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Research article

Approximate mixed type quadratic-cubic functional equation

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Abstract: In this paper, we investigate the generalized Hyers-Ulam stability of the following mixed type quadratic-cubic functional equation

$$2f(2x+y)+2f(2x-y)=4f(x+y)+4f(x-y)+4f(2x)+f(2y)-8f(x)-8f(y)$$

in non-Archimedean (n,β)-normed spaces.

Keywords: generalized Hyers-Ulam stability; mixed type quadratic-cubic functional equation; (n,β)-normed spaces; non-Archimedean (n,β)-normed spaces
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1. Introduction

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms and it was affirmatively answered for Banach spaces by Hyers [17]. Hyers' theorem was generalized by Aoki [1] for approximate additive mappings and by Rassias [37] for approximate linear mappings by considering an unbounded Cauchy difference. Furthermore, a generalization of the Rassias' theorem was obtained by Găvruţă [12] by replacing the unbounded Cauchy difference by a general control function. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2–4, 8, 18, 20, 21, 38, 41] and references therein). The stability problems in non-Archimedean Banach spaces were studied in [13, 14, 28, 30–32].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional Eq

(1.1) was proved by Skof [42] for mappings from a normed space to a Banach space. Cholewa [5] noticed that Skof's theorem remains true if the domain is replaced by an Abelian group. In 1992, Czerwik [7] gave a generalization of the Skof–Cholewa's result. Later, Lee et al. [26] proved Hyers-Ulam-Rassias stability of quadratic functional Eq (1.1) in fuzzy Banach spaces.

In 2008, Ravi et al. [39] introduced the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x+y) + 2f(x-y) + 4f(x) - 2f(y) \tag{1.2}$$

and solved the generalized Hyers-Ulam stability of this Eq (1.2). Jun and Kim [19] considered the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers-Ulam stability of the functional Eq (1.3) in Banach spaces. The functional Eq (1.3) and its pexiderized version

$$f_1(2x+y) + f_2(2x-y) = f_3(x+y) + f_4(x-y) + f_5(x)$$

were studied by Sahoo [40] on commutative groups using an elementary method quite different from Jun and Kim [19]. The function $f(x) = cx^3$ satisfies the functional Eq (1.3), which is thus called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

In 2010, Wang and Liu [44] considered the following mixed type functional equation

$$2f(2x+y) + 2f(2x-y) = 4f(x+y) + 4f(x-y) + 4f(2x) + f(2y) - 8f(x) - 8f(y). \tag{1.4}$$

It is easy to show that the function $f(x) = ax^2 + bx^3$ is a solution of the functional Eq (1.4), where a, b are arbitrary constants. They established the general solution of the functional Eq (1.4), and then proved the generalized Hyers-Ulam stability of the Eq (1.4) in quasi-β-normed spaces.

In 2011, Park [34] investigated the approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces. Later, the stability problems of additive functional inequalities, approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces were also studied [6, 36], respectively. In 2012, Xu and Rassias [48] determined the generalized Hyers-Ulam stability of the mixed additive-cubic functional equation in n-Banach spaces. In 2013, Xu [47] investigated approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces.

Kim and Park [24] proved the generalized Hyers-Ulam stability of additive functional inequalities in non-Archimedean 2-normed spaces. Park et al. [35] proved the generalized Hyers-Ulam stability of the system of additive-cubic-quartic functional equations with constant coefficients in non-Archimedean 2-normed spaces. In 2015, Yang et al. [49] proved the generalized Hyers-Ulam stability of the Cauchy functional equation and the Jensen functional equation in non-Archimedean (n, β)-normed spaces and that of the Pexiderized Cauchy functional equation in (n, β)-normed spaces.

The main purpose of this paper is to establish the generalized Hyers-Ulam stability of the mixed type quadratic-cubic functional Eq (1.4) in non-Archimedean (n, β)-normed spaces.

Throughout this paper, let $\mathbb{N}$ denote the set of positive integers and $i, j, m, n \in \mathbb{N}$, and let $n \geq 2$ be fixed.

2. Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [9,10] in the middle of the 1960s. Then the concept of 2-Banach spaces was introduced by Gähler [11] and White [45,46]. A systematic development of linear n-normed spaces is due to Kim and Cho [25], Malceski [27], Misiak [29] and Gunawan and Mashadi [15]. Following [48,49], we recall some basic facts concerning (n,β)-normed space and some preliminary results.

Definition 2.1. (cf. [49]) *Let $n \in \mathbb{N}$, and let X be a real linear space with* $\dim X \geq n$ *and* $0 < \beta \leq 1$, *let* $\|\cdot,\ldots,\cdot\|_\beta : X^n \to \mathbb{R}$ *be a function satisfying the following properties:*
(N1) $\|x_1, x_2, \ldots, x_n\|_\beta = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*
(N2) $\|x_1, x_2, \ldots, x_n\|_\beta$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$*;*
(N3) $\|\alpha x_1, x_2, \ldots, x_n\|_\beta = |\alpha|^\beta \|x_1, x_2, \ldots, x_n\|_\beta$*;*
(N4) $\|x+y, x_2, \ldots, x_n\|_\beta \leq \|x, x_2, \ldots, x_n\|_\beta + \|y, x_2, \ldots, x_n\|_\beta$ *for all* $x, y, x_1, x_2, \ldots, x_n \in X$ *and* $\alpha \in \mathbb{R}$.
Then the function $\|\cdot,\ldots,\cdot\|_\beta$ *is called an* (n,β)*-norm on* X *and the pair* $(X, \|\cdot,\ldots,\cdot\|_\beta)$ *is called a linear* (n,β)*-normed space or an* (n,β)*-normed space.*

Note that the concept of an (n,β)-normed space is a generalization of an n-normed space ($\beta = 1$) and of a β-normed space ($n = 1$). For some examples of n-normed space, we can refer to [48,49].

Definition 2.2. (cf. [49]) *A sequence* $\{x_k\}$ *in an* (n,β)*-normed space* X *is called a convergent sequence if there exists* $x \in X$ *such that*

$$\lim_{k\to\infty} \|x_k - x, y_2, \ldots, y_n\|_\beta = 0$$

for all $y_2, \ldots, y_n \in X$. *In this case, we call that* $\{x_k\}$ *converges to* x *or that* x *is the limit of* $\{x_k\}$*, write* $x_k \to x$ *as* $k \to \infty$ *or* $\lim_{k\to\infty} x_k = x$.

