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Research article

Nonlocal coupled hybrid fractional system of mixed fractional derivatives via an extension of Darbo's theorem

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Abstract: In this work a new existence result is established for a coupled fractional system consisting of one Caputo and one Riemann-Liouville fractional derivatives and nonlocal hybrid boundary conditions. A new generalization of Darbo's theorem associated with measures of noncompactness is the main tool in our approach. An example is constructed to justify the theoretical result.

Keywords: fractional derivative; Hybrid differential equation; measure of noncompactness; coupled system; fixed point
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1. Introduction

In the past years, fractional differential equations and coupled systems of those equations have attracted a lot of regard from many researches as they have played a key role in many basic sciences such as chemistry, control theory, biology and other arenas [1–3]. In addition, boundary conditions of differential models are strongest tools to extend applications of those equations. In fact, differential equations of fractional models can be extended by creating different types of boundary conditions. Newly, many authors have studied various types of boundary conditions to obtain new results of differential models. The following hybrid differential equation was studied by Dhage and

Lakshmikantham [4]:

$$\begin{cases} \dfrac{d}{dt}\Big[\dfrac{x(t)}{h(t,x(t))}\Big] = \omega(t,x(t)), \quad a.e\ t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

in which h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$, respectively. Based on the above work, the Caputo hybrid boundary value problem of the form:

$$\begin{cases} {}^{C}D^{p}_{0^+}\Big[\dfrac{x(t)}{h(t,x(t))}\Big] = \omega(t,x(t)), \quad a.e\ t \in J := [0,L], \\ a_1\dfrac{x(0)}{h(0,x(0))} + a_2\dfrac{x(L)}{h(L,x(L))} = d, \end{cases}$$

was studied by Hilal and Kajouni [5] in which $0 < p < 1$, h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$, respectively and a_1, a_2, d are real constants with $a_1 + a_2 \neq 0$. For some more results on hybrid boundary value problems see [6–9] and references therein.

The fractional hybrid modeling is of great significance in different engineering fields, and it can be a unique idea for the future combined research between various applied sciences, for example see [10] in which a fractional hybrid modeling of a thermostat is simulated. For some recent results on hybrid fractional differential equations we refer to [11–13].

As in modern mathematics coupled fractional system have been applied to develop differential models of high complexity system, Ntouyas and Al-Sulami [14] have considered the following coupled system:

$$\begin{cases} {}^{C}D^{\alpha_1}u(t) = f_1(t,u(t),v(t)), \quad t \in [0,L],\ 0 < \alpha_1 \leq 1, \\ {}^{RL}D^{\alpha_2}v(t) = f_2(t,u(t),v(t)) \quad t \in [0,L]\ 1 < \alpha_2 \leq 2, \\ u(0) = \lambda\, {}^{C}D^{p}v(\eta), \quad 0 < p < 1, \\ v(0) = 0, \quad v(L) = \gamma I^{q}u(\xi), \end{cases}$$

where ${}^{C}D^{\alpha_1}$ and ${}^{RL}D^{\alpha_2}$ indicate Caputo and Riemann-Liouville fractional derivitves of orders α_1 and α_2 respectively, $f_1, f_2 : [0,L] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, I^q is the Riemann-Liouville fractional integral, $\lambda, \gamma \in \mathbb{R}$ and $\eta, \xi \in (0,L)$. They have applied Banach's fixed point theorem and Leray-Schauder alternative to obtain main results. Nonlocal boundary value problems involving mixed fractional derivatives have been considered in [15] and references cited therein.

Here, we combine mixed fractional derivatives and hybrid fractional differential equations. More precisely, in this paper the existence of solutions for the coupled hybrid system

$$\begin{cases} {}^{C}D^{\alpha_1}\dfrac{u(t)}{f_1(t,u(t),v(t))} = f_2(t,u(t),v(t)), \quad t \in J := [0,L],\ 0 < \alpha_1 \leq 1, \\ {}^{RL}D^{\alpha_2}\dfrac{v(t)}{g_1(t,v(t),u(t))} = g_2(t,u(t),v(t)) \quad t \in J := [0,L]\ 1 < \alpha_2 \leq 2, \end{cases} \tag{1.1}$$

is investigated supplemented with boundary conditions:

$$\begin{cases} \dfrac{u(0)}{f_1(0,u(0),v(0))} = \theta\, {}^{C}D^{p}\dfrac{v(\eta)}{g_1(\eta,u(\eta),v(\eta))}, \quad 0 < p < 1, \\ \dfrac{v(0)}{g_1(0,u(0),v(0))} = 0, \quad \dfrac{v(L)}{g_1(L,u(L),v(L))} = \gamma I^{q}\dfrac{u(\xi)}{f_1(\xi,u(\xi),v(\xi))}. \end{cases} \tag{1.2}$$

where ${}^{C}D^{\alpha_1}$, ${}^{RL}D^{\alpha_2}$ are the Caputo and Riemann-Liouville fractional derivatives of orders $\alpha_1 \in (0,1]$ and $\alpha_2 \in (1,2]$ respectively, I^q is the Riemann-Liouville fractional integral, $f_2, g_2 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f_1, g_1 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\theta, \gamma \in \mathbb{R}$ and $\eta, \xi \in (0, L)$. An existence result is obtained via a new extension of Darbo's theorem associated to measures of noncompactness.

