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Research article

Caputo-Fabrizio fractional differential equations with instantaneous impulses

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Abstract: The subject of this paper is the existence of solutions for a class of Caputo-Fabrizio fractional differential equations with instantaneous impulses. Our results are based on Schauder's and Monch's fixed point theorems and the technique of the measure of noncompactness. Two illustrative examples are the subject of the last section.

Keywords: Fractional differential equation; Caputo-Fabrizio integral of fractional order; Caputo-Fabrizio fractional derivative; instantaneous impulse; measure of noncompactness; fixed point
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1. Introduction

Fractional differential equations have recently been applied in various areas. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to [1, 3, 4, 19, 25, 27, 29], and the references therein. Some new aspects of the Caputo-Fabrizio derivative can be seen in [11, 21] and the references therein. For some applications of a such derivative, we refer to [5, 15].

Differential equations involving impulses effects; appear as a natural description of observed evolution phenomena of several real world problems [12, 16, 18, 26]. Many physical situations are modeled by impulsive differential equations, for example problems in optimal control theory and problems in threshold theory in Biology. Major developments in the theory of impulsive fractional differential equations have been developed in the last years; see the books [1, 26], the papers [1, 2, 7, 6, 17, 20, 24, 28], and the references therein.

Recently, in [4, 8, 13], the measure of noncompactness was applieded to some classes of functional Riemann-Liouville or Caputo fractional differential equations in Banach spaces. See also the classical monographs [9, 10].

In this paper first we investigate the existence of solutions for the following Cauchy problem of Caputo-Fabrizio impulsive fractional differential equations

$$\begin{cases} ({}^{CF}D^r_{t_k}u)(t) = f(t, u(t)); \ t \in I_k, \ k = 0, \cdots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \cdots, m, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $I_0 = [0, t_1], \ I_k = (t_k, t_{k+1}]; \ k = 1, \cdots, m; \ 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \ u_0 \in \mathbb{R}, \ f : I_k \times \mathbb{R} \to \mathbb{R}; \ k = 0, \ldots, m, \ L_k : \mathbb{R} \to \mathbb{R}; \ k = 1, \ldots, m$ are given continuous functions, ${}^{CF}D^r_{t_k}$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$.

Next, by using the measure of noncompactness, we discuss the existence of solutions for problem (1.1), when $u_0 \in E, \ f : I_k \times E \to E; \ ; \ k = 0, \ldots, m, \ L_k : E \to E; \ k = 1, \ldots, m$ are given functions, and $(E\| \cdot \|,)$ is a real or complex Banach space.

2. Preliminaries

Let $I := [0, T]; \ T > 0$, and $C(I) := C(I, \mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{t \in I} |u(t)|.$$

By $L^1(I)$ we denote the Banach space of measurable function $u : I \to \mathbb{R}$ with are Lebesgue integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T |u(t)|dt.$$

As usual, $AC(I)$ denotes the space of all absolutely continuous functions from I into $\mathbb{R}$.

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.1. *[10] Let X be a complete metric space. A map $\mu : \mathcal{M}_X \to [0, \infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.*

(a) $\mu(B) = 0$ if and only if B is precompact (Regularity),
(b) $\mu(B) = \mu(\overline{B})$ (Invariance under closure),
(c) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$ (Semi-additivity).

Definition 2.2 ([10])**.** *Let X be a Banach space and let Ω_X be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0, \infty)$ defined by*

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^m M_j, \mathrm{diam}(M_j) \leq \epsilon\},$$

where $M \in \Omega_E$.

The measure μ satisfies the following properties

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M),\ c \in \mathbb{R}$.
(6) $\mu(\text{conv } M) = \mu(M)$.