Definition 2.3. (cf. [49]) *A sequence* $\{x_k\}$ *in an* (n,β)*-normed space* X *is called a Cauchy sequence if*

$$\lim_{k,m\to\infty} \|x_k - x_m, y_2, \ldots, y_n\|_\beta = 0$$

for all $y_2, \ldots, y_n \in X$. *A linear* (n,β)*-normed space in which every Cauchy sequence is convergent is called a complete* (n,β)*-normed space.*

Remark 2.1. (cf. [49]) *Let* $(X, \|\cdot,\ldots,\cdot\|_\beta)$ *be a linear* (n,β)*-normed space,* $0 < \beta \leq 1$. *One can show that conditions* (N2) *and* (N4) *in Definition 2.1 imply that*

$$\left|\|x, y_2, \ldots, y_n\|_\beta - \|y, y_2, \ldots, y_n\|_\beta\right| \leq \|x - y, y_2, \ldots, y_n\|_\beta$$

for all $x, y, y_2, \ldots, y_n \in X$.

Lemma 2.1. (cf. [49]). *Let* $(X, \|\cdot,\ldots,\cdot\|_\beta)$ *be a linear* (n,β)*-normed space,* $n \geq 2$, $0 < \beta \leq 1$. *If* $x \in X$ *and* $\|x, y_2, \ldots, y_n\|_\beta = 0$ *for all* $y_2, \ldots, y_n \in X$, *then* $x = 0$.

Lemma 2.2. (cf. [48,49]). *Let* $(X, \|\cdot,\ldots,\cdot\|_\beta)$ *be a linear* (n,β)*-normed space,* $n \geq 2$, $0 < \beta \leq 1$. *For a convergent sequence* $\{x_k\}$ *in a linear* (n,β)*-normed space* X,

$$\lim_{k\to\infty} \|x_k, y_2, \ldots, y_n\|_\beta = \|\lim_{k\to\infty} x_k, y_2, \ldots, y_n\|_\beta$$

for all $y_2, \ldots, y_n \in X$.

In 1897, Hensel [16] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [22, 23, 33].

Definition 2.4. (cf. [30]) *By a non-Archimedean field we mean a field* $\mathbb{K}$ *equipped with a function (valuation)* $|\cdot| : \mathbb{K} \to [0, \infty)$ *such that for* $r, s \in \mathbb{K}$*, the following conditions hold:*
(1) $|r| = 0$ *if and only if* $r = 0$*;*
(2) $|rs| = |r||s|$*;*
(3) $|r + s| \leq \max\{|r|, |s|\}$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the function $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$ (i.e., the function $|\cdot|$ is called the trivial valuation if $|r| = 1, \forall r \in \mathbb{K}, r \neq 0$, and $|0| = 0$).

Definition 2.5. (cf. [30]) *Let X be a vector space over a scalar field* $\mathbb{K}$ *with a non-Archimedean non-trivial valuation* $|\cdot|$*. A function* $\|\cdot\| : X \to \mathbb{R}$ *is called a non-Archimedean norm (valuation) if it satisfies the following conditions:*
(i) $\|x\| = 0$ *if and only if* $x = 0$*;*
(ii) *For any* $r \in \mathbb{K}$ *and* $x \in X$*,* $\|rx\| = |r|\|x\|$*;*
(iii) *For all* $x, y \in X$*,* $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ *(the strong triangle inequality).*
Then $(X, \|\cdot\|)$ *is called a non-Archimedean normed space.*

Now, we give the definition of a non-Archimedean (n,β)-normed space which has been introduced in [49].

Definition 2.6. (cf. [49]) *Let X be a real vector space with* $\dim X \geq n$ *over a scalar field* $\mathbb{K}$ *with a non-Archimedean non-trivial valuation* $|\cdot|$*, where n is a positive integer and* β *is a constant with* $0 < \beta \leq 1$*. A real-valued function* $\|\cdot, \ldots, \cdot\|_\beta : X^n \to \mathbb{R}$ *is called a non-Archimedean* (n,β)*-norm on X if the following conditions hold:*
(N1′) $\|x_1, x_2, \ldots, x_n\|_\beta = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*
(N2′) $\|x_1, x_2, \ldots, x_n\|_\beta$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$*;*
(N3′) $\|\alpha x_1, x_2, \ldots, x_n\|_\beta = |\alpha|^\beta \|x_1, x_2, \ldots, x_n\|_\beta$*;*
(N4′) $\|x + y, x_2, \ldots, x_n\|_\beta \leq \max\{\|x, x_2, \ldots, x_n\|_\beta, \|y, x_2, \ldots, x_n\|_\beta\}$ *for all* $x, y, x_1, x_2, \ldots, x_n \in X$ *and* $\alpha \in \mathbb{K}$*. Then* $(X, \|\cdot, \ldots, \cdot\|_\beta)$ *is called a non-Archimedean* (n,β)*-normed space.*

It follows from the preceding definition that the non-Archimedean (n,β)-normed space is a non-Archimedean n-normed space if $\beta = 1$, and a non-Archimedean β-normed space if $n = 1$, respectively.

Remark 2.2. (cf. [49]) *A sequence* $\{x_k\}$ *in a non-Archimedean* (n,β)*-normed space X is a Cauchy sequence if and only if* $\{x_{k+1} - x_k\}$ *converges to zero.*

3. Main results

In this section, we will assume that X is an n-normed space vector space and Y is a complete non-Archimedean (n,β)-normed space, where $n \geq 2$ and $0 < \beta \leq 1$. We prove the generalized Hyers-Ulam stability of the mixed type quadratic-cubic functional Eq (1.4) in non-Archimedean (n,β)-normed spaces. For the sake of convenience, given mapping $f : X \to Y$, we define the difference operator $D_f(x, y) : X \to Y$ of the functional Eq (1.4) by

$$D_f(x, y) = 2f(2x + y) + 2f(2x - y) - 4f(x + y) - 4f(x - y) - 4f(2x) - f(2y) + 8f(x) + 8f(y)$$

for all $x, y \in X$.

Before proceeding to the proof of the main results, we first introduce the following lemmas which will be used in this paper.