Here we emphasize that the proposed coupled hybrid system includes:

- *different orders of fractional derivatives ($\alpha_1 \in (0,1]$ and $\alpha_2 \in (1,2]$);*
- *two different kinds of fractional derivatives (Caputo and Riemann-Liouville);*
- *nonlocal type boundary conditions which contain both fractional derivatives and integrals.*

We have organized the structure of the paper as follows. Section 2 presents some main concepts which will be applied in the future. In the next section, we prove an existence result for the problem (1.1) and (1.2). Finally, we construct an example to illustrate the obtained result.

2. Preliminaries

Now some basic notations are recalled from [2].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $\xi : (0,\infty) \to \mathbb{R}$, is defined as*

$$I^{\alpha}\xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

Definition 2.2. *For a continuous function $\xi : (0,\infty) \to \mathbb{R}$, the Reimann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as:*

$${}^{RL}D^{\alpha}_{0_+}\xi(t) = \frac{1}{\Gamma(n-\sigma)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\sigma-1}\xi(s)ds.$$

Definition 2.3. *Given a continuous function $\xi : [0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of order $\alpha > 0$ is defined as*

$${}^{C}D^{\alpha}_{0_+}\xi(t) ={}^{RL} D^{\alpha}_{0_+}\Big(\xi(t) - \sum_{k=0}^{n-1} \frac{t^k}{ki}\xi^{(k)}(0)\Big), \quad t > 0, n-1 < \alpha < n.$$

Now we present some basic facts about the notion measure of noncompactness.

Assume that Z is the real Banach space with the norm $\|\cdot\|$ and zero element θ. For a nonempty subset X of Z, the closure and the closed convex hull of X will be denoted by $\overline{X}$ and $Conv(X)$, respectively. Also, M_Z and N_Z denote the family of all nonempty and bounded subsets of Z and its subfamily consisting of all relatively compact sets, respectively.

Definition 2.4. *[16, 19] We say that a mapping $h : M_Z \longrightarrow [0,\infty)$ is a measure of noncompactness, if the following conditions hold true:*

(1) The family $Ker h = \{X \in M_Z : h(X) = 0\}$ *is nonempty and* $Ker\ h \subseteq N_Z$.
(2) $X_1 \subseteq Y_1 \Longrightarrow h(X_1) \leq h(Y_1)$.
(3) $h(\overline{X}) = h(X)$.
(4) $h(Conv(X)) = h(X)$.
(5) $h(\alpha X + (1-\alpha)Y) \leq \alpha h(X) + (1-\alpha)h(Y)$ *for* $\alpha \in [0,1]$.
(6) For the sequence $\{X_n\}$ *of closed sets from* M_Z *in which* $X_{n+1} \subseteq X_n$ *for* $n = 1, 2, \dots$ *and* $\lim_{n\to\infty} h(X_n) = 0$*, we have* $\bigcap_{n=1}^{\infty} X_n \neq \varnothing$.

In [17], some generalizations of Darbo's theorem have been proved by Samadi and Ghaemi. Also, in [18], Darbo's theorem was extended and the following result was presented which is basic for our main result.

Theorem 2.1. *Let T be a continuous self-mapping on the set D where D denotes a nonempty bounded, closed and convex subset of a Banach space Z. Assume that for all nonempty subset X of D we have*

$$\theta_1((h(X)) + \theta_2(h(T(X))) \leq \theta_2(h(X)) \tag{2.1}$$

where h is an arbitrary measure of noncompactness defined in Z and $(\theta_1, \theta_2) \in U$*. Then T has a fixed point in D.*

In Theorem 2.1, let U indicate the set of all pairs (θ_1, θ_2) where the following conditions hold true:

(U_1) $\theta_1(t_n) \nrightarrow 0$ for each strictly increasing sequence $\{t_n\}$;
(U_2) θ_2 is strictly increasing function;
(U_3) If $\{\alpha_n\}$ is a sequence of positive numbers, then $\lim_{n\to\infty} \alpha_n = 0 \longleftrightarrow \lim_{n\to\infty} \theta_2(\alpha_n) = -\infty$.
(U_4) Let $\{l_n\}$ be a decreasing sequence in which $l_n \to 0$ and $\theta_1(l_n) < \theta_2(l_n) - \theta_2(l_{n+1})$, then we have $\sum_{n=1}^{\infty} l_n < \infty$.

Next, the definition of a measure of noncompactness in the space $C([0,1])$ is recalled which will be applied later. Fix $Y \in M_{C[0,1]}$ and for $\varepsilon > 0$ and $y \in Y$ we define

$$\begin{aligned} &\varphi(y,\epsilon) = \sup\Big\{|y(t) - y(s)| : t, s \in [0,1], |t-s| \leq \epsilon\Big\}, \\ &\varphi(Y,\epsilon) = \sup\Big\{\varphi(y,\epsilon) : y \in Y\Big\}, \\ &\varphi_0(Y) = \lim_{\epsilon\to 0} \varphi(Y,\epsilon). \end{aligned} \tag{2.2}$$

Banas and Goebel [16] proved that $\varphi_0(Y)$ is a measure of noncompactness in the space $C([0,1])$.

3. Existence results

Now the coupled system (1.1) and (1.2) is investigated in the space $C([0,1])$.