Definition 2.3. *[14] The Caputo-Fabrizio fractional integral of order* $0 < r < 1$ *for a function* $h \in L^1(I)$ *is defined by*

$$({}^{CF}I_0^r h)(\tau) = \frac{2(1-r)}{M(r)(2-r)}h(\tau) + \frac{2r}{M(r)(2-r)}\int_0^\tau h(x)dx, \quad \tau \geq 0$$

where $M(r)$ is normalization constant depending on r. For example, taking $M(r) = \frac{2}{2-r}$, we have

$$({}^{CF}I_0^r h)(\tau) = (1-r)h(\tau) + r\int_0^\tau h(x)dx, \quad \tau \geq 0.$$

Definition 2.4. *[14] The Caputo-Fabrizio fractional derivative of order* $0 < r < 1$ *for a function* $h \in AC(I)$ *is defined by*

$$({}^{CF}D_0^r h)(\tau) = \frac{(2-r)M(r)}{2(1-r)}\int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))h'(x)dx; \ \tau \in I.$$

Note that ${}^{CF}D_0^r h = 0$ if and only if h is a constant function.
For $M(r) = \frac{2}{2-r}$, one has

$$({}^{CF}D_0^r h)(\tau) = \frac{1}{1-r}\int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))h'(x)dx; \ \tau \in I.$$

Lemma 2.5. *Let* $h \in L^1(I)$*. Then the linear Cauchy problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0, T] \\ u(0) = u_0, \end{cases} \tag{2.1}$$

has a unique solution given by

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds, \tag{2.2}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}, \ C = u_0 - a_r h(0).$$

Proof. Suppose that u satisfies (2.1). From Proposition 1 in [22]; the equation

$$({}^{CF}D_0^r u)(t) = h(t)$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we obtain

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

Hence we get (2.2).

For our purpose we will need the following fixed point theorems:

Theorem 2.6. *(Schauder's fixed point theorem [9]). Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 2.7. *(Monch's fixed point theorem [23]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \overline{V} \text{ is compact}, \tag{2.3}$$

holds for every subset V of D, then N has a fixed point.

3. Main results

In this section, we present some results concerning the existence of solutions for the problem (1.1). Consider the Banach space

$$\begin{aligned} PC = \{ & u : I \to E : u \in C(I_k);\ k = 0, \dots, m, \text{ and there exist } u(t_k^-) \\ & \text{and } u(t_k^+);\ k = 1, \dots, m, \text{ with } u(t_k^-) = u(t_k)\}, \end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{t \in I} \|u(t)\|.$$

In the case when $E = \mathbb{R}$, we get

$$\|u\|_{PC} = \sup_{t \in I} |u(t)|.$$

Definition 3.1. *By a solution of the problem (1.1) we mean a function $u \in PC$ that satisfies $u(0) = u_0$, $({}^{CF}D^r_{t_k} u)(t) = f(t, u(t))$; for $t \in I_k,\ k = 0, \cdots, m$, and*

$$u(t_k^+) = u(t_k^-) + L_k(u(t_k^-));\ k = 1, \cdots, m.$$

Lemma 3.2. *Let $h : I \to E$ be a continuous function. A function $u \in PC$ is a solution of the fractional integral equation*

$$\begin{cases} u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds;\ if\ t \in I_0, \\ u(t) = u_0 - a_r h(0) + \sum_{i=1}^{k} L_i(u(t_i^-)) + a_r h(t) \\ \qquad + b_r \int_0^t h(s)ds;\ if\ t \in I_k,\ k = 1, \dots, m, \end{cases} \tag{3.1}$$

if and only if u is a solution of the following problem

$$\begin{cases}({}^{CF}D^r_{t_k}u)(t) = h(t);\ t \in I_k,\ k = 0, \dots, m,\\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-));\ k = 1, \dots, m,\\ u(0) = u_0.\end{cases} \tag{3.2}$$