Lemma 3.1. (cf. [44]). *Let V and W be real vector spaces. If an even mapping $f : V \to W$ satisfies (1.4), then f is quadratic.*

Lemma 3.2. (cf. [44]). *Let V and W be real vector spaces. If an odd mapping $f : V \to W$ satisfies (1.4), then f is cubic.*

Theorem 3.1. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|4|^{m\beta}} = 0 \tag{3.1}$$

for all $x, y, u_2, \ldots, u_n \in X$. The limit

$$\lim_{m\to\infty} \max\left\{|4|^{-j\beta}\varphi(0, 2^j x, u_2, \ldots, u_n) : 0 \le j < m\right\} \tag{3.2}$$

denoted by $\tilde{\varphi}_Q(x, u_2, \ldots, u_n)$, exists for all $x, u_2, \ldots, u_n \in X$. Suppose that $f : X \to Y$ is an even function satisfying $f(0) = 0$ and

$$\|D_f(x, y), u_2, \ldots, u_n\|_\beta \le \varphi(x, y, u_2, \ldots, u_n) \tag{3.3}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x), u_2, \ldots, u_n\|_\beta \le \frac{1}{|4|^\beta}\,\tilde{\varphi}_Q(x, u_2, \ldots, u_n) \tag{3.4}$$

for all $x, u_2, \ldots, u_n \in X$, and if, in addition,

$$\lim_{i\to\infty}\lim_{m\to\infty} \max\left\{|4|^{-j\beta}\varphi(0, 2^j x, u_2, \ldots, u_n) : i \le j < m + i\right\} = 0$$

then Q is the unique quadratic function satisfying (3.4).

Proof. Putting $x = 0$ in (3.3), and by the evenness of f, we get

$$\|f(2y) - 4f(y), u_2, \ldots, u_n\|_\beta \le \varphi(0, y, u_2, \ldots, u_n) \tag{3.5}$$

for all $y, u_2, \ldots, u_n \in X$. If we replace y by x in (3.5) and divide both sides of (3.5) by $|4|^\beta$, then we have

$$\left\|\frac{f(2x)}{4} - f(x), u_2, \ldots, u_n\right\|_\beta \le |4|^{-\beta}\,\varphi(0, x, u_2, \ldots, u_n) \tag{3.6}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing x by $2^m x$ in (3.6) and dividing both sides of (3.6) by $|4|^{m\beta}$, we obtain

$$\left\|\frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m}, u_2, \ldots, u_n\right\|_\beta \le |4|^{-m\beta}|4|^{-\beta}\,\varphi(0, 2^m x, u_2, \ldots, u_n) \tag{3.7}$$

for all $x, u_2, \dots, u_n \in X$ and $m \in \mathbb{N}$. Taking the limit as $m \to \infty$ and using (3.1), we have

$$\lim_{m\to\infty} \left\| \frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m}, u_2, \dots, u_n \right\|_\beta = 0 \tag{3.8}$$

for all $x, u_2, \dots, u_n \in X$. By Remark 2.2, we know that the sequence $\{\frac{f(2^m x)}{4^m}\}$ is Cauchy. Since Y is a complete space, we conclude that $\{\frac{f(2^m x)}{4^m}\}$ is convergent. So we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{m\to\infty} \frac{f(2^m x)}{4^m}$$

for all $x \in X$. It follows from (3.1) and (3.3) that

$$\|D_Q(x,y), u_2, \dots, u_n\|_\beta = \lim_{m\to\infty} \frac{1}{|4|^{m\beta}} \|D_f(2^m x, 2^m y), u_2, \dots, u_n\|_\beta \le \lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \dots, u_n)}{|4|^{m\beta}} = 0$$

for all $x, y, u_2, \dots, u_n \in X$. By Lemma 2.1, we get $D_Q(x,y) = 0$ for all $x, y \in X$. Therefore the function $Q : X \to Y$ satisfies (1.4). Since f is an even function, Q is an even function. By Lemma 3.1 (see also [44, Corollary 2.2]), Q is quadratic. Then Q satisfies

$$Q(2x + y) + Q(2x - y) = 2Q(x + y) + 2Q(x - y) + 4Q(x) - 2Q(y) \tag{3.9}$$

for all $x, y \in X$. Letting $x = 0$ in (3.9), and by the evenness of Q, we get $Q(2x) = 4Q(x)$, so $Q(2^m x) = 4^m Q(x)$.

Replacing x by $2x$ in (3.6) and dividing both sides by $|4|^\beta$, we obtain

$$\left\| \frac{f(2^2 x)}{4^2} - \frac{f(2x)}{4}, u_2, \dots, u_n \right\|_\beta \le |4|^{-2\beta} \varphi(0, 2x, u_2, \dots, u_n) \tag{3.10}$$

for all $x, u_2, \dots, u_n \in X$. It follows from (3.6) and (3.10) that

$$\left\| f(x) - \frac{f(2^2 x)}{4^2}, u_2, \dots, u_n \right\|_\beta \le \max\{|4|^{-\beta} \varphi(0, x, u_2, \dots, u_n), |4|^{-2\beta} \varphi(0, 2x, u_2, \dots, u_n)\}$$

for all $x, u_2, \dots, u_n \in X$.

By induction on m, we get

$$\left\| f(x) - \frac{f(2^m x)}{4^m}, u_2, \dots, u_n \right\|_\beta \le \max\left\{ \frac{\varphi(0, 2^i x, u_2, \dots, u_n)}{|4|^{(i+1)\beta}} : 0 \le i < m \right\} \tag{3.11}$$

for all $x, u_2, \dots, u_n \in X$. Replacing x by $2x$ in (3.11) and dividing both sides by $|4|^\beta$, we get

$$\left\| \frac{f(2x)}{4} - \frac{f(2^{m+1} x)}{4^{m+1}}, u_2, \dots, u_n \right\|_\beta \le \max\left\{ \frac{\varphi(0, 2^{i+1} x, u_2, \dots, u_n)}{|4|^{(i+2)\beta}} : 0 \le i < m \right\} \tag{3.12}$$

for all $x, u_2, \dots, u_n \in X$. By (3.6) and (3.12), we obtain

$$\left\| f(x) - \frac{f(2^{m+1} x)}{4^{m+1}}, u_2, \dots, u_n \right\|_\beta \le \max\left\{ \frac{\varphi(0, x, u_2, \dots, u_n)}{|4|^\beta}, \frac{\varphi(0, 2^{i+1} x, u_2, \dots, u_n)}{|4|^{(i+2)\beta}} : 0 \le i < m \right\}$$

$$= \max\left\{\frac{\varphi(0, 2^i x, u_2, \ldots, u_n)}{|4|^{(i+1)\beta}} : 0 \le i < m+1\right\}$$

for all $x, u_2, \ldots, u_n \in X$ and $m \in \mathbb{N}$. This completes the proof of (3.11). By taking the limit as $m \to \infty$ in (3.11) and using (3.2), one obtains (3.4).