First following Lemma 2.5 of [14], the following lemma is presented which will be applied later.

Lemma 3.1. *Assume that the functions* ϕ *and h are continuous real-valued functions on* $[0, L]$*. Then,* *the functions u and v satisfy the system*

$$\begin{cases} {}^{C}D^{\alpha_1}\dfrac{u(t)}{f_1(t,u(t),v(t))} = \phi(t), \quad t\in[0,L], 1<\alpha_1\le 2,\\ {}^{RL}D^{\alpha_2}\dfrac{v(t)}{g_1(t,u(t),v(t))} = h(t), \quad t\in[0,L], 1<\alpha_2\le 2,\\ \dfrac{u(0)}{f_1(0,u(0),v(0))} = \theta\, {}^{C}D^{p}\dfrac{v(\eta)}{g_1(\eta,u(\eta),v(\eta))},\\ \dfrac{v(0)}{g_1(0,u(0),v(0))} = 0, \quad \dfrac{v(L)}{g_1(L,u(L),v(L))} = \gamma I^{q}\dfrac{u(\xi)}{f_1(\xi,u(\xi),v(\xi))}, \end{cases} \tag{3.1}$$

if and only if u and v satisfy the system

$$\begin{aligned} u(t) &= f_1(t,u(t),v(t))\Big[I^{\alpha_1}\phi(t) + \frac{\theta}{\Delta}\Big\{-T^{\alpha_2-1}I^{\alpha_2-p}h(\eta)\\ &\quad + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(\gamma I^{q+\alpha_1}\phi(\xi) - I^{\alpha_2}h(L)\Big\}\Big],\\ v(t) &= g_1(t,u(t),v(t))\Big[I^{\alpha_1}h(t) + \frac{t^{\alpha_2-1}}{\Delta}\Big\{I^{\alpha_2}h(L) - \gamma I^{q+\alpha_1}\phi(\xi)\\ &\quad -\theta\gamma\frac{\xi^{q}}{\Gamma(1+q)}I^{\alpha_2-p}h(\eta)\Big\}\Big], \end{aligned} \tag{3.2}$$

where $\Delta = L^{\alpha_2-1} + \theta\gamma\dfrac{\Gamma(\alpha_2)\xi^{p}\eta^{\alpha_2-p-1}}{\Gamma(1+q)\Gamma(\alpha_2-p)}$. For convenience we set the notations:

$$M_1 = \frac{L^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{1}{|\Delta|}|\theta||\gamma|\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\frac{\eta^{\alpha_2-p-1}\xi^{q+\alpha_1}}{\Gamma(q+\alpha_1+1)}, \tag{3.3}$$

$$M_2 = \frac{T^{\alpha_2-1}\eta^{\alpha_2-p-1}|\theta|}{|\Delta|}\Big[\frac{L\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)\Gamma(\alpha_2+1)} + \frac{\eta}{\Gamma(1+\alpha_2)}\Big], \tag{3.4}$$

$$M_3 = \frac{L^{\alpha_2-1}|\gamma|\xi^{q+\alpha_1}}{|\Delta|\Gamma(q+\alpha_1+1)}, \tag{3.5}$$

$$M_4 = \frac{L^{\alpha_2}}{\Gamma(1+\alpha_2)}\Big(1+\frac{L^{\alpha_2-1}}{|\Delta|}\Big) + \frac{L^{\alpha_2-1}}{|\Delta|}|\theta||\gamma|\frac{\xi^{q}\eta^{\alpha_2-p}}{\Gamma(1+q)\Gamma(\alpha_2-p+1)}. \tag{3.6}$$

Now we present the main result of this section as follows:

Theorem 3.1. *Suppose that we have the following assumptions:*

(D_1) *The functions $f_2, g_2 : I\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous provided that*

$$\begin{aligned} &|f_2(t,u,v)| \le l_1, \quad |g_2(t,u,v)| \le l_2,\\ &|f_2(t,u_1,v_1) - f_2(t,u_2,v_2)| \le k_1|u_1-u_2| + k_2|v_1-v_2|,\\ &|g_2(t,u_1,v_1) - g_2(t,u_2,v_2)| \le \gamma_1|u_1-u_2| + \gamma_2|v_1-v_2|, \end{aligned}$$

where $l_1, l_2, k_1, k_2, \gamma_1, \gamma_2 \ge 0$, $t\in I$ and $u, v, u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(D_2) *The continuous functions* f_1, g_1 *have been defined from* $I \times \mathbb{R} \times \mathbb{R}$ *into* $\mathbb{R} \setminus \{0\}$ *provided that*

$$\begin{aligned}
&|f_1(t,u_1,v_1) - f_1(t,u_2,v_2)| \le \frac{1}{6}e^{-d}(|u_1-u_2| + |v_1-v_2|),\\
&|g_1(t,u_1,v_1) - g_1(t,u_2,v_2)| \le \frac{1}{6}e^{-d}(|u_1-u_2| + |v_1-v_2|),\\
&|f_1(t_2,u,v) - f_1(t_1,u,v)| \le |t_2 - t_1|,\\
&|g_1(t_2,u,v) - g_1(t_1,u,v)| \le |t_2 - t_1|.
\end{aligned}$$

where $d > 0$*,* $t, t_1, t_2 \in [0, L]$ *and* $u, v, u_1, u_2, v_1, v_2 \in \mathbb{R}$*. Moreover, assume that* $\overline{M} = \sup\{|f_1(t,0,0)| : t \in [0,L]\}$ *and* $\overline{N} = \sup\{|g_1(t,0,0)| : t \in [0,L]\}$*.*

