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Research article

Well-posedness and stability for Bresse-Timoshenko type systems with thermodiffusion effects and nonlinear damping

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Abstract: Nonlinear Bresse-Timoshenko beam model with thermal, mass diffusion and theormoelastic effects is studied. We state and prove the well-posedness of problem. The global existence and uniqueness of solution is proved by using the classical Faedo-Galerkin approximations along with two a priori estimates. We prove an exponential stability estimate under assumption $(2.3)_1$ and polynomial decay rate for solution under $(2.3)_2$, by using a multiplier technique combined with an appropriate Lyapuniv functions.

Keywords: Bresse-Timoshenko type systems; thermodiffusion effects; well-possedness; exponential stability; polynomial decay; nonlinear damping
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1. Introduction and position of problem

In the present paper, a nonlinear Bresse-Timoshenko system with thermodiffusion effects is considered. The beam is modeled by the following system

$$\begin{cases} \rho_1\partial_{tt}\varphi - \kappa\partial_x(\partial_x\varphi + \psi) + \sigma_1\partial_t\varphi = 0 \\ -\rho_2\partial_{tt}\varphi_x - b\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma\theta_x - \beta C_x + \sigma_2\mathcal{G}(\partial_t\psi) = 0 \\ \rho_3\partial_t\theta + \varpi\partial_t C - \kappa\theta_{xx} - \gamma\partial_t\psi_x = 0 \\ \partial_t C - h(\beta\psi_x + \rho C - \varpi\theta)_{xx} = 0, \end{cases} \tag{1.1}$$

where

$$(x,t) \in (0,L) \times (0,\infty),$$

here L is the distance between the ends of the center line of the beam. The function C denotes the concentration of diffusive material in the elastic body. The constant $h > 0$ is the diffusion coefficient, ϖ is a measure of the thermo-diffusion effect. To simplify the system, we use the next relation between chemical potential P and the concentration of the diffusion material C

$$C = \frac{1}{\varrho}(P - \beta\psi_x + \varpi\theta).$$

Here ϱ is a measure of the diffusive effect, we put

$$\alpha = b - \frac{\beta^2}{\varrho},\ \ \xi_1 = \gamma + \frac{\beta\varpi}{\varrho},\ \ \xi_2 = \frac{\beta}{\varrho},\ \ c = \rho_3 + \frac{\varpi}{\varrho},\ \ r = \frac{1}{\varrho}.$$

Substitute in (1.1), the problem becomes

$$\begin{cases} \rho_1\partial_{tt}\varphi - \kappa(\varphi_x + \psi)_x + \sigma_1\partial_t\varphi = 0 \\ -\rho_2\partial_{tt}\varphi_x - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \xi_1\theta_x - \xi_2 P_x + \sigma_2\mathcal{G}(\partial_t\psi) = 0 \\ c\partial_t\theta + d\partial_t P - \kappa\theta_{xx} - \xi_1\partial_t\psi_x = 0 \\ d\partial_t\theta + r\partial_t P - hP_{xx} - \xi_2\partial_t\psi_x = 0. \end{cases} \tag{1.2}$$

The aim of this paper is to study the system (1.2) with following initial data

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \partial_t\varphi(x,0) = \varphi_1(x), \partial_{tt}\varphi(x,0) = \varphi_2(x) \\ \psi(x,0) = \psi_0(x), \partial_t\psi(x,0) = \psi_1(x) \\ \theta(x,0) = \theta_0(x), P(x,0) = P_0(x), x \in (0,L), \end{cases} \tag{1.3}$$

where $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, P_0$ are given functions, and the Dirichlet boundary conditions

$$\varphi(x,t) = \psi(x,t) = \theta(x,t) = P(x,t) = 0,\ \ x = 0, L, t > 0. \tag{1.4}$$

In engineering practice, when solving problems of the dynamics of composite mechanical structures, which are various kinds of connections, questions arise on determining the characteristics of natural vibrations of such coupled systems. Note that problems related to the category of non-classical problems of mathematical physics, when we talk about the combination of elements, the behavior of which is described by equations of different type. This causes certain difficulties in solving them, therefore, in practice, models of real structures are used, simplified by introducing additional hypotheses and assumptions into consideration. We mention som references dealing with dynamics of engineering structures and non-classical problems of mathematical physics [7, 8, 16, 17].

