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Research article

Qualitative analysis of nonlinear implicit neutral differential equation of fractional order

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Abstract: In this paper, we discuss sufficient conditions for the existence of solutions for a class of Initial value problem for an neutral differential equation involving Caputo fractional derivatives. Also, we discuss some types of Ulam stability for this class of implicit fractional-order differential equation. Some applications and particular cases are presented. Finally, the existence of at least one mild solution for this class of implicit fractional-order differential equation on an infinite interval by applying Schauder fixed point theorem and the local attractivity of solutions are proved.

Keywords: neutral differential equation; Caputo fractional derivative; Ulam stability; Ulam-Hyers stability; mild solution; attractivity of solutions
Mathematics Subject Classification: 26A33, 34K37, 35B35

1. Introduction

Fractional differential equations have played an important role and have presented valuable tools in the modeling of many phenomena in various fields of science and engineering [6–16]. There has been a significant development in fractional differential equations in recent decades [2–5, 23, 26, 33, 37]. On the other hand, many authors studied the stability of functional equations and established some types of Ulam stability [1, 17–22, 24, 27–37] and references there in. Moreover, many authors discussed local and global attractivity [8–11, 34].

Benchohra et al. [13] established some types of Ulam-Hyers stability for an implicit fractional-order differential equation.

A. Baliki et al. [11] have given sufficient conditions for existence and attractivity of mild solutions for second order semi-linear functional evolution equation in Banach spaces using Schauder's fixed point theorem.

Benchohra et al. [15] studied the existence of mild solutions for a class of impulsive semilinear fractional differential equations with infinite delay and non-instantaneous impulses in Banach spaces. This results are obtained using the technique of measures of noncompactness.

Motivated by these works, in this paper, we investigate the following initial value problem for an implicit fractional-order differential equation

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}\left[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))\right]=\mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))) \quad \mathfrak{t}\in J,\ 1<\alpha\leq 2,\ \alpha\geq\beta, \\ \\ \Big(\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(t))\Big)\Big|_{\mathfrak{t}=0}=0 \quad \text{and} \quad \frac{d}{dt}\left[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))\right]_{\mathfrak{t}=0}=0 \end{cases} \tag{1.1}$$

where ${}^{C}\mathfrak{D}^{\alpha}$ is the Caputo fractional derivative, $\mathfrak{h}: J\times R\longrightarrow R$, $\mathfrak{g}_1: J\times R\times R\longrightarrow R$ and $\mathfrak{g}_2: J\times R\longrightarrow R$ are given functions satisfy some conditions and $J=[0,T]$.

we give sufficient conditions for the existence of solutions for a class of initial value problem for an neutral differential equation involving Caputo fractional derivatives. Also, we establish some types of Ulam-Hyers stability for this class of implicit fractional-order differential equation and some applications and particular cases are presented.

Finally, existence of at least one mild solution for this class of implicit fractional-order differential equation on an infinite interval $J=[0,+\infty)$, by applying Schauder fixed point theorem and proving the attractivity of these mild solutions.

By a solution of the Eq (1.1) we mean that a function $x\in C^2(J,R)$ such that

(i) the function $\mathfrak{t}\rightarrow[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))]\in C^2(J,R)$ and
(ii) $\mathfrak{x}$ satisfies the equation in (1.1).

2. Preliminaries

Definition 1. *[23] The Riemann-Liouville fractional integral of the function* $\mathfrak{f}\in L^1([\mathfrak{a},\mathfrak{b}])$ *of order* $\alpha\in R_+$ *is defined by*

$$\mathfrak{I}^{\alpha}_{\mathfrak{a}}\,\mathfrak{f}(\mathfrak{t})=\int_{\mathfrak{a}}^{\mathfrak{t}}\frac{(\mathfrak{t}-\mathfrak{s})^{\alpha-1}}{\Gamma(\alpha)}\,\mathfrak{f}(\mathfrak{s})\,d\mathfrak{s}.$$

and when $\mathfrak{a}=0$, *we have* $\mathfrak{I}^{\alpha}\,\mathfrak{f}(\mathfrak{t})=\mathfrak{I}^{\alpha}_{0}\,\mathfrak{f}(\mathfrak{t})$.

Definition 2. *[23] For a function* $\mathfrak{f}:[\mathfrak{a},\mathfrak{b}]\rightarrow R$ *the Caputo fractional-order derivative of* $\mathfrak{f}$, *is defined by*

$${}^{C}\mathfrak{D}^{\alpha}\mathfrak{h}(\mathfrak{t})=\frac{1}{\Gamma(\mathfrak{n}-\alpha)}\int_{\mathfrak{a}}^{\mathfrak{t}}\frac{\mathfrak{h}^{(\mathfrak{n})}(\mathfrak{s})}{(\mathfrak{t}-\mathfrak{s})^{\mathfrak{n}-\alpha-1}}\,d\mathfrak{s},$$

where where $\mathfrak{n}=[\alpha]+1$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Lemma 1. *[23]. Let* $\alpha\geq 0$ *and* $\mathfrak{n}=[\alpha]+1$. *Then*

$$\mathfrak{I}^{\alpha}({}^{C}\mathfrak{D}^{\alpha}\mathfrak{f}(\mathfrak{t}))=\mathfrak{f}(\mathfrak{t})-\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{f}^{k}(0)}{k!}\mathfrak{t}^{k}$$

Lemma 2. *Let* $\mathfrak{f}\in L^1([\mathfrak{a},\mathfrak{b}])$ *and* $\alpha\in(0,1]$, *then*

(i) ${}^{C}\mathfrak{D}^{\alpha}\mathfrak{I}^{\alpha}\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$.
(ii) The operator $\mathfrak{I}^{\alpha}$ *maps* $L^1([\mathfrak{a},\mathfrak{b}])$ *into itself continuously.*
(iii) For $\gamma, \beta > 0$, *then*

$$\mathfrak{I}_{\mathfrak{a}}^{\beta}\,\mathfrak{I}_{\mathfrak{a}}^{\gamma}\,\mathfrak{f}(\mathfrak{t}) \;=\; \mathfrak{I}_{\mathfrak{a}}^{\gamma}\mathfrak{I}_{\mathfrak{a}}^{\beta}\,\mathfrak{f}(\mathfrak{t}) \;=\; \mathfrak{I}_{\mathfrak{a}}^{\gamma+\beta}\,\mathfrak{f}(\mathfrak{t}),$$

For further properties of fractional operators (see [23, 25, 26]).