(D_3) *There exists a positive solution* r_0 *of the following inequality:*

$$e^{-d}2r_0(M_1l_1 + M_2l_2 + M_3l_1 + M_4l_2) + \overline{M}(M_1l_1 + M_2l_2) + \overline{N}(M_3l_1 + M_4l_2) \le r_0.$$

exists. Moreover, assume that

$$\overline{M}(k_1 + k_2) < \frac{1}{6}e^{-d} \ \text{ and } \ 2r_0(k_1+k_2) + M_1l_1 + M_2l_2 + M_3l_1 + M_4l_2 < 1.$$

Then the coupled hybrid fractional system (1.1) and (1.2) has a solution on $[0, L]$*.*

Proof. Define $G : C([0,L],\mathbb{R}) \times C([0,L],\mathbb{R}) \longrightarrow C([0,L],\mathbb{R}) \times C([0,L],\mathbb{R})$ by

$$G(u,v)(t) = (G_1(u,v)(t), G_2(u,v)(t)),$$

where

$$\begin{aligned}
G_1(u,v)(t) &= f_1(t,u(t),v(t))\Big[I^{\alpha_1}\overline{f}(t) + \frac{\theta}{\Delta}\Big\{ - L^{\alpha_2-1}I^{\alpha_2-p}\overline{g}(\eta)\\
&\quad + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(\gamma I^{q+\alpha_1}\overline{f}(\xi) - I^{\alpha_2}\overline{g}(L)\Big\}\Big],\\
G_2(u,v)(t) &= g_1(t,u(t),v(t))\Big[I^{\alpha_1}\overline{g}(t)\\
&\quad + \frac{t^{\alpha_2-1}}{\Delta}\Big\{I^{\alpha_2}\overline{g}(L) - \gamma I^{q+\alpha_1}\overline{f}(\xi) - \theta\gamma\frac{\xi^q}{\Gamma(1+q)}I^{\alpha_2-p}\overline{g}(\eta)\Big\}\Big],
\end{aligned}$$

with $\overline{f}(t) = f_2(t,u(t),v(t))$ and $\overline{g}(t) = g_2(t,u(t),v(t))$. Assume that the space $C([0,L],\mathbb{R}) \times C([0,L],\mathbb{R})$ has been equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup\{|u(t)| : t \in [0,L]\}$. Define $D_{r_0} = \{u \in C([0,L],\mathbb{R}) : \|u\| \le r_0\}$. First we show that $G(D_{r_0} \times D_{r_0}) \subseteq D_{r_0} \times D_{r_0}$. Given $t \in [0,L]$ and $u, v \in D_{r_0}$, we earn

$$\begin{aligned}
|G_1(u,v)(t)| &\le |f_1(t,u(t),v(t))|(M_1l_1 + M_2l_2)\\
&\le |f_1(t,u(t),v(t)) - f_1(t,0,0)|(M_1l_1 + M_2l_2) + |f_1(t,0,0)|(M_1l_1 + M_2l_2)\\
&\le e^{-d}(|u(t)| + |v(t)|)(M_1l_1 + M_2l_2) + \overline{M}(M_1l_1 + M_2l_2)\\
&\le e^{-d}2r_0(M_1l_1 + M_2l_2) + \overline{M}(M_1l_1 + M_2l_2).
\end{aligned}$$

The above estimate yields that

$$\|G_1(u,v)\| \le e^{-d}2r_0(M_1l_1 + M_2l_2) + \overline{M}(M_1l_1 + M_2l_2). \tag{3.7}$$

Similarly we can prove that

$$\|G_2(u,v)\| \le e^{-d}2r_0(M_3l_1 + M_4l_2) + \overline{N}(M_3l_1 + M_4l_2). \tag{3.8}$$

Consequently, we have

$$\begin{aligned} \|G(u,v)\| \quad &\le e^{-d}2r_0(M_1l_1 + M_2l_2 + M_3l_1 + M_4l_2) \\ &+\overline{N}(M_3l_1 + M_4l_2) + \overline{M}(M_1l_1 + M_2l_2). \end{aligned} \tag{3.9}$$

Due to (3.9) and (D_3) we derive that $G(D_{r_0}\times D_{r_0}) \subseteq D_{r_0}\times D_{r_0}$. Now we verify the continuity property of G on $D_{r_0}\times D_{r_0}$. Let $(x_1,y_1),(u_1,v_1) \in D_{r_0}\times D_{r_0}$ and $\varepsilon > 0$ be arbitrarily such that $\|(x_1,y_1)-(u_1,v_1)\| < \frac{\varepsilon}{2}$. Given $t \in [0,L]$ we get