This new kind of problem is due to a mixture of Timoshenko system [20] and Bresse system or the curved beam [9]. The coupled system from where one gets the Bresse-Timoshenko comes from Elishakoff [11] by combining d'Alembert's principle for dynamic equilibrium from Timoshenko hypothesis, resulting the coupled system

$$\begin{cases} \rho_1 \partial_{tt}\varphi - \kappa(\varphi_x + \psi)_x = 0 \\ -\rho_2 \partial_{tt}\varphi_x - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0. \end{cases} \tag{1.5}$$

One most famous thermoelasticity is the Cattaneo's law, which is unable to account for some physical properties and it cannot answer all questions, its uses are limited, this let us think to couple the fields of strain, temperature, and mass diffusion according to the Gurtin-Pinkin model. The stabilization of the Bresse-Timoshenko model is studied only by few authors.

When $\mathcal{G} \equiv 0$, the problem (1.2) has been studied in [5], where a new Timoshenko system with thermal and mass diffusion effects according to the Gurtin-Pinkin model is proposed. The authors proved global well-posedness of system by using the semigroup theory and also the quasistability. Despite the fact that a sufficient number of works have been devoted to the study of natural vibrations of a Breese-Timoshenko beam, the problem of determining qualitative properties with thermal, mass diffusion and theormoelastic effects remains unsolved. [2, 3, 10, 13, 19].

In [6], the authors studied stability of thermoviscoelastic Bresse beam system. The exponential decay of energy is proved and implicit Euler type scheme based on finite differences in time and finite elements in spaces is introduced to show that the discrete energy decreases in time and an error estimates are obtained.

Without thermodiffusion effects, in [13], Feng and *al.*, considered a Bresse-Timoshenko type system with time-dependent delay terms

$$\begin{cases} \rho_1 \partial_{tt} y - \kappa(y_x + \psi)_x = 0 \\ -\rho_2 \partial_{tt} y_x - b\psi_{xx} + \kappa(y_x + \psi) + \mu_1 \partial_t \psi + \mu_2 \partial_t \psi(t - \tau(t)) = 0, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \rho_1 \partial_{tt} y - \kappa(y_x + \psi)_x + \mu_1 \partial_t y + \mu_2 \partial_t y(t - \tau(t)) = 0 \\ -\rho_2 \partial_{tt} y_x - b\psi_{xx} + \kappa(y_x + \psi) = 0. \end{cases} \tag{1.7}$$

In both systems (1.6) and (1.7), the authors used an appropriate Lyapunov functional to prove an exponential decay results. (See [1–3, 19]). The present article is a logical continuation of works [5, 10, 13] for nonlinear case with thermal, mass diffusion and thermoelastic effects.

The rest of work is organized as follows: In section 2, we recall some preliminaries and assumptions. In section 3, we state and prove the well-posedness of solution. In section 4, we prove the main stability result in both cases where $\mathcal{H}$ is linear and nonlinear.

2. Preliminary

We assume that the symmetric matrix

$$\Lambda = \begin{pmatrix} c & d \\ d & r \end{pmatrix}, \tag{2.1}$$

is positive definite, and thus for all θ, P

$$rP^2 + c\theta^2 + 2dP\theta > 0. \tag{2.2}$$

In recent years, there has been an increase in interest in the use of nonlinear properties. The value of the nonlinearity is influenced by nonlinear damping. It is associated with the development of a wave process of diffusion of the fundamental wave by waves that are far from it in frequency. To date, such nonlinear processes have not been studied fairly well in thermodiffusion effects.