3. Main results

Consider the initial value problem for the implicit fractional-order differential Eq (1.1) under the following assumptions:

(i) $\mathfrak{h} : J \times R \longrightarrow R$ is a continuous function and there exists a positive constant K_1 such that:

$$|\,\mathfrak{h}(\mathfrak{t},\mathfrak{x}) - \mathfrak{h}(\mathfrak{t},\mathfrak{y})\,| \leqslant K_1\,|\,\mathfrak{x}-\mathfrak{y}\,| \quad \text{for each} \quad \mathfrak{t}\in J \ \text{ and } \ \mathfrak{x},\mathfrak{y}\in R.$$

(ii) $\mathfrak{g}_1 : J\times R\times R \longrightarrow R$ is a continuous function and there exist two positive constants K, H such that:

$$|\,\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y}) - |\,\mathfrak{g}_1(\mathfrak{t},\widetilde{\mathfrak{x}},\widetilde{\mathfrak{y}})\,| \leqslant K\,|\,\mathfrak{x}-\widetilde{\mathfrak{x}}\,| + H\,|\,\mathfrak{y}-\widetilde{\mathfrak{y}}\,| \ \text{ for each } \mathfrak{t}\in J \text{ and } \ \mathfrak{x},\widetilde{\mathfrak{x}},\mathfrak{y},\widetilde{\mathfrak{y}}\in R$$

(iii) $\mathfrak{g}_2 : J\times R \longrightarrow R$ is a continuous function and there exists a positive constant K_2 such that:

$$|\,\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}) - \mathfrak{g}_2(\mathfrak{t},\mathfrak{y})\,| \leqslant K_2\,|\,\mathfrak{x}-\mathfrak{y}\,| \quad \text{for each } \mathfrak{t}\in J \ \text{ and } \mathfrak{x},\mathfrak{y}\in R.$$

Lemma 3. *Let assumptions (i)–(iii) be satisfied. If a function* $\mathfrak{x}\in C^2(J,R)$ *is a solution of initial value problem for implicit fractional-order differential equation* (1.1)*, then it is a solution of the following nonlinear fractional integral equation*

$$\mathfrak{x}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s} \tag{3.1}$$

Proof. Assume first that $\mathfrak{x}$ is a solution of the initial value problem (1.1). From definition of Caputo derivative, we have

$$\mathfrak{I}^{2-\alpha}\mathfrak{D}^2\,(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))) = \mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))).$$

Operating by $\mathfrak{I}^{\alpha-1}$ on both sides and using Lemma 2, we get

$$\mathfrak{I}^1\mathfrak{D}^2\,(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))) = \mathfrak{I}^{\alpha-1}\mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))).$$

Then

$$\frac{d}{d\mathfrak{t}}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))) - \frac{d}{d\mathfrak{t}}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})))\bigg|_{\mathfrak{t}=0} = \mathfrak{I}^{\alpha-1}\mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))).$$

Using initial conditions, we have

$$\frac{d}{d\mathfrak{t}}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))) = \mathfrak{I}^{\alpha-1}\mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))).$$

Integrating both sides of (1.1), we obtain

$$(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))) - (\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0} = \mathfrak{I}^{\alpha}\mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))).$$

Then

$$\mathfrak{x}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}$$

Conversely, assume that $\mathfrak{x}$ satisfies the nonlinear integral Eq (3.1). Then operating by ${}^{C}\mathfrak{D}^{\alpha}$ on both sides of Eq (3.1) and using Lemma 2, we obtain

$$\begin{aligned} {}^{C}\mathfrak{D}^{\alpha}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))) &= {}^{C}\mathfrak{D}^{\alpha}\mathfrak{I}^{\alpha}\mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))) \\ &= \mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))). \end{aligned}$$

Putting $\mathfrak{t} = 0$ in (3.1) and since $\mathfrak{g}_1$ is a continuous function, then we obtain

$$(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0} = I^{\alpha}\mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0} = 0.$$

Also,

$$\frac{d}{d\mathfrak{t}}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))) = \mathfrak{I}^{\alpha-1}\mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))).$$

Then we have

$$\frac{d}{d\mathfrak{t}}(\mathfrak{x}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0} = \mathfrak{I}^{\alpha-1}\mathfrak{g}_1(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0} = 0.$$

Hence the equivalence between the initial value problem (1.1) and the integral Eq (3.1) is proved. Then the proof is completed. □

Definition 3. *The Eq (1.1) is Ulam-Hyers stable if there exists a real number* $\mathfrak{c}_{\mathfrak{f}} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $\mathfrak{z} \in C^2(J, R)$ *of the inequality*

$$|{}^{C}\ \mathfrak{D}^{\alpha}\ [\mathfrak{z}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] - \mathfrak{g}_1(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})))\ | \leqslant \epsilon,\ \mathfrak{t} \in J,$$

there exists a solution $\mathfrak{y} \in C^2(J, R)$ *of Eq (1.1) with*

$$|\ \mathfrak{z}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})| \leqslant \mathfrak{c}_{\mathfrak{f}}\epsilon,\ \mathfrak{t} \in J.$$

Definition 4. *The Eq (1.1) is generalized Ulam-Hyers stable if there exists* $\psi_{\mathfrak{f}} \in C(R_+, R_+)$, $\psi_{\mathfrak{f}}(0) = 0$, *such that for each solution* $\mathfrak{z} \in C^2(J, R)$ *of the inequality*

$$|{}^{C}\ \mathfrak{D}^{\alpha}\ [\mathfrak{z}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] - \mathfrak{g}_1(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})))\ | \leqslant \epsilon,\ \mathfrak{t} \in J,$$

there exists a solution $\mathfrak{y} \in C^2(J, R)$ *of Eq (1.1) with*

$$|\ \mathfrak{z}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})| \leqslant \psi_{\mathfrak{f}}(\epsilon),\ \mathfrak{t} \in J.$$

Definition 5. *The Eq (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, R_+)$ if there exists a real number $\mathfrak{c}_{\mathfrak{f}} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{z} \in C^2(J, R)$ of the inequality*

$$|^C\ \mathfrak{D}^\alpha\,[\mathfrak{z}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] - \mathfrak{g}_1(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}), \mathfrak{I}^\beta \mathfrak{g}_2(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})))\ |\leqslant \epsilon\varphi(\mathfrak{t}),\ \ \mathfrak{t} \in J,$$

there exists a solution $\mathfrak{y} \in C^2(J, R)$ of Eq (1.1) with

$$|\ \mathfrak{z}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})| \leqslant \mathfrak{c}_{\mathfrak{f}}\epsilon\varphi(\mathfrak{t}),\ \ \mathfrak{t} \in J.$$

Definition 6. *The Eq (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, R_+)$ if there exists a real number $\mathfrak{c}_{\mathfrak{f},\varphi} > 0$ such that for each solution $\mathfrak{z} \in C^2(J, R)$ of the inequality*

$$|^C\ \mathfrak{D}^\alpha\,[\mathfrak{z}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] - \mathfrak{g}_1(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}), \mathfrak{I}^\beta \mathfrak{g}_2(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})))\ |\leqslant \varphi(\mathfrak{t}),\ \ \mathfrak{t} \in J,$$

there exists a solution $\mathfrak{y} \in C^2(J, R)$ of Eq (1.1) with

$$|\ \mathfrak{z}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})| \leqslant \mathfrak{c}_{\mathfrak{f},\varphi}\varphi(\mathfrak{t}),\ \ \mathfrak{t} \in J.$$

Now, our aim is to investigate the existence of unique solution for (1.1). This existence result will be based on the contraction mapping principle.

