be operators in $B(X)$ and $V \in C_2(X)$. Then*

$$\|\delta_{U,T}(H) + V\|_2^2 = \|\delta_{U,T}(H)\|_2^2 + \|V\|_2^2$$

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}(H)\|_2^2 + \|V\|_2^2$$

if either of the following hold

(i) U is a class $\mathcal{A}_k$ and $(T^*)^{-1}$ is a class $\mathcal{A}$;
(ii) U is a class $\mathcal{A}$ and $(T^*)^{-1}$ is a class $\mathcal{A}$;
(iii) U is hyponormal and $(T^*)^{-1}$ is a class $\mathcal{A}$;
(iv) U is a class $\mathcal{A}_k$ and $(T^*)^{-1}$ is hyponormal.

4. Discussions

The basic properties of class $\mathcal{A}_k$ are studied and discussed. The Putnam-Fuglede Theorem plays an important role in operator theory. We proved that the Putnam-Fuglede Theorem for class $\mathcal{A}_k$ operators holds in the Hilbert-Schmidt case. Also, range-kernel results for the generalized derivations induced by certain $\mathcal{A}_k$ classes are obtained.

5. Conclusions

The questions which logically arise after this study are as follows:

1. Is the Putnam-Fuglede Theorem remains true for $\mathcal{A}_k$ class in any Hilbert space H?
2. Is the Putnam-Fuglede Theorem remains true for $\mathcal{A}_k$ class in any bilateral ideal in $B(H)$?

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Conflict of interest

The author declares no conflict of interest.

Authors' contributions

All authors drafted the manuscript, and they read and approved the final manuscript.

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