The function $\mathcal{G} \in C^1(\mathbb{R}, \mathbb{R})$ is assumed to be a non-decreasing function (can be taken as $\mathcal{G}(y) = |y|^{m-2}y, m \geq 2$) such that there exist $\varepsilon, c_1, c_2 > 0$ and a convex increasing function $\mathcal{H} \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\begin{gathered} 1)\mathcal{H}(0) = 0 \text{ and } \mathcal{H} \text{ is linear on } [0, \varepsilon] \text{ or} \\ 2)\partial_t\mathcal{H}(0) = 0 \text{ and } \partial_{tt}\mathcal{H} > 0 \text{ on } [0, \varepsilon] \text{ such that} \\ \mathcal{G}(s)| \leq c_2|s| \quad \text{if } |s| > \varepsilon \\ s^2 + \mathcal{G}^2(s) \leq \mathcal{H}^{-1}(s\mathcal{G}(s)) \quad \text{if } |s| \leq \varepsilon \\ |\partial_s\mathcal{G}(s)| \leq \tau. \end{gathered} \tag{2.3}$$

The energy of solution is defined as

$$\begin{aligned} 2\mathcal{E}(t) &= \int_0^L \left[\rho_1\partial_t\varphi^2 + \alpha\psi_x^2 + \kappa(\varphi_x + \psi)^2 + \frac{\rho_1\rho_2}{\kappa}\partial_{tt}\varphi^2 + \rho_2\partial_t\varphi_x^2\right] dx \\ &\quad + \int_0^L \left[\tau_0\theta^2 + rP^2 + 2d\theta P\right] dx. \end{aligned} \tag{2.4}$$

Lemma 2.1. *The functional (2.4) satisfies*

$$\begin{aligned} \mathcal{E}'(t) &= -\delta\int_0^L \theta_x^2 dx - h\int_0^L P_x^2 dx - \sigma_1\int_0^L \partial_t\varphi^2 dx \\ &\quad -\frac{\sigma_1\rho_2}{\kappa}\int_0^L \partial_{tt}\varphi^2 dx - \sigma_2\int_0^L \partial_t\psi\mathcal{G}(\partial_t\psi)dx \\ &\leq 0. \end{aligned} \tag{2.5}$$

Proof. Multiplying the equations of (1.2) by $\partial_t\varphi, \partial_t\psi, \theta, P$ respectively, using integration by parts, and (1.4), we get

$$\begin{cases} \dfrac{\rho_1}{2}\partial_t\displaystyle\int_0^L \partial_t\varphi^2 dx + \kappa\int_0^L (\varphi_x + \psi)\partial_t\varphi_x dx + \sigma_1\int_0^L \partial_t\varphi^2 dx = 0 \\ +\rho_2\displaystyle\int_0^L \partial_{tt}\varphi\partial_t\psi_x dx + \frac{\alpha}{2}\partial_t\int_0^L \psi_x^2 dx + \kappa\int_0^L (\varphi_x + \psi)\partial_t\psi dx \\ \quad -\xi_1\displaystyle\int_0^L \theta_x\partial_t\psi dx - \xi_2\int_0^L P_x\partial_t\psi dx + \sigma_2\int_0^L \partial_t\psi\mathcal{G}(\partial_t\psi)dx = 0 \\ \dfrac{\tau_0}{2}\partial_t\displaystyle\int_0^L \theta^2 dx + d\int_0^L \partial_t P\theta dx + \delta\int_0^L \theta_x^2 dx - \xi_1\int_0^L \partial_t\psi_x\theta dx = 0 \\ \dfrac{r}{2}\partial_t\displaystyle\int_0^L P^2 dx + d\int_0^L \partial_t\theta P dx + h\int_0^L P_x^2 dx - \xi_2\int_0^L \partial_t\psi_x P dx = 0. \end{cases} \tag{2.6}$$