Theorem 1. *Let assumptions (i)-(iii) be satisfied. If* $K_1 + \dfrac{K\ T^\alpha}{\Gamma(\alpha+1)} + \dfrac{K_2\ H\ T^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)} < 1,$ *then there exists a unique solution for the nonlinear neutral differential equation of fractional order .*

Proof. Define the operator N by:

$$N\mathfrak{x}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\right)d\mathfrak{s},\ \ \mathfrak{t} \in J.$$

In view of assumptions (i)-(iii), then $N : C^2(J, R) \to C^2(J, R)$ is continuous operator.

Now, let $\mathfrak{x}$ and $,\widetilde{\mathfrak{x}} \in C^2(J, R),$ be two solutions of (1.1)then

$$\begin{aligned}
|\ N\mathfrak{x}(\mathfrak{t}) - N\widetilde{\mathfrak{x}}(\mathfrak{t})\ | &= \Bigg|\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\Big)d\mathfrak{s} \\
&- \mathfrak{h}(\mathfrak{t}, \widetilde{\mathfrak{x}}(\mathfrak{t})) \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\Big(\mathfrak{s}, \widetilde{\mathfrak{x}}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \widetilde{\mathfrak{x}}(\theta))d\theta\Big)d\mathfrak{s}\Bigg| \\
&\leqslant K_1|\mathfrak{x}(\mathfrak{t}) - \widetilde{\mathfrak{x}}(\mathfrak{t})| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1}\Bigg|\mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\Big) \\
&- g_1\Big(\mathfrak{s}, \widetilde{\mathfrak{x}}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \widetilde{\mathfrak{x}}(\theta))d\theta\Big)\Bigg|d\mathfrak{s} \\
&\leqslant K_1|\mathfrak{x}(\mathfrak{t}) - \widetilde{\mathfrak{x}}(\mathfrak{t})|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}K \mid \mathfrak{x}(\mathfrak{s})-\widetilde{\mathfrak{x}}(\mathfrak{s})|d\mathfrak{s}\\
&+ H\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mid \mathfrak{g}_2(\theta,\mathfrak{x}(\theta))-\mathfrak{g}_2(\theta,\widetilde{\mathfrak{x}}(\theta))|d\theta d\mathfrak{s}\\
&\leqslant K_1|\mathfrak{x}(\mathfrak{t})-\widetilde{\mathfrak{x}}(\mathfrak{t})|+\frac{K}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mid \mathfrak{x}(\mathfrak{s})-\widetilde{\mathfrak{x}}(\mathfrak{s})|d\mathfrak{s}\\
&+ \frac{H}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\frac{K_2}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mid \mathfrak{x}(\theta)-\widetilde{\mathfrak{x}}(\theta)|d\theta d\mathfrak{s}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|N\mathfrak{x}(\mathfrak{t})-N\widetilde{\mathfrak{x}}(\mathfrak{t})\| &\leqslant K_1\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|+\frac{K\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}d\mathfrak{s}\\
&+ \|\mathfrak{x}-\widetilde{\mathfrak{x}}\|\frac{H}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\frac{K_2}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}d\theta d\mathfrak{s}\\
&\leqslant K_1\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|+\frac{K\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|\,T^{\alpha}}{\Gamma(\alpha+1)}\\
&+ \|\mathfrak{x}-\widetilde{\mathfrak{x}}\|\frac{K_2\,T^{\beta}}{\Gamma(\beta+1)}\frac{H}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}d\mathfrak{s}\\
&\leqslant K_1\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|+\frac{K\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|\,T^{\alpha}}{\Gamma(\alpha+1)}\\
&+ \|\mathfrak{x}-\widetilde{\mathfrak{x}}\|\frac{K_2\,T^{\beta}}{\Gamma(\beta+1)}\frac{H\,T^{\alpha}}{\Gamma(\alpha+1)}\\
&\leq \left[K_1+\frac{K\,T^{\alpha}}{\Gamma(\alpha+1)}+\frac{K_2\,H\,T^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right]\|\mathfrak{x}-\widetilde{\mathfrak{x}}\|
\end{aligned}
$$

Since $K_1+\frac{K\,T^{\alpha}}{\Gamma(\alpha+1)}+\frac{K_2\,H\,T^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)}<1$. It follows that N has a unique fixed point which is a solution of the initial value problem (1.1) in $C^2(J,R)$. □

4. Stability of solutions of the IVP (1.1)

4.1. Ulam-Hyers stability

Theorem 2. *Let assumptions of Theorem 1 be satisfied. Then the fractional order differential Eq (1.1) is Ulam-Hyers stable.*

Proof. Let $\mathfrak{y}\in C^2(J,R)$ be a solution of the inequality

$$
|{}^{C}\,\mathfrak{D}^{\alpha}\left[\mathfrak{y}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))\right]-\mathfrak{g}_1(\mathfrak{t},\mathfrak{y}(\mathfrak{t}),\mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t},\mathfrak{y}(\mathfrak{t})))\,|\leqslant\epsilon,\ \epsilon>0,\ \mathfrak{t}\in J. \tag{4.1}
$$

Let $\mathfrak{x}\in C^2(J,R)$ be the unique solution of the initial value problem for implicit fractional-order differential Eq (1.1). By using Lemma 3, The Cauchy problem (1.1) is equivalent to

$$
\mathfrak{x}(\mathfrak{t})=\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))+\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}.
$$

Operating by $\mathfrak{I}^{\alpha-1}$ on both sides of (4.1) and then integrating, we get

$$\begin{aligned}\Bigg|\mathfrak{y}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t})) &- \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{y}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\Bigg|\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\epsilon\, d\mathfrak{s},\\
&\leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