Proof. Assume u satisfies (3.2). If $t \in I_0$, then

$$({}^{CF}D^r_0 u)(t) = h(t).$$

Lemma 2.5 implies that

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

If $t \in I_1$, then

$$({}^{CF}D^r_{t_1} u)(t) = h(t).$$

Lemma 2.5 implies that

$$u(t) = u(t_1) - a_r h(t_1) + a_r h(t) + b_r \int_{t_1}^t h(s)ds.$$

Thus

$$\begin{aligned}u(t) &= L_1(u(t_1^-)) + u(t_1^-) - a_r h(t_1) + a_r h(t) + b_r \int_{t_1}^t h(s)ds\\ &= L_1(u(t_1^-)) + u_0 - a_r h(0) + a_r h(t_1^-)\\ &+ b_r \int_0^{t_1^-} h(s)ds - a_r h(t_1) + a_r h(t) + b_r \int_{t_1}^t h(s)ds\\ &= L_1(u(t_1^-)) + u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.\end{aligned}$$

If $t \in I_2$, then

$$({}^{CF}D^r_{t_2} u)(t) = h(t).$$

Then, we obtain

$$\begin{aligned}u(t) &= u(t_2) - a_r h(t_2) + a_r h(t) + b_r \int_{t_2}^t h(s)ds\\ &= L_2(u(t_2^-)) + u(t_2^-) - a_r h(t_2) + a_r h(t) + b_r \int_{t_2}^t h(s)ds\\ &= L_2(u(t_2^-)) + L_1(u(t_1^-)) + u_0 - a_r h(0) + a_r h(t_2^-) + b_r \int_0^{t_2^-} h(s)ds\\ &- a_r h(t_2) + a_r h(t) + b_r \int_{t_2}^t h(s)ds\\ &= L_2(u(t_2^-)) + L_1(u(t_1^-)) + u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.\end{aligned}$$

If $t \in I_k$, then again from Lemma 3.3 we get (3.1).

Conversely, suppose that u satisfies (3.1). If $t \in I_0$, then

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

Thus, $u(0) = u_0$ and using the fact that ${}^{CF}D^r_{t_k}$ is the left inverse of $({}^{CF}I^r_0$ we get $({}^{CF}D^r_0 u)(t) = h(t)$. Now, if $t \in I_k;\ k = 1, \dots, m$, we get $({}^{CF}D^r_{t_k} u)(t) = h(t)$. Also, we can easily show that

$$u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)).$$

Hence, if u satisfies (3.1) then we get (3.2).

As in the prove of the above Lemma, we can show the following one:

Lemma 3.3. *A function $u \in PC$ is a solution of problem (1.1), if and only if u satisfies the following integral equation*

$$\begin{cases} u(t) = c + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds;\ if\ t \in I_0, \\ u(t) = c + \sum_{i=1}^{k} L_i(u(t_i^-)) + a_r f(t, u(t)) \\ \qquad +b_r \int_0^t f(s, u(s))ds;\ if\ t \in I_k,\ k = 1, \dots, m, \end{cases} \tag{3.3}$$

where $c = u_0 - a_r f(0, u_0)$.

3.1. Existence results in the scalar case

The following hypotheses will be used in the sequel.

(H_1) There exists a positive continuous function $p \in C(I_k);\ k = 0, \dots, m$, such that

$$|f(t, u)| \leq p(t)(1 + |u|);\quad t \in I_k,\quad u \in \mathbb{R}.$$

(H_2) There exists $q^* \geq 0$ such that

$$|L_k(u)| \leq q^*(1 + |u|);\quad u \in \mathbb{R}.$$

(H_3) For each bounded set $B \subset PC$, the set $\{t \mapsto f(t, u(t)) : u \in B,\ t \in I_k;\ k = 0, \dots, m\}$ is equicontinuous.

Set

$$p^* = \sup_{t \in I} p(t).$$

Theorem 3.4. *Assume that the hypotheses $(H_1) - (H_3)$ hold. If*

$$mq^* + p^*(a_r + Tb_r) < 1, \tag{3.4}$$

then the problem (1.1) has at least one solution defined on I.