Now we proceed to prove the uniqueness property of Q. Let Q' be another quadratic function satisfying (3.4). Since

$$\begin{aligned}\lim_{i\to\infty} \frac{\tilde{\varphi}_Q(2^i x, u_2, \ldots, u_n)}{|4|^{i\beta}} &= \lim_{i\to\infty}\lim_{m\to\infty} \frac{1}{|4|^{i\beta}} \max\left\{\frac{\varphi(0, 2^{i+j} x, u_2, \ldots, u_n)}{|4|^{j\beta}} : 0 \le j < m\right\} \\ &= \lim_{i\to\infty}\lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^{j} x, u_2, \ldots, u_n)}{|4|^{j\beta}} : i \le j < m+i\right\} \qquad (3.13)\end{aligned}$$

for all $x, u_2, \ldots, u_n \in X$. So we have

$$\begin{aligned}\|Q(x) - Q'(x), u_2, \ldots, u_n\|_\beta &= \lim_{i\to\infty} |4|^{-i\beta}\, \|Q(2^i x) - Q'(2^i x), u_2, \ldots, u_n\|_\beta \\ &\le \lim_{i\to\infty} |4|^{-i\beta} \max\left\{\|Q(2^i x) - f(2^i x), u_2, \ldots, u_n\|_\beta, \|f(2^i x) - Q'(2^i x), u_2, \ldots, u_n\|_\beta\right\} \\ &\le \frac{1}{|4|^\beta} \lim_{i\to\infty} |4|^{-i\beta}\, \tilde{\varphi}_Q(2^i x, u_2, \ldots, u_n) = 0\end{aligned}$$

for all $x, u_2, \ldots, u_n \in X$. If

$$\lim_{i\to\infty}\lim_{m\to\infty} \max\left\{|4|^{-j\beta}\varphi(0, 2^j x, u_2, \ldots, u_n) : i \le j < m+i\right\} = 0,$$

then $\|Q(x) - Q'(x), u_2, \ldots, u_n\|_\beta = 0$. By Lemma 2.1, $Q = Q'$, and the proof is complete. □

Corollary 3.1. *Let $\rho : [0, \infty) \to [0, \infty)$ be a function satisfying*
(i) $\rho(|2|t) \le \rho(|2|)\rho(t)$ for all $t \ge 0$,
(ii) $\rho(|2|) \le |2|^{r\beta}$, where r is a fixed real number in $r \in [2, \infty)$.
Let $\delta > 0$, X be an n-normed space with norm $\|\cdot, \ldots, \cdot\|$, let $f : X \to Y$ be an even function with $f(0) = 0$ and satisfying the inequality

$$\|D_f(x, y), u_2, \ldots, u_n\|_\beta \le \delta\left[\rho(\|x, u_2, \ldots, u_n\|) + \rho(\|y, u_2, \ldots, u_n\|)\right] \qquad (3.14)$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x), u_2, \ldots, u_n\|_\beta \le \frac{\delta}{|4|^\beta} \rho(\|x, u_2, \ldots, u_n\|) \qquad (3.15)$$

for all $x, u_2, \ldots, u_n \in X$.

Proof. Define $\varphi : X^{n+1} \to [0, \infty)$ by

$$\varphi(x, y, u_2, \ldots, u_n) := \delta[\rho(\|x, u_2, \ldots, u_n\|) + \rho(\|y, u_2, \ldots, u_n\|)].$$

Since $|4|^{-\beta}\rho(|2|) < |2|^{(r-2)\beta} \le 1$, we have

$$\lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|4|^{m\beta}} \le \lim_{m\to\infty} \left(\frac{\rho(|2|)}{|4|^\beta}\right)^m \varphi(x, y, u_2, \ldots, u_n) = 0$$

for all $x, y, u_2, \ldots, u_n \in X$. Also

$$\tilde{\varphi}_Q(x, u_2, \ldots, u_n) = \lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|4|^{j\beta}} : 0 \le j < m\right\} = \varphi(0, x, u_2, \ldots, u_n)$$

and

$$\lim_{i\to\infty} \lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|4|^{j\beta}} : i \le j < m+i\right\} = \lim_{i\to\infty} \frac{\varphi(0, 2^i x, u_2, \ldots, u_n)}{|4|^{i\beta}} = 0$$

for all $x, u_2, \ldots, u_n \in X$. Hence the result follows by Theorem 3.1. □

Theorem 3.2. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|8|^{m\beta}} = 0 \tag{3.16}$$

for all $x, y, u_2, \ldots, u_n \in X$. The limit

$$\lim_{m\to\infty} \max\left\{|8|^{-j\beta}\varphi(0, 2^j x, u_2, \ldots, u_n) : 0 \le j < m\right\} \tag{3.17}$$

denoted by $\tilde{\varphi}_C(x, u_2, \ldots, u_n)$, exists for all $x, u_2, \ldots, u_n \in X$. Suppose that $f : X \to Y$ is an odd function satisfying

$$\|D_f(x, y), u_2, \ldots, u_n\|_\beta \le \varphi(x, y, u_2, \ldots, u_n) \tag{3.18}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a cubic function $C : X \to Y$ such that

$$\|f(x) - C(x), u_2, \ldots, u_n\|_\beta \le \frac{1}{|8|^\beta} \tilde{\varphi}_C(x, u_2, \ldots, u_n) \tag{3.19}$$

for all $x, u_2, \ldots, u_n \in X$. And if, in addition,

$$\lim_{i\to\infty} \lim_{m\to\infty} \max\left\{|8|^{-j\beta}\varphi(0, 2^j x, u_2, \ldots, u_n) : i \le j < m+i\right\} = 0$$

then C is the unique cubic function satisfying (3.19).