Also, we have

$$\begin{aligned}|\mathfrak{y}(\mathfrak{t})-\mathfrak{x}(\mathfrak{t})| &= \left|\mathfrak{y}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\right|\\
&= \Bigg|\mathfrak{y}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\\
&+ \mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))+\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{y}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\\
&- \mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{y}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\Bigg|\\
&\leq \left|\mathfrak{y}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{y}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\right|\\
&+ |\mathfrak{h}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))|\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\Bigg|\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{y}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))d\theta\right)\\
&- \mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)\Bigg|d\mathfrak{s}\\
&\leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}+K_1|\mathfrak{y}(\mathfrak{t})-\mathfrak{x}(\mathfrak{t})|\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\left[K|\mathfrak{y}(\mathfrak{t})-\mathfrak{x}(\mathfrak{t})|+\frac{H}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}|\mathfrak{g}_2(\theta,\mathfrak{y}(\theta))-\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))|d\theta\right]d\mathfrak{s}\end{aligned}$$

$$\begin{aligned}\|\mathfrak{y}-\mathfrak{x}\| &\leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}+K_1\|\mathfrak{y}-\mathfrak{x}\|\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\left[K\|\mathfrak{y}-\mathfrak{x}\|+\frac{H\, K_2\|\mathfrak{x}-\mathfrak{y}\|T^{\beta}}{\Gamma(\beta+1)}\right]d\mathfrak{s}\\
&\leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}+K_1\|\mathfrak{y}-\mathfrak{x}\|\\
&+ \frac{K\, T^{\alpha}\|\mathfrak{y}-\mathfrak{x}\|}{\Gamma(\alpha+1)}+\frac{H\, K_2T^{\beta+\alpha}\|\mathfrak{x}-\mathfrak{y}\|}{\Gamma(\beta+1)\Gamma(\alpha+1)}.\end{aligned}$$

Then

$$\|\mathfrak{y}-\mathfrak{x}\|\leq \frac{\epsilon\, T^{\alpha}}{\Gamma(\alpha+1)}\left[1-\left(K_1+\frac{K\, T^{\alpha}}{\Gamma(\alpha+1)}+\frac{H\, K_2\, T^{\beta+\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right)\right]^{-1}=\mathfrak{c}\,\epsilon,$$

thus the intial value problem (1.1) is Ulam-Heyers stable, and hence the proof is completed. □

By putting $\psi(\varepsilon) = \mathfrak{c}\,\varepsilon,\ \psi(0) = 0$ yields that the Eq (1.1) is generalized Ulam-Heyers stable.

4.2. Ulam-Hyers-Rassias stability

Theorem 3. *Let assumptions of Theorem 1 be satisfied, there exists an increasing function $\varphi \in C(J, R)$ and there exists $\lambda_\varphi > 0$ such that for any $\mathfrak{t} \in J$, we have*

$$\mathfrak{I}^{\alpha}\varphi(\mathfrak{t}) \leqslant \lambda_\varphi\varphi(\mathfrak{t}),$$

then the Eq (1.1) is Ulam-Heyers-Rassias stable.

Proof. Let $\mathfrak{y} \in C^2(J, R)$ be a solution of the inequality

$$|^{C}\ \mathfrak{D}^{\alpha}\,[\mathfrak{y}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}))] - \mathfrak{g}_1(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}), \mathfrak{I}^{\beta}\mathfrak{g}_2(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})))\ | \leqslant \epsilon\varphi(\mathfrak{t}),\ \epsilon > 0,\ \mathfrak{t} \in J. \tag{4.2}$$

Let $\mathfrak{x} \in C^2(J, R)$ be the unique solution of the initial value problem for implicit fractional-order differential Eq (1.1). By using Lemma 3, The Cauchy problem (1.1) is equivalent to

$$\mathfrak{x}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}.$$

Operating by $\mathfrak{I}^{\alpha-1}$ on both sides of (4.2) and then integrating, we get

$$\begin{aligned}
\Bigg|\mathfrak{y}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})) \ &- \ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\Bigg| \\
&\leqslant \ \frac{\epsilon}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\varphi(\mathfrak{s})d\mathfrak{s}, \\
&\leq \ \epsilon\,\mathfrak{I}^{\alpha}\varphi(\mathfrak{t}) \\
&\leq \ \epsilon\,\lambda_\varphi\varphi(\mathfrak{t}).
\end{aligned}$$

Also, we have

$$\begin{aligned}
|\mathfrak{y}(\mathfrak{t}) - \mathfrak{x}(\mathfrak{t})| \ &= \ \left|\mathfrak{y}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\right| \\
&= \ \Bigg|\mathfrak{y}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))d\theta\right)d\mathfrak{s} \\
&+ \ \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{y}(\theta))d\theta\right)d\mathfrak{s} \\
&- \ \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\Bigg| \\
&\leq \ \left|\mathfrak{y}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{y}(\theta))d\theta\right)d\mathfrak{s}\right| \\
&+ \ |\mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))| \\
&+ \ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\Bigg|\mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta, \mathfrak{y}(\theta))d\theta\right)
\end{aligned}$$

$$
\begin{aligned}
&- \quad \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) \Bigg| d\mathfrak{s} \\
&\leq \quad \epsilon\, \lambda_\varphi \varphi(\mathrm{t}) + K_1 |\mathfrak{y}(\mathrm{t}) - \mathfrak{x}(\mathrm{t})| \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{t}} (\mathrm{t}-\mathfrak{s})^{\alpha-1} \left[K|\mathfrak{y}(\mathrm{t}) - \mathfrak{x}(\mathrm{t})| + \frac{H}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} |\mathfrak{g}_2(\theta, \mathfrak{y}(\theta)) - \mathfrak{g}_2(\theta, \mathfrak{x}(\theta))| d\theta \right] d\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| \quad &\leq \quad \epsilon\, \lambda_\varphi \varphi(\mathrm{t}) + K_1 \|\mathfrak{y} - \mathfrak{x}\| \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{t}} (\mathrm{t}-\mathfrak{s})^{\alpha-1} \left[K\|\mathfrak{y}(\mathrm{t}) - \mathfrak{x}(\mathrm{t})\| + \frac{H\, K_2 \|\mathfrak{x} - \mathfrak{y}\| T^\beta}{\Gamma(\beta+1)} \right] d\mathfrak{s} \\
&\leq \quad \epsilon \lambda_\varphi \varphi(\mathrm{t}) + K_1 \|\mathfrak{y} - \mathfrak{x}\| \\
&+ \quad \frac{K\, T^{\ \alpha} \|\mathfrak{y} - \mathfrak{x}\|}{\Gamma(\alpha+1)} + \frac{H\, K_2\, T^{\beta+\alpha} \|\mathfrak{x} - \mathfrak{y}\|}{\Gamma(\beta+1)\Gamma(\alpha+1)}.
\end{aligned}
$$

Then

$$
\|\mathfrak{y} - \mathfrak{x}\| \leq \epsilon\, \lambda_\varphi\, \varphi(\mathrm{t}) \left[1 - \left(K_1 + \frac{K\, T^\alpha}{\Gamma(\alpha+1)} + \frac{H\, K_2\, T^{\beta+\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} \right) \right]^{-1} = \mathfrak{c}\ \epsilon \varphi(\mathrm{t}),
$$

then the initial problem (1.1) is Ulam-Heyers-Rassias stable, and hence the proof is completed. □