$$\begin{aligned}
&|G_1(x_1,y_1)(t) - G_1(u_1,v_1)(t)| \\
&\le |f_1(t,x_1(t),y_1(t)) - f_1(t,u_1(t),v_1(t))|\Big[I^{\alpha_1}|f_2(t,x_1(t),y_1(t))| \\
&+\frac{|\theta|}{|\Delta|}\Big\{L^{\alpha_2-1}I^{\alpha_2-p}|g_2(\eta,x_1(\eta),y_1(\eta))| + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(|\gamma|I^{q+\alpha_1}|f_2(\xi,x_1(\xi),y_1(\xi))| \\
&+I^{\alpha_2}|g_2(L,x_1(L),y_1(L))|\Big\}\Big] + |f_1(t,u_1(t),v_1(t))|\Big[I^{\alpha_1}\Big|f_2(t,x_1(t),y_1(t)) - f_2(t,u_1(t),v_1(t))\Big| \\
&+\frac{|\theta|}{|\Delta|}\Big\{L^{\alpha_2-1}I^{\alpha_2-p}\Big|g_2(\eta,x_1(\eta),y_1(\eta)) - g_2(\eta,u_1(\eta),v_1(\eta))\Big| \\
&+\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(|\gamma|I^{q+\alpha_1}\Big|f_2(\xi,x_1(\xi),y_1(\xi)) - f_2(\xi,u_1(\xi),v_1(\xi))\Big| \\
&+I^{\alpha_2}\Big|g_2(L,x_1(L),y_1(L)) - g_2(L,u_1(L),v_1(L))\Big|\Big\}\Big] \\
&\le e^{-d}(\|x_1-u_1\| + \|y_1-v_1\|)(M_1l_1 + M_2l_2) + |f_1(t,x_1(t),y_1(t))|\Big[(k_1\|x_1-u_1\| \\
&+k_2\|y_1-v_1\|)I^{\alpha_1}(1) + \frac{|\theta|}{\Delta|}\Big\{L^{\alpha_2-1}(\gamma_1\|x_1-u_1\| + \gamma_2\|y_1-v_1\|)I^{\alpha_2-p}(1) \\
&+\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(|\gamma|k_1\|x_1-u_1\| + k_2\|y_1-v_1\|)I^{q+\alpha_1}(1) \\
&+(\gamma_1\|x_1-u_1\| + \gamma_2\|y_1-v_1\|)I^{\alpha_2}(1)\Big\}\Big] \\
&\le e^{-d}\varepsilon(M_1l_1 + M_2l_2) + |f_1(t,x_1(t),y_1(t))|\Big[(k_1\varepsilon + k_2\varepsilon)I^{\alpha_1}(1) \\
&+\frac{|\theta|}{|\Delta|}\Big\{L^{\alpha_2-1}(\gamma_1\varepsilon + \gamma_2\varepsilon)I^{\alpha_2-p}(1) + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(|\gamma|k_1\varepsilon + k_2\varepsilon)I^{q+\alpha_1}(1) \\
&+(\gamma_1\varepsilon + \gamma_2\varepsilon)I^{\alpha_2}(1)\Big\}\Big].
\end{aligned}$$

Due to the above estimate we conclude that

$$
\begin{aligned}
&\|G_1(x_1,y_1)-G_1(u_1,v_1)\| \\
&\le e^{-d}\varepsilon(M_1 l_1+M_2 l_2)+(e^{-d}r_0+\overline{M})\Big[(k_1\varepsilon+k_2\varepsilon)I^{\alpha_1}(1) \\
&+\frac{|\theta|}{\Delta}\Big\{L^{\alpha_2-1}(\gamma_1\varepsilon+\gamma_2\varepsilon I^{\alpha_2-p}(1) \\
&+\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(|\gamma|k_1\varepsilon+k_2\varepsilon)I^{q+\alpha_1}(1)-(\gamma_1\varepsilon+\gamma_2\varepsilon)I^{\alpha_2}(1)\Big\}\Big].
\end{aligned}
\tag{3.10}
$$

Similarly, for $(x_1,y_1),(u_1,v_1)\in D_{r_0}\times D_{r_0}$ and $t\in[0,L]$ we conclude that

$$
\begin{aligned}
&|G_2(x_1,y_1)(t)-G_2(u_1,v_1)(t)| \\
&\le |g_1(t,x_1(t),y_1(t))-g_1(t,u_1(t),v_1(t))|\Big[I^{\alpha_1}|g_2(t,x_1(t),y_1(t))| \\
&+\frac{L^{\alpha_2-1}}{|\Delta|}\Big\{I^{\alpha_2}|g_2(L,x_1(L),y_1(L))|+|\gamma|I^{q+\alpha_1}|f_2(\xi,x_1(\xi),y_1(\xi))| \\
&+\theta\gamma\frac{\xi^q}{\Gamma(1+q)}I^{\alpha_2-p}|g_2(\eta,x_1(\eta),y_1(\eta))|\Big\}\Big] \\
&+|g_1(t,u_1(t),v_1(t))|\Big[I^{\alpha_1}|g_2(t,x_1(t),y_1(t))-g_2(t,u_1(t),v_1(t))| \\
&+\frac{L^{\alpha_2-1}}{|\Delta|}\Big\{I^{\alpha_2}|g_2(L,x_1(L),y_1(L))-g_2(L,u_1(L),v_1(L))| \\
&+|\gamma|I^{q+\alpha_1}|f_2(\xi,x_1(\xi),y_1(\xi))-f_2(\xi,u_1(\xi),v_1(\xi))| \\
&+|\theta||\gamma|\frac{\xi^q}{\Gamma(1+q)}I^{\alpha_2-p}|g_2(\eta,x_1(\eta),y_1(\eta))-g_2(\eta,u_1(\eta),v_1(\eta))|\Big\}\Big].
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\|G_2(x_1,y_1)-G_2(u_1,v_1)\| \\
&\le e^{-d}\varepsilon(M_3\ell_1+M_4\ell_2)+(e^{-d}r_0+\overline{N})\Big[(\gamma_1\varepsilon+\gamma_2\varepsilon)I^{\alpha_1}(1) \\
&+\frac{L^{\alpha_2-1}}{|\Delta|}\Big\{(\gamma_1\varepsilon+\gamma_2\varepsilon)I^{\alpha_2}(1)+|\gamma|(k_1\varepsilon+k_2\varepsilon)I^{q+\alpha_1}(1) \\
&+|\theta||\gamma|\frac{\xi^q}{\Gamma(1+q)}(\gamma_1\varepsilon+\gamma_2\varepsilon)I^{\alpha_2-p}(1)\Big\}\Big].
\end{aligned}
\tag{3.11}
$$