Proof. Putting $x = 0$ in (3.18), and by the oddness of f, we get

$$\|f(2y) - 8f(y), u_2, \ldots, u_n\|_\beta \le \varphi(0, y, u_2, \ldots, u_n) \tag{3.20}$$

for all $y, u_2, \ldots, u_n \in X$. If we replace y by x in (3.20) and divide both sides of (3.20) by $|8|^\beta$, then we have

$$\left\|\frac{f(2x)}{8} - f(x), u_2, \ldots, u_n\right\|_\beta \le |8|^{-\beta}\varphi(0, x, u_2, \ldots, u_n) \tag{3.21}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing x by $2^j x$ in (3.21) and dividing both sides of (3.21) by $|8|^{m\beta}$, we obtain

$$\left\|\frac{f(2^{m+1}x)}{8^{m+1}} - \frac{f(2^m x)}{8^m}, u_2, \ldots, u_n\right\|_\beta \le |8|^{-m\beta}\, |8|^{-\beta}\, \varphi(0, 2^m x, u_2, \ldots, u_n) \tag{3.22}$$

for all $x, u_2, \ldots, u_n \in X$ and $m \in \mathbb{N}$. Taking the limit as $m \to \infty$ and using (3.16), we have

$$\lim_{m\to\infty} \left\| \frac{f(2^{m+1}x)}{8^{m+1}} - \frac{f(2^m x)}{8^m}, u_2, \ldots, u_n \right\|_\beta = 0 \tag{3.23}$$

for all $x, u_2, \ldots, u_n \in X$. By Remark 2.2, we know that the sequence $\{\frac{f(2^m x)}{8^m}\}$ is Cauchy. Since Y is a complete space, we conclude that $\{\frac{f(2^m x)}{8^m}\}$ is convergent. So we can define the function $C : X \to Y$ by

$$C(x) = \lim_{m\to\infty} \frac{f(2^m x)}{8^m}$$

for all $x \in X$.

Similar to the proof of Theorem 3.1, using induction one can show that

$$\left\| f(x) - \frac{f(2^m x)}{8^m}, u_2, \ldots, u_n \right\|_\beta \leq \max \left\{ \frac{\varphi(0, 2^i x, u_2, \ldots, u_n)}{|8|^{(i+1)\beta}} : 0 \leq i < m \right\} \tag{3.24}$$

for all $x, u_2, \ldots, u_n \in X$ and $m \in \mathbb{N}$. By taking m to approach infinity in (3.24) and using (3.17), one obtains (3.19). It follows from (3.16) and (3.18) that

$$\|D_C(x,y), u_2, \ldots, u_n\|_\beta = \lim_{m\to\infty} \frac{1}{|8|^{m\beta}} \|D_f(2^m x, 2^m y), u_2, \ldots, u_n\|_\beta \leq \lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|8|^{m\beta}} = 0$$

for all $x, y, u_2, \ldots, u_n \in X$. By Lemma 2.1, we get $D_C(x,y) = 0$ for all $x, y \in X$. Therefore the function $C : X \to Y$ satisfies (1.4). Since f is an odd function, C is an odd function. By Lemma 3.2 (see also [44, Corollary 2.2]), C is cubic. Then C satisfies

$$C(2x + y) + C(2x - y) = 2\,C(x + y) + 2\,C(x - y) + 12\,C(x) \tag{3.25}$$

for all $x, y \in X$. Letting $x = 0$ in (3.25), and by the oddness of C, we get $C(2x) = 8C(x)$, so $C(2^m x) = 8^m C(x)$. Let

$$\lim_{i\to\infty} \lim_{m\to\infty} \max \left\{ |8|^{-j\beta} \varphi(0, 2^j x, u_2, \ldots, u_n) : i \leq j < m + i \right\} = 0,$$

for all $x, u_2, \ldots, u_n \in X$ and let C' be another cubic function satisfying (3.19). Then

$$\begin{aligned}
\|C(x) - C'(x), u_2, \ldots, u_n\|_\beta &= \lim_{i\to\infty} |8|^{-i\beta} \|C(2^i x) - C'(2^i x), u_2, \ldots, u_n\|_\beta \\
&\leq \lim_{i\to\infty} |8|^{-i\beta} \max \left\{ \|C(2^i x) - f(2^i x), u_2, \ldots, u_n\|_\beta, \|f(2^i x) - C'(2^i x), u_2, \ldots, u_n\|_\beta \right\} \\
&\leq \frac{1}{|8|^\beta} \lim_{i\to\infty} \lim_{m\to\infty} \max \left\{ \frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|8|^{j\beta}} : i \leq j < m + i \right\} = 0
\end{aligned}$$

for all $x, u_2, \ldots, u_n \in X$. Therefore $\|C(x) - C'(x), u_2, \ldots, u_n\|_\beta = 0$. By Lemma 2.1, we have $C = C'$. This completes the proof of the uniqueness of C. □

Corollary 3.2. *Let* $\rho : [0, \infty) \to [0, \infty)$ *be a function satisfying*
(i) $\rho(|2|t) \leq \rho(|2|)\rho(t)$ *for all* $t \geq 0$,
(ii) $\rho(|2|) \leq |2|^{\lambda\beta}$, *where* λ *a fixed real number in* $\lambda \in [3, \infty)$.

Let $\delta > 0$, X be an n-normed space with norm $\|\cdot, \ldots, \cdot\|$, let $f : X \to Y$ be an odd function satisfying the inequality

$$\|D_f(x, y), u_2, \ldots, u_n\|_\beta \leq \delta[\rho(\|x, u_2, \ldots, u_n\|) + \rho(\|y, u_2, \ldots, u_n\|)] \tag{3.26}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x), u_2, \ldots, u_n\|_\beta \leq \frac{\delta}{|8|^\beta} \rho(\|x, u_2, \ldots, u_n\|) \tag{3.27}$$

for all $x, u_2, \ldots, u_n \in X$.

Proof. The proof is similar to the proof of Corollary 3.1 and the result follows from Theorem 3.2. □

Combining Theorems 3.1 and 3.2, we obtain the following theorem.