Proof. Consider the operator $N : PC \to PC$ defined by:

$$\begin{cases}(Nu)(t) = c + a_r f(t,u(t)) + b_r \int_0^t f(s,u(s))ds; \ if \ t \in I_0,\\ (Nu)(t) = c + \sum_{i=1}^{k} L_i(u(t_i^-)) + a_r f(t,u(t)) \\ \qquad\qquad +b_r \int_0^t f(s,u(s))ds; \ if \ t \in I_k, \ k = 1,\dots,m.\end{cases} \tag{3.5}$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1).
Let $R > 0$, such that

$$R > \frac{|c| + mq^* + p^*(a_r + Tb_r)}{1 - mq^* - p^*(a_r + Tb_r)},$$

and consider the ball $B_R := B(0,R) = \{w \in \|w\|_{PC} \leq R\}$.
For each $t \in I_0$, and $u \in PC$, we have

$$\begin{aligned}|(Nu)(t)| &= \left|c + a_r f(t,u(t)) + b_r \int_0^t f(s,u(s))ds\right| \\ &\leq |c| + a_r|f(t,u(t))| + b_r \int_0^t |f(s,u(s))|ds \\ &\leq |c| + p^*(a_r + Tb_r)(1 + R) \\ &\leq R.\end{aligned}$$

On the other hand, for each $t \in I_k : \ k = 1,\dots,m$, and $u \in PC$, we have

$$\begin{aligned}|(Nu)(t)| &\leq \sum_{i=1}^{k} |L_i(u(t_i^-))| + |c| + a_r|f(t,u(t))| + b_r \int_0^t |f(s,u(s))|ds \\ &\leq mq^*(1 + R) + |c| + p^*(a_r + Tb_r)(1 + R) \\ &\leq R.\end{aligned}$$

Hence, for $t \in I$, and $u \in PC$, we get

$$\|N(u)\|_{PC} \leq mq^* + |c| + p^*(a_r + Tb_r) := R.$$

This proves that $N(B_R) \subset B_R$. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.6. The proof will be given in two steps.

Step 1. *$N : B_R \to B_R$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I_0$, we have

$$|(Nu_n)(t) - (Nu)(t)| \leq a_r|f(t,u_n(t)) - f(t,u(t))| + b_r \int_0^t |f(s,u_n(s)) - f(s,u(s))|ds. \tag{3.6}$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by using the Lebesgue dominated convergence theorem, (3.6) implies

$$\|N(u_n) - N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Also, for each $t \in I_k;\ k = 1, \dots, m$, we have

$$
\begin{aligned}
|(Nu_n)(t) - (Nu)(t)| &\leq \sum_{i=1}^{k} |L_i(u_n(t_i^-)) - L_i(u(t_i^-))| \\
&+ a_r|f(t, u_n(t)) - f(t, u(t))| \\
&+ b_r \int_0^t |f(s, u_n(s)) - f(s, u(s))| ds.
\end{aligned}
$$

Again, we get the continuity of our operator N.

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.

Next, let $\tau_1, \tau_2 \in I_k;\ k = 0, \dots, m$; such that $t_k \leq \tau_1 < t \leq \tau_2 \leq t_{k+1}$ and let $u \in B_R$. Then, from the continuity of f, and (H_3), we get

$$
\begin{aligned}
|(Nu)(\tau_2) - (Nu)(\tau_1)| &\leq a_r|f(\tau_2, u(\tau_2)) - f(\tau_1, u(\tau_1))| + b_r \int_{\tau_1}^{\tau_2} |f(s, u(s))| ds \\
&\leq a_r|f(\tau_2, u(\tau_2)) - f(\tau_1, u(\tau_1))| + (1 + R)p^* b_r(\tau_2 - \tau_1) \\
&\longrightarrow 0 \ as\ \tau_1 \longrightarrow \tau_2.
\end{aligned}
$$

Hence, $N(B_R)$ is bounded and equicontinuous.