In view of (3.10) and (3.11) we earn

$$
\|G(x_1,y_1)-G(u_1,v_1)\|\le E_1(\varepsilon)+E_2(\varepsilon), \tag{3.12}
$$

where $E_1(\varepsilon),E_2(\varepsilon)\longrightarrow 0$. Hence G is continuous on $D_{r_0}\times D_{r_0}$.

Next we show that the condition (2.1) of Theorem 2.1 is fulfilled. Let $X_1,X_2\subseteq D_{r_0}$ and $\overline{\varphi}(X)=\varphi_0(X_1)+\varphi_0(X_2)$ where $X_i, i=1,2$ indicate the natural projection of X into $C(I)$.

For convenience we put

$$H_1(x,y)(t) = I^{\alpha_1}\overline{f}(t) + \frac{\theta}{\Delta}\Big\{ - L^{\alpha_2-1}I^{\alpha_2-p}\overline{g}(\eta) + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-p)}\eta^{\alpha_2-p-1}(\gamma I^{q+\alpha_1}\overline{f}(\xi) - I^{\alpha_2}\overline{g}(L)\Big\},$$
$$H_2(x,y)(t) = I^{\alpha_1}\overline{g}(t) + \frac{t^{\alpha_2-1}}{\Delta}\Big\{I^{\alpha_2}\overline{g}(L) - \gamma I^{q+\alpha_1}\overline{f}(\xi) - \theta\gamma\frac{\xi^q}{\Gamma(1+q)}I^{\alpha_2-p}\overline{g}(\eta)\Big\}.$$

Let $t_1, t_2 \in [0, L]$ and $\varepsilon > 0$ be arbitrarily such that $|t_2 - t_1| \leq \varepsilon$. Thus for $(x_1, y_1) \in X_1 \times X_2$ we get

$$\begin{aligned}
&|G_1(x_1,y_1)(t_2) - G_1(x_1,y_1)(t_1)| \\
&= |f_1(t_2,x_1(t_2),y_1(t_2))H_1(x_1,y_1)(t_2) - f_1(t_1,x_1(t_1),y_1(t_1))H_1(x_1,y_1)(t_1)| \\
&\leq |f_1(t_2,x_1(t_2),y_1(t_2))H_1(x_1,y_1)(t_2) - f_1(t_2,x_1(t_1),y_1(t_1))H_1(x_1,y_1)(t_2)| \\
&+|f_1(t_2,x_1(t_1),y_1(t_1))H_1(x_1,y_1)(t_2) - f_1(t_1,x_1(t_1),y_1(t_1))H_1(x_1,y_1)(t_2)| \\
&+|f_1(t_1,x_1(t_1),y_1(t_1))||H_1(x_1,y_1)(t_2) - H_1(x_1,y_1)(t_1)| \\
&\leq e^{-d}\frac{1}{6}(|x_1(t_2) - x_1(t_1)| + |y_1(t_2) - y_1(t_1)|)(M_1l_1 + M_2l_2) + |t_2 - t_1|(M_1l_1 + M_2l_2) \\
&+\Big[\frac{1}{6}e^{-d}(|x_1(t_1)| + |y_1(t_1)|) + \overline{M}\Big](k_1|x_1(t_2) - x_1(t_1)| + k_2|y_1(t_2) - y_1(t_1)|) \\
&\leq \frac{1}{6}e^{-d}(\varphi(X_1,\epsilon) + \varphi(X_2,\varepsilon))(M_1l_1 + M_2l_2) \\
&+\varepsilon(M_1l_1 + M_2l_2) + \Big[\frac{1}{6}e^{-d}2r_0 + \overline{M}\Big](k_1\varphi(X_1,\varepsilon) + k_2\varphi(X_2,\varepsilon)).
\end{aligned}$$

Consequently, from assumption (D_3) and the above estimate we conclude that

$$\begin{aligned}
\varphi(G_1(X_1 \times X_2),\varepsilon) \;\leq\; &\frac{1}{6}e^{-d}(\varphi(X_1,\varepsilon) + \varphi(X_2,\varepsilon))(M_1l_1 + M_2l_2) + \varepsilon(M_1l_1 + M_2l_2) \\
&+\Big[\frac{1}{6}e^{-d}2r_0 + \overline{M}\Big](k_1\varphi(X_1,\varepsilon) + k_2\varphi(X_2,\varepsilon)).
\end{aligned}$$