Theorem 3.3. *Let $\varphi : X^{n+1} \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|4|^{m\beta}} = \lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y, u_2, \ldots, u_n)}{|8|^{m\beta}} = 0 \tag{3.28}$$

for all $x, y, u_2, \ldots, u_n \in X$. The limit

$$\lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|4|^{j\beta}} : 0 \leq j < m\right\} \tag{3.29}$$

denoted by $\tilde{\varphi}_Q(x, u_2, \ldots, u_n)$, and

$$\lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|8|^{j\beta}} : 0 \leq j < m\right\} \tag{3.30}$$

denoted by $\tilde{\varphi}_C(x, u_2, \ldots, u_n)$, exists for all $x, u_2, \ldots, u_n \in X$. Suppose that $f : X \to Y$ is a function satisfying $f(0) = 0$ and

$$\|D_f(x, y), u_2, \ldots, u_n\|_\beta \leq \varphi(x, y, u_2, \ldots, u_n) \tag{3.31}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exist a quadratic function $Q : X \to Y$ and a cubic function $C : X \to Y$ such that

$$\begin{aligned} \|f(x) - Q(x) - C(x), u_2, \ldots, u_n\|_\beta \leq & \frac{1}{|8|^\beta} \max\Big\{ \max\{\tilde{\varphi}_Q(x, u_2, \ldots, u_n), \tilde{\varphi}_Q(-x, u_2, \ldots, u_n)\}, \\ & \frac{1}{|2|^\beta} \max\{\tilde{\varphi}_C(x, u_2, \ldots, u_n), \tilde{\varphi}_C(-x, u_2, \ldots, u_n)\}\Big\} \end{aligned} \tag{3.32}$$

for all $x, u_2, \ldots, u_n \in X$, and if, in addition,

$$\begin{aligned} &\lim_{i\to\infty} \lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|4|^{j\beta}} : i \leq j < m + i\right\} \\ &= \lim_{i\to\infty} \lim_{m\to\infty} \max\left\{\frac{\varphi(0, 2^j x, u_2, \ldots, u_n)}{|8|^{j\beta}} : i \leq j < m + i\right\} = 0 \end{aligned}$$

then Q is the unique quadratic function and C is the unique cubic function.

Proof. Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ for all $x \in X$, and

$$\|D_{f_e}(x,y), u_2, \ldots, u_n\|_\beta \le \frac{1}{|2|^\beta} \max\{\varphi(x,y,u_2,\ldots,u_n), \varphi(-x,-y,u_2,\ldots,u_n)\}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Theorem 3.1, then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f_e(x) - Q(x), u_2, \ldots, u_n\|_\beta \le \frac{1}{|2^3|^\beta} \max\{\tilde{\varphi}_Q(x,u_2,\ldots,u_n), \tilde{\varphi}_Q(-x,u_2,\ldots,u_n)\} \tag{3.33}$$

for all $x, u_2, \ldots, u_n \in X$.

Let $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\|D_{f_o}(x,y), u_2, \ldots, u_n\|_\beta \le \frac{1}{|2|^\beta} \max\{\varphi(x,y,u_2,\ldots,u_n), \varphi(-x,-y,u_2,\ldots,u_n)\}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Theorem 3.2, then there exists a unique cubic function $C : X \to Y$ satisfying

$$\|f_o(x) - C(x), u_2, \ldots, u_n\|_\beta \le \frac{1}{|2^4|^\beta} \max\{\tilde{\varphi}_C(x,u_2,\ldots,u_n), \tilde{\varphi}_C(-x,u_2,\ldots,u_n)\} \tag{3.34}$$

for all $x, u_2, \ldots, u_n \in X$. Hence, (3.32) follows from (3.33) and (3.34). This completes the proof of the theorem. □

From now on, assume that $|2| \neq 1$, X is a non-Archimedean (n,β_1)-normed space and Y is a complete non-Archimedean (n,β)-normed space, where $n \ge 2$ and $0 < \beta, \beta_1 \le 1$. We can formulate our results as follows.

Theorem 3.4. *Let $\theta \in [0,\infty)$, $p, q \in (0,\infty)$ with $(p+q)\beta_1 > 2\beta$. Suppose that $f : X \to Y$ is an even function satisfying $f(0) = 0$ and*

$$\begin{aligned} \|D_f(x,y), u_2, \ldots, u_n\|_\beta \le \theta(&\|x,u_2,\ldots,u_n\|_{\beta_1}^p \|y,u_2,\ldots,u_n\|_{\beta_1}^q \\ &+ \|x,u_2,\ldots,u_n\|_{\beta_1}^{p+q} + \|y,u_2,\ldots,u_n\|_{\beta_1}^{p+q}) \end{aligned} \tag{3.35}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x), u_2, \ldots, u_n\|_\beta \le \frac{\theta}{|4|^\beta} \|x,u_2,\ldots,u_n\|_{\beta_1}^{p+q} \tag{3.36}$$

for all $x, u_2, \ldots, u_n \in X$.

Proof. Putting $x = 0$ in (3.35), and by the evenness of f, we get

$$\|f(2y) - 4f(y), u_2, \ldots, u_n\|_\beta \le \theta \|y,u_2,\ldots,u_n\|_{\beta_1}^{p+q} \tag{3.37}$$

for all $y, u_2, \ldots, u_n \in X$. If we replace y by x in (3.37) and divide both sides of (3.37) by $|4|^\beta$, then we have

$$\left\|\frac{f(2x)}{4} - f(x), u_2, \ldots, u_n\right\|_\beta \le \theta |4|^{-\beta} \|x,u_2,\ldots,u_n\|_{\beta_1}^{p+q} \tag{3.38}$$

for all $x, u_2, \dots, u_n \in X$. Replacing x by $2^m x$ in (3.38) and dividing both sides of (3.38) by $|4|^{m\beta}$, we obtain

$$\begin{aligned}\left\|\frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m}, u_2, \dots, u_n\right\|_\beta &\le \theta\,|4|^{-m\beta}\,|4|^{-\beta}\,|2^{m(p+q)\beta_1}|\,\|x, u_2, \dots, u_n\|_{\beta_1}^{p+q} \\ &= \theta\,|4|^{-\beta}\,|2^{(p+q)\beta_1-2\beta}|^m\,\|x, u_2, \dots, u_n\|_{\beta_1}^{p+q} \end{aligned} \tag{3.39}$$