As a consequence of the above two steps, together with the Ascoli-Arzelá theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 2.6, we deduce that N has a fixed point u which is a solution of problem (1.1).

3.2. Existence results in Banach spaces

The following hypotheses will be used in the sequel.

(H_4) The function $t \mapsto f(t, u)$ is measurable on $I_k;\ k = 0, \dots, m$, for each $u \in E$, and the function $u \mapsto f(t, u)$ is continuous on E for each $t \in I_k;\ k = 0, \dots, m$,

(H_5) There exists a positive continuous function $\overline{p} \in C(I_k);\ k = 0, \dots, m$, such that

$$\|f(t, u)\| \leq \overline{p}(t)(1 + \|u\|);\ \ t \in I_k,\ \ u \in E,$$

(H_6) For each bounded set $B \subset E$ and for each $t \in I_k;\ k = 0, \dots, m$, we have

$$\mu(f(t, B)) \leq \overline{p}(t)\mu(B);\ t \in I_k,\ k = 0, \dots, m,$$

(H_7) There exists $\overline{q}^* \geq 0$, such that

$$\|L_k(u)\| \leq \overline{q}^*(1 + \|u\|);\ u \in E,$$

and, for each bounded set $B \subset E;\ k = 0, \dots, m$, we have

$$\mu(L_k(B)) \leq \overline{q}^*\mu(B),$$

(H_8) For each bounded set $B_1 \subset PC$, the set $\{t \mapsto f(t, u(t)) : u \in B_1,\ t \in I_k;\ k = 0, \ldots, m\}$ is equicontinuous.

Set

$$\overline{p}^* = \sup_{t \in I} \overline{p}(t).$$

Theorem 3.5. *Assume that the hypotheses* $(H_4) - (H_8)$ *hold. If*

$$\rho := m\overline{q}^* + a_r\overline{p}^* + Tb_r\overline{p}^* < 1, \tag{3.7}$$

then the problem (1.1) has at least one solution defined on I.

Proof. Consider the operator $N : PC \to PC$ defined in (3.5), and let $B_R \subset PC$ be the ball centered at the origin with radius $R \geq \frac{\|c\|+\rho}{1-\rho}$. For each $t \in I_0$, and $u \in PC$, we have

$$\begin{aligned} \|((Nu)(t)\| &= \left\| c + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds \right\| \\ &\leq \|c\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\|ds \\ &\leq \|c\| + \overline{p}^*(a_r + Tb_r)(1 + R) \\ &\leq \|c\| + m\overline{q}^*(1 + R) + \overline{p}^*(a_r + Tb_r)(1 + R) \\ &\leq R. \end{aligned}$$

Next, for each $t \in I_k :\ k = 1, \ldots, m$, and $u \in PC$, we get

$$\begin{aligned} \|(Nu)(t)\| &\leq \sum_{i=1}^{k} \|L_i(u(t_i^-))\| + \|c\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\|ds \\ &\leq \|c\| + m\overline{q}^*(1 + R) + \overline{p}^*(a_r + Tb_r)(1 + R) \\ &\leq R. \end{aligned}$$

Hence, for $t \in I$, and $u \in PC$, we get

$$\|N(u)\|_{PC} \leq R.$$

This proves that N transforms the ball B_R into itself.
We prove in three steps that $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.7.

Step 1. $N : B_R \to B_R$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I_0$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \leq a_r\|f(t, u_n(t)) - f(t, u(t))\| + b_r \int_0^t \|f(s, u_n(s)) - f(s, u(s))\|ds.$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then (3.6) implies

$$\|N(u_n) - N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty,$$

by the Lebesgue dominated convergence theorem.
Also, for each $t \in I_k;\ k = 1, \dots, m$, we get

$$\begin{aligned}\|(Nu_n)(t) - (Nu)(t)\| &\leq \sum_{i=1}^{k} \|L_i(u_n(t_i^-)) - L_i(u(t_i^-))\| \\ &+ a_r\|f(t, u_n(t)) - f(t, u(t))\| \\ &+ b_r \int_0^t \|f(s, u_n(s)) - f(s, u(s))\| ds.\end{aligned}$$

Hence, we get the continuity of our operator N.