Hence we have

$$\varphi_0(G_1(X_1 \times X_2)) \leq e^{-d}\frac{1}{2}(\varphi_0(X_1) + \varphi_0(X_2)). \tag{3.13}$$

Similarly, we prove that

$$\varphi_0(G_2(X_1 \times X_2)) \leq \frac{1}{2}e^{-d}(\varphi_0(X_1) + \varphi_0(X_2)). \tag{3.14}$$

Combining (3.13) and (3.14) we conclude that

$$\begin{aligned}
\overline{\varphi}(G(X) &\leq \overline{\varphi}(G(X_1 \times X_2)) \times G(X_2 \times X_1)) \\
&= \varphi_0(G(X_1 \times X_2)) + \varphi_0(G(X_2 \times X_1)) \\
&\leq \frac{1}{2}e^{-d}((\varphi_0(X_1) + \varphi_0(X_2)) + \frac{1}{2}e^{-d}(\varphi_0(X_2) + \varphi_0(X_1)).
\end{aligned} \tag{3.15}$$

By taking logarithms, we earn

$$d + \ln(\overline{\varphi}(G(X_1 \times X_2))) \leq \ln(\overline{\varphi}(X)). \tag{3.16}$$

Hence we obtain all conditions of Theorem 2.1 with $\theta_1(t) = d$ and $\theta_2(t) = \ln(t)$. Consequently, from Theorem 2.1 a fixed point of the mapping G is obtained which implies that the coupled system (1.1) and (1.2) has a solution on $[0, L]$ and the proof is completed.

□

Now an example is presented to show the applicability of the obtained result.

Example 3.1. *We consider the following hybrid nonlocal system of mixed fractional derivatives:*

$$\begin{cases} {}^{C}D^{\frac{1}{2}} \dfrac{u(t)}{\frac{e^{-d}(|u(t)|+|v(t)|+1)}{6(1+\frac{t}{8})}} = \dfrac{e^{-td}}{100}\cos\Big(\dfrac{u(t)+v(t)}{100}\Big), \quad t \in [0,1], \\ {}^{RL}D^{\frac{3}{2}} \dfrac{v(t)}{\frac{e^{-d}}{6(|u(t)|+|v(t)|+1)(1+t)}} = \dfrac{e^{-t}}{100}\sin\Big(\dfrac{u(t)+v(t)}{50}\Big), \quad t \in [0,1]. \\ \dfrac{u(0)}{f_1(0,u(0),v(0))} = \sqrt{3}\,{}^{C}D^{\frac{1}{2}} \dfrac{v(\frac{1}{3})}{g_1(\frac{1}{3}, u(\frac{1}{3}), v(\frac{1}{3}))}, \\ \dfrac{v(0)}{g_1(0,u(0),v(0))} = 0, \quad \dfrac{v(1)}{g_1(1,u(1),v(1))} = \sqrt{2} I^{\frac{1}{2}} \dfrac{u(\frac{1}{2})}{f_1(\frac{1}{2}, u(\frac{1}{2}), v(\frac{1}{2}))}. \end{cases} \tag{3.17}$$

Here, we have

$$\begin{aligned} &\alpha_1 = \frac{1}{2}, \theta = \sqrt{3}, p = \frac{1}{2}, \eta = \frac{1}{3}, \alpha_2 = \frac{3}{2}, \gamma = \sqrt{2}, q = \frac{1}{2}, \xi = \frac{1}{2}, d \text{ a positive real number,} \\ &f_2(t,u(t),v(t)) = \frac{e^{-td}}{100}\cos\Big(\frac{u(t)+v(t)}{100}\Big), \quad g_2(t,u(t),v(t)) = \frac{e^{-t}}{100}\sin\Big(\frac{u(t)+v(t)}{50}\Big), \\ &f_1(t,u(t),v(t)) = \frac{e^{-d}(|u(t)|+|v(t)|+1)}{6(1+\frac{t}{8})}, \\ &g_1(t,u(t),v(t)) = \frac{e^{-d}}{6(|u(t)|+|v(t)|+1)(1+t)}. \end{aligned}$$

The above system is a special case of the system (1.1) and (1.2). Now we show that the conditions of Theorem 3.1 are satisfied. Due to the definitions of f_2 and g_2, given $t \in [0,1]$ and $u, v, u_1, u_2, v_1, v_2 \in \mathbb{R}$ we have

$$\begin{aligned} &|f_2(t,u,v)| \le \frac{1}{100}, \quad |g_2(t,u,v)| \le \frac{1}{100}, \\ &|f_2(t,u_1,v_1) - f_2(t,u_2,v_2)| \le \frac{1}{10000}|u_1 - u_2| + \frac{1}{10000}|v_1 - v_2|, \\ &|g_2(t,u_1,v_1) - g_2(t,u_2,v_2)| \le \frac{1}{10000}|u_1 - u_2| + \frac{1}{10000}|v_1 - v_2|. \end{aligned}$$