for all $x, u_2, \dots, u_n \in X$ and $m \in \mathbb{N}$. Since $(p+q)\beta_1 > 2\beta$ and $|2| \neq 1$, we have

$$\lim_{m\to\infty}\left\|\frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m}, u_2, \dots, u_n\right\|_\beta = 0 \tag{3.40}$$

for all $x, u_2, \dots, u_n \in X$. By Remark 2.2, we know that the sequence $\{\frac{f(2^m x)}{4^m}\}$ is Cauchy. Since Y is a complete space, we conclude that $\{\frac{f(2^m x)}{4^m}\}$ is convergent. So we can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{m\to\infty}\frac{f(2^m x)}{4^m}$$

for all $x \in X$. Similar to the proof of Theorem 3.1, using induction one can show that

$$\left\|f(x) - \frac{f(2^m x)}{4^m}, u_2, \dots, u_n\right\|_\beta \le \theta\,|4|^{-\beta}\,\|x, u_2, \dots, u_n\|_{\beta_1}^{p+q} \tag{3.41}$$

for all $x, u_2, \dots, u_n \in X$ and $m \in \mathbb{N}$. By taking the limit as $m \to \infty$ in (3.41), we obtain (3.36).

It follows from (3.35) and Lemma 2.2 that

$$\begin{aligned}\|D_Q(x,y), u_2, \dots, u_n\|_\beta &= \lim_{m\to\infty}\frac{1}{|4|^{m\beta}}\,\|D_f(2^m x, 2^m y), u_2, \dots, u_n\|_\beta \\ &\le \lim_{m\to\infty}\frac{\theta}{|4|^{m\beta}}\left(\|2^m x, u_2, \dots, u_n\|_{\beta_1}^p\,\|2^m y, u_2, \dots, u_n\|_{\beta_1}^q + \|2^m x, u_2, \dots, u_n\|_{\beta_1}^{p+q} + \|2^m y, u_2, \dots, u_n\|_{\beta_1}^{p+q}\right) \\ &= \lim_{m\to\infty}\theta\,|2^{(p+q)\beta_1-2\beta}|^m\left(\|x, u_2, \dots, u_n\|_{\beta_1}^p\,\|y, u_2, \dots, u_n\|_{\beta_1}^q + \|x, u_2, \dots, u_n\|_{\beta_1}^{p+q} + \|y, u_2, \dots, u_n\|_{\beta_1}^{p+q}\right)\end{aligned}$$

for all $x, y, u_2, \dots, u_n \in X$. Since $(p+q)\beta_1 > 2\beta$ and $|2| \neq 1$, we get

$$\|D_Q(x,y), u_2, \dots, u_n\|_\beta = 0$$

for all $x, y, u_2, \dots, u_n \in X$. By Lemma 2.1, we get $D_Q(x,y) = 0$ for all $x, y \in X$. Therefore the function $Q : X \to Y$ satisfies (1.4). Since f is an even function, Q is an even function. By Lemma 3.1 (see also [44, Corollary 2.2]), Q is quadratic. Then, we get $Q(2x) = 4Q(x)$ and $Q(2^m x) = 4^m Q(x)$.

To prove the uniqueness property of Q. Let Q' be another quadratic function satisfying (3.36). Then

$$\begin{aligned}\|Q(x) - Q'(x), u_2, \dots, u_n\|_\beta &= \lim_{m\to\infty}|4|^{-m\beta}\,\|Q(2^m x) - Q'(2^m x), u_2, \dots, u_n\|_\beta \\ &\le \lim_{m\to\infty}|4|^{-m\beta}\max\left\{\|Q(2^m x) - f(2^m x), u_2, \dots, u_n\|_\beta, \|f(2^m x) - Q'(2^m x), u_2, \dots, u_n\|_\beta\right\} \\ &\le \frac{\theta}{|4|^\beta}\lim_{m\to\infty}|2^{(p+q)\beta_1-2\beta}|^m\,\|x, u_2, \dots, u_n\|_{\beta_1}^{p+q} = 0\end{aligned}$$

for all $x, u_2, \dots, u_n \in X$. By Lemma 2.1, we get $Q = Q'$ for all $x \in X$. So Q is the unique quadratic function satisfying (3.36). □

Theorem 3.5. *Let $\theta \in [0,\infty)$, $p,q \in (0,\infty)$ with $(p+q)\beta_1 > 3\beta$. Suppose that $f : X \to Y$ is an odd function satisfying*

$$\|D_f(x,y), u_2, \ldots, u_n\|_\beta \le \theta\Big(\|x, u_2, \ldots, u_n\|_{\beta_1}^p \|y, u_2, \ldots, u_n\|_{\beta_1}^q + \|x, u_2, \ldots, u_n\|_{\beta_1}^{p+q} + \|y, u_2, \ldots, u_n\|_{\beta_1}^{p+q} \Big)$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x), u_2, \ldots, u_n\|_\beta \le \frac{\theta}{|8|^\beta} \|x, u_2, \ldots, u_n\|_{\beta_1}^{p+q}$$

for all $x, u_2, \ldots, u_n \in X$.

Proof. The proof is similar to the proof of Theorem 3.4. □

Next, combining Theorems 3.4 and 3.5, we obtain the following result.

Theorem 3.6. *Let $\theta \in [0,\infty)$, $p,q \in (0,\infty)$ with $(p+q)\beta_1 > 3\beta$. Suppose that $f : X \to Y$ is a function satisfying $f(0) = 0$ and*

$$\|D_f(x,y), u_2, \ldots, u_n\|_\beta \le \theta\Big(\|x, u_2, \ldots, u_n\|_{\beta_1}^p \|y, u_2, \ldots, u_n\|_{\beta_1}^q + \|x, u_2, \ldots, u_n\|_{\beta_1}^{p+q} + \|y, u_2, \ldots, u_n\|_{\beta_1}^{p+q} \Big)$$

for all $x, y, u_2, \ldots, u_n \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - Q(x) - C(x), u_2, \ldots, u_n\|_\beta \le \frac{\theta}{|8|^\beta} \|x, u_2, \ldots, u_n\|_{\beta_1}^{p+q}$$

for all $x, u_2, \ldots, u_n \in X$.