5. Existence and attractivity of solutions on half line

In this section, we prove some results on the existence of mild solutions and attractivity for the neutral fractional differential equation (1.1) by applying Schauder fixed point theorem. Denote $\mathfrak{B}C = \mathfrak{B}C(J)$, $J = [0, +\infty)$ and consider the following assumptions:

(I) $\mathfrak{h} : J \times R \longrightarrow R$ is a continuous function and there exists a continuous function $K_{\mathfrak{h}}(\mathrm{t})$ such that:

$$
|\ \mathfrak{h}(\mathrm{t}, \mathfrak{x}) - \mathfrak{h}(\mathrm{t}, \mathfrak{y})\ | \leqslant K_{\mathfrak{h}}(\mathrm{t})\ |\ \mathfrak{x} - \mathfrak{y}\ | \quad \text{for each } \mathrm{t} \in J \text{ and } \mathfrak{x}, \mathfrak{y} \in R,
$$

where $K_{\mathfrak{h}}^* = \sup_{\mathrm{t}\geq 0} K_{\mathfrak{h}}(\mathrm{t}) < 1$, $\lim_{\mathrm{t}\to\infty} K_{\mathfrak{h}}(\mathrm{t}) = 0$, and $\lim_{\mathrm{t}\to\infty} \mathfrak{h}(\mathrm{t}, 0) = 0$.

(II) $\mathfrak{g}_1 : J \times R \times R \longrightarrow R$ satisfies Carathéodory condition and there exist an integrable function $\mathfrak{a}_1 : R_+ \longrightarrow R_+$ and a positive constant $\mathfrak{b}$ such that:

$$
|\ \mathfrak{g}_1(\mathrm{t}, \mathfrak{x}, \mathfrak{y})\ | \leq \frac{\mathfrak{a}_1(\mathrm{t})}{1 + |\mathfrak{x}|} + \mathfrak{b}|\mathfrak{y}| \quad \text{for each } \mathrm{t} \in J \text{ and } \mathfrak{x},\ \mathfrak{y} \in R.
$$

(III) $\mathfrak{g}_2 : J \times R \longrightarrow R$ satisfies Carathéodory condition and there exists an integrable function $\mathfrak{a}_2 : R_+ \longrightarrow R_+$ such that:

$$
|\ \mathfrak{g}_2(\mathrm{t}, \mathfrak{x})\ | \leq \frac{\mathfrak{a}_2(\mathrm{t})}{1 + |\mathfrak{x}|} \quad \text{for each } \mathrm{t} \in J \text{ and } \mathfrak{x} \in R.
$$

(IV) Let

$$
\lim_{\mathrm{t}\to\infty} \int_0^{\mathrm{t}} \frac{(\mathrm{t}-\mathfrak{s})^{\alpha-1}}{\Gamma(\alpha)} \mathfrak{a}_1(\mathfrak{s})\, d\mathfrak{s} = 0
$$

$$\mathfrak{a}_1^* = \sup_{\mathfrak{t}\in J} \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha-1}}{\Gamma(\alpha)} \mathfrak{a}_1(\mathfrak{s})\, d\mathfrak{s}$$

$$\lim_{\mathfrak{t}\to\infty} \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \mathfrak{a}_2(\mathfrak{s})\, d\mathfrak{s} = 0$$

$$\mathfrak{a}_2^* = \sup_{\mathfrak{t}\in J} \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \mathfrak{a}_2(\mathfrak{s})\, d\mathfrak{s}$$

By a mild solution of the Eq (1.1) we mean that a function $x \in C(J, R)$ such that $\mathfrak{x}$ satisfies the equation in (3.1).

Theorem 4. *Let assumptions (I)–(IV) be satisfied. Then there exists at least one mild solution for the nonlinear implicit neutral differential equation of fractional order (1.1). Moreover, mild solutions of IVP (1.1) are locally attractive.*

Proof. For any $\mathfrak{x} \in \mathfrak{B}C$, define the operator $\mathbb{A}$ by

$$\mathbb{A}\mathfrak{x}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) d\mathfrak{s}.$$

The operator $\mathbb{A}$ is well defined and maps $\mathfrak{B}C$ into $\mathfrak{B}C$. Obviously, the map $\mathbb{A}(\mathfrak{x})$ is continuous on J for any $\mathfrak{x} \in \mathfrak{B}C$ and for each $\mathfrak{t} \in J$, we have

$$\begin{aligned}
|\mathbb{A}\mathfrak{x}(\mathfrak{t})| &\leq |\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))| + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left| \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right)\right| d\mathfrak{s} \\
&\leq |\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \mathfrak{h}(\mathfrak{t}, 0)| + |\mathfrak{h}(\mathfrak{t}, 0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left[\frac{\mathfrak{a}_1(\mathfrak{s})}{1+|\mathfrak{x}(\mathfrak{s})|} + \mathfrak{b} \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} |\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))| d\theta\right] d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}(\mathfrak{t})|\mathfrak{x}(\mathfrak{t})| + |\mathfrak{h}(\mathfrak{t}, 0)| + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left[\mathfrak{a}_1(\mathfrak{s}) + \mathfrak{b} \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \frac{\mathfrak{a}_2(\theta)}{1+|\mathfrak{x}(\theta)|} d\theta\right] d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}^*|\mathfrak{x}(\mathfrak{t})| + |\mathfrak{h}(\mathfrak{t}, 0)| + \mathfrak{a}_1^* + \mathfrak{b} \frac{1}{\Gamma(\alpha+\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha+\beta-1} \frac{\mathfrak{a}_2(\theta)}{1+|\mathfrak{x}(\theta)|} d\theta \\
&\leq K_{\mathfrak{h}}^* M + |\mathfrak{h}(\mathfrak{t}, 0)| + \mathfrak{a}_1^* + \mathfrak{b} \frac{1}{\Gamma(\alpha+\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha+\beta-1} \mathfrak{a}_2(\theta) d\theta \\
&\leq K_{\mathfrak{h}}^* M + |\mathfrak{h}(\mathfrak{t}, 0)| + \mathfrak{a}_1^* + \mathfrak{b}\mathfrak{a}_2^* \leq M.
\end{aligned}$$

Then

$$\|\mathbb{A}\mathfrak{x}(\mathfrak{t})\|_{\mathfrak{B}C} \leq M, \quad M = (|\mathfrak{h}(\mathfrak{t}, 0)| + \mathfrak{a}_1^* + \mathfrak{b}\mathfrak{a}_2^*)(1 - K_{\mathfrak{h}}^*)^{-1}. \tag{5.1}$$

Thus $\mathbb{A}(\mathfrak{x}) \in \mathfrak{B}C$. This clarifies that operator $\mathbb{A}$ maps $\mathfrak{B}C$ into itself.