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.

Next, let $\tau_1, \tau_2 \in I_k;\ k = 0, \dots, m$; such that $t_k \leq \tau_1 < t \leq \tau_2 \leq t_{k+1}$ and let $u \in B_R$. Then, we have

$$\begin{aligned}\|(Nu)(\tau_2) - (Nu)(\tau_1)\| &\leq a_r\|f(\tau_2, u(\tau_2)) - f(\tau_1, u(\tau_1))\| + b_r \int_{\tau_1}^{\tau_2} \|f(s, u(s))\| ds \\ &\leq a_r\|f(\tau_2, u(\tau_2)) - f(\tau_1, u(\tau_1))\| + b_r\overline{p}^*(1 + R)(\tau_2 - \tau_1).\end{aligned}$$

From the continuity of f, and (H_8) the right-hand side of the above inequality tends to zero as $\tau_1 \longrightarrow \tau_2$, and such convergence is uniform in $u \in B_R$. Hence, $N(B_R)$ is bounded and equicontinuous.

Step 3. *The implication (2.3) holds.*
Now let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$, V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on I. By (H_6) and the properties of the measure μ, for each $t \in I_0$, we have

$$\begin{aligned}v(t) &\leq \mu((NV)(t) \cup \{0\}) \\ &\leq \mu((NV)(t)) \\ &\leq a_r v(t) + b_r \int_0^t v(s) ds \\ &\leq a_r\overline{p}(t)\mu(V(t) + b_r \int_0^t \overline{p}(s)\mu(V(s)) ds \\ &\leq a_r\overline{p}^*\mu(V(t) + b_r\overline{p}^* \int_0^t \mu(V(s)) ds \\ &\leq (a_r + Tb_r)\overline{p}^*\|v\|_{PC}.\end{aligned}$$

Thus

$$\|v\|_{PC} \leq \rho\|v\|_{PC}.$$

Also, for each $t \in I_k;\ k = 1, \dots, m$, we get

$$\begin{aligned} v(t) &\le \mu((NV)(t)\cup\{0\}) \\ &\le \mu((NV)(t)) \\ &\le \overline{q}^* \sum_{i=1}^{k} \mu(V(s)) + a_r\overline{p}(t)\mu(V(t) + b_r \int_0^t \overline{p}(s)\mu(V(s))ds \\ &\le \overline{q}^* \sum_{i=1}^{k} \mu(V(t)) + a_r\overline{p}^*\mu(V(t) + b_r\overline{p}^* \int_0^t \mu(V(s))ds \\ &\le (m\overline{q}^* + a_r\overline{p}^* + Tb_r\overline{p}^*)\|v\|_{PC}. \end{aligned}$$

Hence

$$\|v\|_{PC} \le \rho\|v\|_{PC}.$$

From (3.7), we get $\|v\|_{PC} = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. From the Ascoli-Arzelà theorem, V is relatively compact in B_R. We conclude by Theorem 2.7 that N has a fixed point which is a solution of (1.1).