Proof. The proof is similar to the proof of Theorem 3.3 and the result follows from Theorems 3.4 and 3.5. □

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan,* **2** (1950), 64–66.
2. Y. J. Cho, P. C. S. Lin, S. S. Kim, A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2001.

3. Y. J. Cho, Th. M. Rassias, R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Springer Science, New York, 2013.
4. Y. J. Cho, C. Park, Th. M. Rassias, R. Saadati, *Stability of Functional Equations in Banach Algebras*, Springer Science, New York, 2015.
5. P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.,* **27** (1984), 76–86.
6. K. Ciepliński, T. Z. Xu, Approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces, *Carpathian J. Math.,* **29** (2013), 159–166.
7. S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg,* **62** (1992), 59–64.
8. R. W. Freese, Y. J. Cho, *Geometry of Linear 2-Normed Spaces*, Nova Science Publishers, Inc., New York, 2001.
9. S. Gähler, 2-metrische Räume und ihere topologische struktur, *Math. Nachr.,* **26** (1963), 115–148.
10. S. Gähler, Lineare 2-normierte Räume, *Math. Nachr.,* **28** (1964), 1–43.
11. S. Gähler, Über 2-Banach Räume, *Math. Nachr.,* **42** (1969), 335–347.
12. P. Găvruţă, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.,* **184** (1994), 431–436.
13. M. E. Gordji, M. B. Savadkouhi, Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces, *Appl. Math. Lett.,* **23** (2010), 1198–1202.
14. M. E. Gordji, M. B. Savadkouhi, Stability of cubic and quartic functional equations in non-Archimedean spaces, *Acta Appl. Math.,* **110** (2010), 1321–1329.
15. H. Gunawan, M. Mashadi, On n-normed spaces, *Int. J. Appl. Math. Sci.,* **27** (2001), 631–639.
16. K. Hensel, Über eine neue Begründung der Theorie der algebraischen Zahlen, *Jahresber. Deutsch. Math. Verein*, **6** (1897), 83–88.
17. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. USA,* **27** (1941), 222–224.
18. D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several variables*, Birkh*ä*user, Basel, 1998.
19. K. W. Jun, H. M. Kim, The generalized of the Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.,* **274** (2002), 267–278.
20. S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Science, New York, 2011.
21. Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Science, New York, 2009.
22. A. K. Katsaras, A. Beoyiannis, Tensor products of non-Archimedean weighted spaces of continuous functions, *Georgian Math. J.,* **6** (1999), 33–44.
23. A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models,* Kluwer Academic Publishers, Dordrecht, 1997.

24. C. I. Kim, S. W. Park, The generalized Hyers-Ulam stability of additive functional inequalities in non-Archimedean 2-normed space, *Korean J. Math.,* **22** (2014), 339–348.
25. S. S. Kim, Y. J. Cho, Strict convexity in linear n-normed spaces, *Demonstr. Math.,* **29** (1996), 739–744.
26. J. R. Lee, S. Y. Jang, C. Park, D. Y. Shin, Fuzzy stability of quadratic functional equations, *Adv. Differ. Equ.,* **2010** (2010), 412160.
27. R. Malceski, Strong n-convex n-normed spaces, *Mat. Bilt.,* **21** (1997), 81–102.
28. A. K. Mirmostafaee, Approximately additive mappings in non-Archimedean normed spaces, *Bull. Korean Math. Soc.,* **46** (2009), 387–400.
29. A. Misiak, N-inner product spaces, *Math. Nachr.,* **140** (1989), 299–319.
30. M. S. Moslehian, Th. M. Rassias, Stability of functional equations in non-Archimedean spaces, *Appl. Anal. Discrete Math.,* **1** (2007), 325–334.
31. M. S. Moslehian, Gh. Sadeghi, A Mazur-Ulam theorem in non-Archimedean normed spaces, *Nonlinear Anal.,* **69** (2008), 3405–3408.
32. A. Najati, F. Moradlou, Hyers-Ulam-Rassias stability of the Apollonius type quadratic mapping in non-Archimedean spaces, *Tamsui Oxf. J. Math. Sci.,* **24** (2008), 367–380.
33. P. J. Nyikos, On some non-Archimedean spaces of Alexandroff and Urysohn, *Topol. Appl.,* **91** (1999), 1–23.
34. W. G. Park, Approximate additive mappings in 2-Banach spaces and related topics, *J. Math. Anal. Appl.,* **376** (2011), 193–202.
35. C. Park, M. E. Gordji, M. B. Ghaemi, H. Majani, Fixed points and approximately octic mappings in non-Archimedean 2-normed spaces, *J. Ineq. Appl.,* **2012** (2012), 289.
36. C. Park, Additive functional inequalities in 2-Banach spaces, *J. Ineq. Appl.,* **2013** (2013), 447.
37. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978), 297–300.
38. Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic, Dordrecht, 2003.
39. K. Ravi, R. Murali, M. Arunkumar, The generalized Hyers-Ulam-Rassias stability of a quadratic function equation, *J. Ineq. Pure Appl. Math.,* **9** (2008), 20.
40. P. K. Sahoo, A generalized cubic functional equation, *Acta Math. Sinica (English Series)*, **21** (2005), 1159–1166.
41. P. K. Sahoo, P. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, 2011.
42. F. Skof, Local properties and approximations of operators, *Rend. Sem. Mat. Fis. Milano,* **53** (1983), 113–129.
43. S. M. Ulam, *Problems in Modern Mathematics, Chapter VI*, Science Editions, Wiley, New York, 1964.
44. L. G. Wang, B. Liu, The Hyers-Ulam stability of a functional equation deriving from quadratic and cubic functions in quasi-β-normed spaces, *Acta Math. Sin.,* **26** (2010), 2335–2348.

45. A. White, 2-*Banach spaces,* Doctorial Diss., St. Louis Univ., 1968.

46. A. White, 2-Banach spaces, *Math. Nachr.,* **42** (1969), 43–60.

47. T. Z. Xu, Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces, *Abst. Appl. Anal.,* **2013** (2013), 648709.

48. T. Z. Xu, J. M. Rassias, On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces, *Abst. Appl. Anal.,* **2012** (2012), 926390.

49. X. Z. Yang, L. D. Chang, G. F. Liu, G. N. Shen, Stability of functional equations in (n,β)-normed spaces, *J. Ineq. Appl.,* **2015** (2015), 112.

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