Finding the solutions of IVP (1.1) is reduced to find solutions of the operator equation $\mathbb{A}(\mathfrak{x}) = \mathfrak{x}$. Eq (5.1) implies that $\mathbb{A}$ maps the ball $\mathbb{B}_M := \mathbb{B}(0, M) = \{\mathfrak{x} \in \mathfrak{B}C : \|\mathfrak{x}(\mathfrak{t})\|_{\mathfrak{B}C} \leq M\}$ into itself. Now, our proof will be established in the following steps:

Step 1: $\mathbb{A}$ is continuous.
Let $\{\mathfrak{x}_n\}_{n\in N}$ be a sequence such that $\mathfrak{x}_n \to \mathfrak{x}$ in $\mathbb{B}_M$. Then, for each $\mathfrak{t} \in J$, we have

$$\begin{aligned}
|\ \mathbb{A}\mathfrak{x}_n(\mathfrak{t}) - \mathbb{A}\mathfrak{x}(\mathfrak{t})\ | \ &= \ \bigg| \mathfrak{h}(\mathfrak{t}, \mathfrak{x}_n(\mathfrak{t})) \\
&+ \ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}_n(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}_n(\theta)) d\theta\Big) d\mathfrak{s} \\
&- \ \mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) \\
&- \ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\Big) d\mathfrak{s} \bigg| \\
&\leqslant \ K_{\mathfrak{h}}(\mathfrak{t}) |\mathfrak{x}_n(\mathfrak{t}) - \mathfrak{x}(\mathfrak{t})| \\
&+ \ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\alpha-1} \bigg| \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}_n(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}_n(\theta)) d\theta\Big) \\
&- \ \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\Big) \bigg| d\mathfrak{s} \\
&\leqslant \ K_{\mathfrak{h}}^* |\mathfrak{x}_n(\mathfrak{t}) - \mathfrak{x}(\mathfrak{t})| \\
&+ \ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\alpha-1} \bigg| \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}_n(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}_n(\theta)) d\theta\Big) \\
&- \ \mathfrak{g}_1\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\Big) \bigg| d\mathfrak{s}
\end{aligned}$$

Assumptions (II) and (III) implies that:

$$\mathfrak{g}_1\Big(\mathfrak{t}, \mathfrak{x}_n, \mathfrak{I}^{\beta} \mathfrak{g}_2(\mathfrak{t}, \mathfrak{x}_n)\Big) \to \mathfrak{g}_1\Big(\mathfrak{t}, \mathfrak{x}, \mathfrak{I}^{\beta} \mathfrak{g}_2(\mathfrak{t}, \mathfrak{x})\Big) \text{ as } n \to \infty.$$

Using Lebesgue dominated convergence theorem, we have

$$\|\mathbb{A}\mathfrak{x}_n(\mathfrak{t}) - \mathbb{A}\mathfrak{x}(\mathfrak{t})\|_{\mathfrak{B}C} \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathbb{A}(\mathbb{B}_M)$ is uniformly bounded.
It is obvious since $\mathbb{A}(\mathbb{B}_M) \subset \mathbb{B}_M$ and $\mathbb{B}_M$ is bounded.
Step 3: $\mathbb{A}(\mathbb{B}_M)$ is equicontinuous on every compact subset $[0, T]$ of J, $T > 0$ and $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, T]$, $\mathfrak{t}_2 > \mathfrak{t}_1$ (without loss of generality), we get

$$\begin{aligned}
|\ \mathbb{A}\mathfrak{x}(\mathfrak{t}_2) - \mathbb{A}\mathfrak{x}(\mathfrak{t}_1)\ | \ &\leq \ \bigg| \mathfrak{h}(\mathfrak{t}_2, \mathfrak{x}(\mathfrak{t}_2)) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) d\mathfrak{s} \\
&- \ \mathfrak{h}(\mathfrak{t}_1, \mathfrak{x}(\mathfrak{t}_1)) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1 - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) d\mathfrak{s} \bigg| \\
&\leq \ |\ \mathfrak{h}(\mathfrak{t}_2, \mathfrak{x}(\mathfrak{t}_2)) - \mathfrak{h}(\mathfrak{t}_1, \mathfrak{x}(\mathfrak{t}_1))| \\
&+ \ \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) d\mathfrak{s} \\
&- \ \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1 - \mathfrak{s})^{\alpha-1} \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s} - \theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta\right) d\mathfrak{s} \bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \ |\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_2))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))+\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\Big|\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\\
&-\ \int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\Big|\\
&\le\ K_{\mathfrak{h}}(\mathfrak{t})\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\Big|\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\\
&-\ \int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\mathfrak{s})^{\alpha-1}\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)d\mathfrak{s}\Big|\\
&\le\ K_{\mathfrak{h}}(\mathfrak{t})\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\left|\mathfrak{g}_1\left(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\right)\right|d\mathfrak{s}\\
&\le\ K_{\mathfrak{h}}(\mathfrak{t})\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\left[\frac{\mathfrak{a}_1(\mathfrak{s})}{1+|\mathfrak{x}(\mathfrak{s})|}+\mathfrak{b}\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}|\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))|d\theta\right]d\mathfrak{s}\\
&\le\ K_{\mathfrak{h}}^*\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\left[\mathfrak{a}_1(\mathfrak{s})+\mathfrak{b}\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\frac{\mathfrak{a}_2(\theta)}{1+|\mathfrak{x}(\theta)|}d\theta\right]d\mathfrak{s}\\
&\le\ K_{\mathfrak{h}}^*\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\left[\mathfrak{a}_1(\mathfrak{s})+\mathfrak{b}\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{a}_2(\theta)d\theta\right]d\mathfrak{s}.
\end{aligned}
$$

Thus, for $\mathbf{a}_{\mathfrak{i}}=\sup\limits_{\mathfrak{t}\in[0,T]}\mathfrak{a}_{\mathfrak{i}},\ \mathfrak{i}=1,2$ and from the continuity of the functions $\mathfrak{a}_{\mathfrak{i}}$ we obtain

$$
\begin{aligned}
\mid\mathbb{A}\mathfrak{x}(\mathfrak{t}_2)-\mathbb{A}\mathfrak{x}(\mathfrak{t}_1)\mid\ &\le\ K_{\mathfrak{h}}^*\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{\alpha-1}\left[\mathfrak{a}_1(\mathfrak{s})+\frac{\mathfrak{b}\mathbf{a}_2}{\Gamma(\beta+1)}\mathfrak{s}^{\beta}\right]d\mathfrak{s}.\\
&\le\ K_{\mathfrak{h}}^*\mid\mathfrak{x}(\mathfrak{t}_2)-\mathfrak{x}(\mathfrak{t}_1)\mid+|\mathfrak{h}(\mathfrak{t}_2,\mathfrak{x}(\mathfrak{t}_1))-\mathfrak{h}(\mathfrak{t}_1,\mathfrak{x}(\mathfrak{t}_1))|\\
&+\ \frac{\mathbf{a}_1}{\Gamma(\alpha+1)}(\mathfrak{t}_2-\mathfrak{t}_1)^{\alpha}+\frac{\mathfrak{b}\mathbf{a}_2}{\Gamma(\alpha+\beta+1)}(\mathfrak{t}_2-\mathfrak{t}_1)^{\alpha+\beta}.
\end{aligned}
$$

Continuity of $\mathfrak{h}$ implies that

$$\mid(\mathbb{A}\mathfrak{x})(\mathfrak{t}_2)\ -\ (\mathbb{A}\mathfrak{x})(\mathfrak{t}_1)\mid\ \to 0\quad\text{as}\quad \mathfrak{t}_2\to\mathfrak{t}_1.$$

Step 4: $\mathbb{A}(\mathbb{B}_M)$ is equiconvergent.