Consequently, f_2 and g_2 satisfy condition (D_1) with $l_1 = l_2 = \frac{1}{100}$ and $k_1 = k_2 = \gamma_1 = \gamma_2 = \frac{1}{10000}$. Besides, obviously the functions f_1 and g_1 satisfy the condition (D_2). Furthermore, $\overline{M} = \frac{e^{-d}}{6}$ and $\overline{N} = \frac{e^{-d}}{6}$. To verify condition (D_3), given that $M_1 \approx 1.52$, $M_2 \approx 0.58$, $M_3 \approx 0.25$ and $M_4 \approx 1.26$, so the existent inequality in (D_3) has the form

$$
\begin{aligned}
&e^{-d}2r_0\Big(1.52\times\frac{1}{100}+0.58\times\frac{1}{100}+0.25\times\frac{1}{100}+1.26\times\frac{1}{100}\Big)\\
&+\frac{e^{-d}}{6}\Big(1.52\times\frac{1}{100}+0.58\times\frac{1}{100}\Big)+\frac{e^{-d}}{6}\Big(0.25\times\frac{1}{100}+1.26\times\frac{1}{100}\Big)\le r_0.
\end{aligned}
$$

Obviously, the above inequality has a positive solution r_0*, for example* $r_0 = e^{-d}$*. Moreover, we have*

$$
\overline{M}(k_1+k_2)=\frac{e^{-d}}{6}\Big(\frac{2}{10000}\Big)\le\frac{e^{-d}}{6},
$$

$$
\begin{aligned}
&2r_0(k_1+k_2)+M_1l_1+M_2l_2+M_3l_1+M_4l_2\\
&=2e^{-d}\Big(\frac{2}{10000}\Big)+1.52\times\frac{1}{100}+0.58\times\frac{1}{100}+0.25\times\frac{1}{100}+1.26\times\frac{1}{100}<1.
\end{aligned}
$$

Therefore all conditions of Theorem 3.1 are satisfied. Hence, by Theorem 3.1 the system (3.17) has a solution on $[0, 1]$.

4. Conclusions

We have studied a nonlocal coupled hybrid fractional system consisting from one Caputo and one Riemann-Liouville fractional derivatives and nonlocal hybrid boundary conditions. An existence result is established via a new generalization of Darbo's fixed point theorem associated with measures of noncompactness. The obtained result is well illustrated by a numerical example. The result obtained in this paper is new and significantly contributes to the existing literature on the topic.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, Amesterdam, 2006.
3. B. Ahmad, A. Alsaedi, S. K. Ntouyas, J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*, Springer, Cham, Switzerland, 2017.
4. B. C. Dhage, V. Lakshmikantham, Basic results on hybrid differential equations, *Nonlinear Anal. Hybrid Syst.,* **4** (2010), 414–424.

5. K. Hilal, A. Kajouni, Boundary value problems for hybrid differential equations with fractional order, *Adv. Differ. Equ.,* **2015** (2015), 1–19.
6. E. T. Karimov, B. Lopez, K. Sadarngani, About the existence of solutions for a hybrid nonlinear generalized pantograph equation, *Fract. Differ. Calc.,* **6** (2016), 95–110.
7. N. Mahmudov, M. Matar, Existence of mild solutions for hybrid differential equations with arbitrary fractional order, *TWMS J. Pure Appl. Math.,* **8** (2017), 160–169.
8. S. Sitho, S. K. Ntouyas, J. Tariboon, Existence results for hybrid fractional integro-differential equations, *Bound. Value Probl.,* **2015** (2015), 1–13.
9. S. Sun, Y. Zhao, Z. Han, Y. Li, The existence of solutions for boundary value problem of fractional hybrid differential equations, *Commun. Nonlinear Sci. Numer. Simul.,* **17** (2012), 4961–4967.
10. D. Baleanu, S. Etemad, S. Rezapour, A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions, *Bound. Value Probl.,* **2020** (2020), 1–16.
11. D. Baleanu, S. Etemad, S. Rezapour, On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators, *Alexandria Eng. J.,* **59** (2020), 3019–3027.
12. B. C. Dhage, Existence and attractivity theorems for nonlinear hybrid fractional integrodifferential equations with anticipation and retardation, *CUBO,* **22** (2020), 325–350.
13. D. Ji, W. Ge, A nonlocal boundary value problems for hybrid ϕ-Caputo fractional integro-differential equations, *AIMS Math.,* **5** (2020), 7175–7190.
14. S. K. Ntouyas, H. H. Al-Sulami, A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions, *Adv. Differ. Equ.,* **2020** (2020), 1–21.
15. B. Ahmad, S. K. Ntouyas, A. Alsaedi, Existence theory for nonlocal boundary value problems involving mixed fractional derivatives, *Nonlinear Anal. Model. Control,* **24** (2019), 1–21.
16. J. Banas, K. Goebel, *Measure of noncompactness in Banach spaces*, Lecture notes in Pure and Appl. Math, New York: Dekker, 1980.
17. A. Samadi, M. B. Ghaemi, An extension of Darbo fixed point theorem and its applications to coupled fixed point and integral equations, *Filomat,* **28** (2014), 879–886.
18. A. Samadi, Applications of measure of noncompactness to coupled fixed points and systems of integral equations, *Miskolc Math. Notes,* **19** (2018), 537–553.
19. A. Samadi, S. K. Ntouyas, Solvability for infinite systems of fractional differential equations in Banach sequence spaces ℓ_p and c_0, *Filomat,* **34** (2020), 3943–3955.

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