4. Examples

Example 1. Consider the problem of impulsive Caputo-Fabrizio fractional differential equation

$$\begin{cases} ({}^{CF}D^r_{t_k}u)(t) = f(t,u(t));\ t \in I_k,\ k = 0,\dots,m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-));\ k = 1,\dots,m, \\ u(0) = 0, \end{cases} ;\ r \in (0,1),\ t \in [0,1], \tag{4.1}$$

where

$$f(t,u(t)) = \frac{t^2}{(1+2a_r+2b_r)(1+|u(t)|)}\left(e^{-7} + \frac{1}{e^{t+5}}\right)(1+u(t));\ t \in [0,1],$$

and

$$L_k(u(t_k^-)) = \frac{1+|u(t_k^-)|}{3e^5(1+2a_r+2b_r)};\ k = 1,\dots,m.$$

Clearly, the function f is continuous.
For each $t \in [0,1]$, we have

$$|f(t,u(t))| \le \frac{t^2}{1+2a_r+2b_r}\left(e^{-7} + \frac{1}{e^{t+5}}\right)(1+|u(t)|),$$

and

$$|L_k(u)| \le \frac{e^{-5}(1+|u|)}{3(1+2a_r+2b_r)}.$$

Hence, the hypothesis (H_1) is satisfied with

$$p^* = \frac{2ce^{-5}}{1+2a_r+2b_r},$$

and (H_2) is satisfied with

$$q^* = \frac{e^{-5}}{3(1 + 2a_r + 2b_r)}.$$

We shall show that condition (3.4) holds with $T = 1$. Indeed; if we assume, for instance, that the number of impulses $m = 3$, then we have

$$mq^* + p^*(a_r + Tb_r) = e^{-5} < 1.$$

Simple computations show that all conditions of Theorem 3.4 are satisfied. It follows that the problem (4.1) has at least one solution on $[0, 1]$.

Example 2. Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the problem of Caputo-Fabrizio fractional impulsive differential equation

$$\begin{cases} ({}^{CF}D^r_{t_k} u)(t) = f(t, u(t)); \ t \in I_k, \ k = 0, \dots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \dots, m, \\ u(0) = 0, \end{cases} ; \ r \in (0, 1), \ t \in [0, 1], \tag{4.2}$$

where $u = (u_1, u_2, \dots, u_n, \dots), \ f = (f_1, f_2, \dots, f_n, \dots),$

$${}^{CF}D^r_{t_k} u = ({}^{CF}D^r_{t_k} u_1, \dots, {}^{CF}D^r_{t_k} u_n, \dots); \ k = 0, \dots, m,$$

$$f_n(t, u(t)) = \frac{c_r}{1 + \|u(t)\|_E} \left(e^{-7} + \frac{1}{e^{t+5}} \right) (2^{-n} + u_n(t)); \ t \in [0, 1],$$

$$L_k(u(t_k^-)) = \frac{c_r(1 + u(t_k^-))}{3e^4}; \ k = 1, \dots, m,$$

and $c_r = \frac{1}{1+a_r+b_r}$.
For each $u \in E$ and $t \in [0, 1]$, we have

$$\|f(t, u(t)\|_E \le c_r \left(e^{-7} + \frac{1}{e^{t+5}} \right) (1 + \|u(t)\|_E),$$

and

$$\|L_k(u(t_k^-))\|_E \le \frac{c_r}{3e^4}(1 + \|u(t_k^-)\|_E).$$

Hence, the hypothesis (H_5) is satisfied with $p^* = 2c_r e^{-5}$, and (H_7) is satisfied with $q^* = \frac{c_r}{3} e^{-4}$. We shall show that condition (3.7) holds with $T = 1$. Indeed; if we assume, for instance, that the number of impulses $m = 3$, then we have

$$\rho = mq^* + a_r p^* + Tb_r p^* = c_r(1 + a_r + b_r)e^{-5} = e^{-5} < 1.$$

Simple computations show that all conditions of Theorem 3.5 are satisfied. It follows that the problem (4.2) has at least one solution on $[0, 1]$.

5. Conclusions

In this paper, we provided some sufficient conditions ensuring the existence of solutions for functional fractional differential equations with instantaneous impulses; involving the Caputo-Fabrizio fractional derivative. The techniqued used are the fixed point theory and the measure of noncompactness.

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Conflict of interest

The authors declare no conflict of interests.

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