Let $\mathfrak{t} \in J$ and $\mathfrak{x} \in \mathbb{B}_M$ then we have

$$
\begin{aligned}
|\mathbb{A}\mathfrak{x}(\mathfrak{t})| &\leq |\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \mathfrak{h}(\mathfrak{t}, 0)| + |\mathfrak{h}(\mathfrak{t}, 0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left| \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta \right) \right| d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}(\mathfrak{t})|\mathfrak{x}(\mathfrak{t})| + |\mathfrak{h}(\mathfrak{t}, 0)| + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left[\frac{\mathfrak{a}_1(\mathfrak{s})}{1+|\mathfrak{x}(\mathfrak{s})|} + \mathfrak{b} \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} |\mathfrak{g}_2(\theta, \mathfrak{x}(\theta))| d\theta \right] d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}(\mathfrak{t})|\mathfrak{x}(\mathfrak{t})| + |\mathfrak{h}(\mathfrak{t}, 0)| + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left[\mathfrak{a}_1(\mathfrak{s}) + \mathfrak{b} \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \frac{\mathfrak{a}_2(\theta)}{1+|\mathfrak{x}(\theta)|} d\theta \right] d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}(\mathfrak{t})|\mathfrak{x}(\mathfrak{t})| + |\mathfrak{h}(\mathfrak{t}, 0)| + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \mathfrak{a}_1(\mathfrak{s}) d\mathfrak{s} + \mathfrak{b} \frac{1}{\Gamma(\alpha+\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha+\beta-1} \mathfrak{a}_2(\mathfrak{s}) d\mathfrak{s}.
\end{aligned}
$$

In view of assumptions (I) and (IV), we obtain

$$|\mathbb{A}\mathfrak{x}(\mathfrak{t})| \to 0 \text{ as } \mathfrak{t} \to \infty.$$

Then $\mathbb{A}$ has a fixed point $\mathfrak{x}$ which is a solution of IVP (1.1) on J.

Step 5: Local attactivity of mild solutions. Let $\mathfrak{x}^*$ be a mild solution of IVP (1.1). Taking $\mathfrak{x} \in \mathbb{B}(\mathfrak{x}^*, 2M)$, we have

$$
\begin{aligned}
\left|\mathbb{A}\mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t})\right| &= \left|\mathbb{A}\mathfrak{x}(\mathfrak{t}) - \mathbb{A}\mathfrak{x}^*(\mathfrak{t})\right| \\
&\leq |\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \mathfrak{h}(\mathfrak{t}, \mathfrak{x}^*(t))| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left| \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta \right) \right. \\
&- \left. \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}^*(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}^*(\theta)) d\theta \right) \right| d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}(\mathfrak{t}) \,|\, \mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t}) \,| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left[\left| \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}(\theta)) d\theta \right) \right| \right. \\
&+ \left. \left| \mathfrak{g}_1\left(\mathfrak{s}, \mathfrak{x}^*(\mathfrak{s}), \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{g}_2(\theta, \mathfrak{x}^*(\theta)) d\theta \right) \right| \right] d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}^* \,|\, \mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t}) \,| \\
&+ \frac{2}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\alpha-1} \left| \mathfrak{a}_1(\mathfrak{s}) + \frac{\mathfrak{b}}{\Gamma(\beta)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\beta-1} \mathfrak{a}_2(\theta) d\theta \right| d\mathfrak{s} \\
&\leq K_{\mathfrak{h}}^* \,|\, \mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t}) \,| + 2\mathfrak{a}_1^* + 2\mathfrak{b} \frac{1}{\Gamma(\beta+\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}-\theta)^{\alpha+\beta-1} \mathfrak{a}_2(\theta) d\theta \\
&\leq 2(K_{\mathfrak{h}}^* \,|\, \mathfrak{x}(\mathfrak{t}) \,| + |\mathfrak{h}(\mathfrak{t}, 0)| + \mathfrak{a}_1^* + \mathfrak{b}\mathfrak{a}_2^*) \\
&\leq 2(K_{\mathfrak{h}}^* M + |\mathfrak{h}(t, 0)| + \mathfrak{a}_1^* + \mathfrak{b}\mathfrak{a}_2^*) \leq 2M.
\end{aligned}
$$

We have

$$\|\mathbb{A}\mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t})\|_{\mathfrak{B}C} \leq 2M.$$

Hence $\mathbb{A}$ is a continuous function such that $\mathbb{A}(\mathbb{B}(\mathfrak{x}^*, 2M)) \subset \mathbb{B}(\mathfrak{x}^*, 2M)$.
Moreover, if x is a mild solution of IVP (1.1), then

$$\left|\mathfrak{x}(\mathfrak{t}) - \mathfrak{x}^*(\mathfrak{t})\right| = \left|\mathbb{A}\mathfrak{x}(\mathfrak{t}) - \mathbb{A}\mathfrak{x}^*(\mathfrak{t})\right|$$

$$
\begin{aligned}
&\leq \ |\ \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))-\mathfrak{h}(\mathfrak{t},\mathfrak{x}^*(\mathfrak{t}))\ | \\
&+\ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\Big|\mathfrak{g}_1\Big(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\Big) \\
&-\ \mathfrak{g}_1\Big(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}^*(\theta))d\theta\Big)\Big|d\mathfrak{s} \\
&\leq\ K_{\mathfrak{h}}(\mathfrak{t})\ |\ \mathfrak{x}(\mathfrak{t})-\mathfrak{x}^*(\mathfrak{t})\ | \\
&+\ \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\Big[\Big|\mathfrak{g}_1\Big(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}(\theta))d\theta\Big)\Big| \\
&+\ \Big|\mathfrak{g}_1\Big(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\theta)^{\beta-1}\mathfrak{g}_2(\theta,\mathfrak{x}^*(\theta))d\theta\Big)\Big|\Big]d\mathfrak{s} \\
&\leq\ K_{\mathfrak{h}}^*\ |\ \mathfrak{x}(\mathfrak{t})-\mathfrak{x}^*(\mathfrak{t})\ | \\
&+\ \frac{2}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{a}_1(\mathfrak{s})d\mathfrak{s}+\frac{2\mathfrak{b}}{\Gamma(\alpha+\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\theta)^{\alpha+\beta-1}\mathfrak{a}_2(\theta)d\theta.
\end{aligned}
$$

Then

$$
|\mathfrak{x}(\mathfrak{t})-\mathfrak{x}^*(\mathfrak{t})|\leq(1-K_{\mathfrak{h}}^*)^{-1}\Big[\frac{2}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\alpha-1}\mathfrak{a}_1(\mathfrak{s})d\mathfrak{s}+\frac{2b}{\Gamma(\alpha+\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\theta)^{\alpha+\beta-1}\mathfrak{a}_2(\theta)d\theta\Big]. \tag{5.2}
$$

In view of assumption of (IV) and estimation (5.2), we get

$$
\lim_{t\to\infty}|\mathfrak{x}(\mathfrak{t})-\mathfrak{x}^*(\mathfrak{t})|=0.
$$

Then, all mild solutions of IVP (1.1) are locally attractive. □

6. Applications

As particular cases of the IVP (1.1), we have

- Taking $\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y})=\mathfrak{g}_1(\mathfrak{t},\mathfrak{x})$, we obtain the initial value problem

$$
\begin{cases}
{}^{C}\mathfrak{D}^{\alpha}\,[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))]=\mathfrak{g}_1(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))\quad \mathfrak{t}\in J,\ 1<\alpha\leq 2,\\[2ex]
\Big(\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))\Big)\Big|_{\mathfrak{t}=0}=0\quad\text{and}\quad \frac{d}{dt}[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))]_{\mathfrak{t}=0}=0
\end{cases}
$$

- Letting $\alpha\to 2,\ \beta\to 1,$ as a particular case of Theorem 1 we can deduce an existence result for the initial value problem for implicit second-order differe-integral equation

$$
\begin{cases}
\frac{d^2}{dt^2}\,(\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})))=\mathfrak{g}_1\Big(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\int_0^{\mathfrak{t}}\mathfrak{g}_2(\mathfrak{s},\mathfrak{x}(\mathfrak{s}))d\mathfrak{s}\Big)\quad \mathfrak{t}\in J,\\[2ex]
(\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})))\Big|_{\mathfrak{t}=0}=0\quad\text{and}\quad \frac{d}{dt}\Big[\mathfrak{x}(\mathfrak{t})-\mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}))\Big]_{\mathfrak{t}=0}=0
\end{cases}
$$

As particular cases we can deduce existence results for some initial value problem of second order differential equations (when $\mathfrak{h}=0$) and $\alpha\to 2,$ we get:

- Taking $\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y}) = -\lambda^2\mathfrak{x}(\mathfrak{t}),\ \lambda \in R^+$, then we obtain a second order differential equation of simple harmonic oscillator

$$\begin{cases} \frac{d^2\mathfrak{x}(\mathfrak{t})}{d\mathfrak{t}^2} = -\lambda^2\mathfrak{x}(\mathfrak{t}) \quad \mathfrak{t}\in J, \\ \\ \mathfrak{x}(0)=0 \quad \text{and} \quad \mathfrak{x}'(0)=0 \end{cases}$$

- Taking $\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y}) = \left(\frac{\mathfrak{t}^2-k}{\mathfrak{t}^2}\right)\mathfrak{x} + \mathfrak{q}(\mathfrak{x}),\ k\in R$ where $\mathfrak{q}(\mathfrak{x})$ is continuous function, then we obtain Riccati differential equation of second order

$$\begin{cases} \mathfrak{t}^2\frac{d^2\mathfrak{x}(\mathfrak{t})}{d\mathfrak{t}^2} - (\mathfrak{t}^2-k)\mathfrak{x}(\mathfrak{t}) = \mathfrak{t}^2\mathfrak{q}(\mathfrak{x}(\mathfrak{t})) \quad \mathfrak{t}\in J, \\ \\ \mathfrak{x}(0)=0 \quad \text{and} \quad \mathfrak{x}'(0)=0 \end{cases}$$

- Taking $\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y}) = -(\mathfrak{t}^2-2l\mathfrak{t}-k)\mathfrak{x} + \mathfrak{q}(\mathfrak{x}),\ k\in R$ where $\mathfrak{q}(\mathfrak{x})$ is continuous function and l is fixed, then we obtain Coulomb wave differential equation of second order

$$\begin{cases} \frac{d^2\mathfrak{x}(\mathfrak{t})}{d\mathfrak{t}^2} + (\mathfrak{t}^2-2l\mathfrak{t}-k)\mathfrak{x} = \mathfrak{q}(\mathfrak{x}(\mathfrak{t})) \quad \mathfrak{t}\in J, \\ \\ \mathfrak{x}(0)=0 \quad \text{and} \quad \mathfrak{x}'(0)=0 \end{cases}$$

- Taking $\mathfrak{g}_1(\mathfrak{t},\mathfrak{x},\mathfrak{y}) = \left(\frac{-8\pi^2 m}{\hbar^2}\right)(E\mathfrak{x} - \frac{k\mathfrak{t}^2}{2}\mathfrak{x}) + \mathfrak{q}(\mathfrak{x}),\ k\in R$ where $\mathfrak{q}(\mathfrak{x})$ is continuous function and $\hbar$ is the Planket's constant and E, k are positive real numbers, then we obtain of Schrödinger wave differential equation for simple harmonic oscillator

$$\begin{cases} \frac{d^2\mathfrak{x}(\mathfrak{t})}{d\mathfrak{t}^2} = \left(\frac{-8\pi^2 m}{\hbar^2}\right)(E\mathfrak{x}(\mathfrak{t}) - \frac{k\mathfrak{t}^2}{2}\mathfrak{x}(\mathfrak{t})) + \mathfrak{q}(\mathfrak{x}(\mathfrak{t})) \quad \mathfrak{t}\in J, \\ \\ \mathfrak{x}(0)=0 \quad \text{and} \quad \mathfrak{x}'(0)=0. \end{cases}$$

7. Conclusions

Sufficient conditions for the existence of solutions for a class of neutral integro-differential equations of fractional order (1.1) are discussed which involved many key functional differential equations that appear in applications of nonlinear analysis. Also, some types of Ulam stability for this class of implicit fractional differential equation are established. Some applications and particular cases are presented. Finally, the existence of at least one mild solution for this class of equations on an infinite interval by applying Schauder fixed point theorem and the local attractivity of solutions are proved.

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Conflict of interest

The authors declare that they have no competing interests.

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