Then, taking the derivative $(1.2)_1$, we get

$$\partial_t \psi_x = \rho_1 \frac{\partial_t (\partial_{tt}\varphi)}{\kappa} - \partial_t (\varphi_x)_x + \frac{\sigma_1}{\kappa} \partial_{tt}\varphi. \tag{2.7}$$

Now substituting (2.7) in $(1.2)_2$ using integration by parts and summing, then by using (2.3), we obtain $\mathcal{E}$ is decreasing. □

We introduce the following Hilbert spaces

$$\mathbb{H} = H_0^1(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2(0, L) \times H_0^1(0, L). \tag{2.8}$$

3. Well-posedness of problem

In this section, we prove the existence and the uniqueness of global solution for system (1.2)–(1.4) by using the Faedo-Galerkin method.

Theorem 3.1. *Assume the assumption (2.1), (2.2) hold. If the initial data* $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \psi_0) \in \mathbb{H}, \theta_0, P_0 \in L^2(0, L)$*, then problem (1.2)–(1.4) has a weak solution such that*

$$\begin{aligned} &\varphi, \psi \in C(\mathbb{R}_+, H_0^1(0, L)) \cap C^1(\mathbb{R}_+, L^2(0, L)), \\ &\partial_t\varphi, \partial_{tt}\varphi, \theta, P \in C(\mathbb{R}_+, L^2(0, L)). \end{aligned}$$

In addition, the solution $(\varphi, \partial_t\varphi, \partial_{tt}\varphi, \psi, \theta, P)$ *depends continuously on the initial data in* $\mathbb{H} \times L^2(0, L) \times L^2(0, L)$*. In particular, problem (1.2)–(1.4) has a unique weak solution.*

Proof. By Using Faedo-Galerkin approximations, we prove the existence of unique global solution of (1.2)–(1.4). For more detail, we refer the reader to see [4, 12, 14].

3.1. Approximate problem

Let $\{u_j\}, \{v_j\}, \{\theta_j\}, \{P_j\}$ be the Galerkin basis, For $n \geq 1$, let

$$\begin{aligned} W_n &= span\{u_1, u_2, ..., u_n\} \\ K_n &= span\{v_1, v_2, ..., v_n\} \\ \Theta_n &= span\{\theta_1, \theta_2, ..., \theta_n\} \\ \Gamma_n &= span\{P_1, P_2, ..., P_n\}. \end{aligned}$$

Given initial data $(\varphi_0, \psi_0) \in H_0^1(0, L) \times H_0^1(0, L)$, $\varphi_1, \varphi_2, \varphi_3 \in L^2(0, L)$, and $\theta_0, P_0 \in L^2(0, L)$, we define the approximations

$$
\begin{aligned}
\varphi_n &= \sum_{j=1}^{n} g_{jn}(t)u_j(x)\\
\psi_n &= \sum_{j=1}^{n} \zeta_{jn}(t)v_j(x)\\
\theta_n &= \sum_{j=1}^{n} f_{jn}(t)\theta_j(x)\\
P_n &= \sum_{j=1}^{n} k_{jn}(t)P_j(x),
\end{aligned}
\tag{3.1}
$$

which satisfy the following approximate problem

$$
\begin{cases}
\rho_1(\partial_{tt}\varphi_n, u_j) + \kappa((\varphi_{nx} + \psi_n), u_{jx}) - \sigma_1(\partial_t\varphi_n, u_j) = 0\\
\alpha(\psi_{nx}, v_{jx}) + \rho_2(\partial_{tt}\varphi_n, v_{jx}) + \kappa((\varphi_{nx} + \psi_n), v_j)\\
\quad +\xi_1(\theta_n, v_{jx}) + \xi_2(P_n, v_{jx}) + \sigma_2(\mathcal{G}(\partial_t\psi_n), v_j) = 0\\
\tau_0(\partial_t\theta_n, \theta_j) + d(\partial_t P_n, \theta_j) + \delta(\theta_{nx}, \theta_{jx}) + \xi_1(\partial_t\psi_n, \theta_{jx}) = 0\\
d(\partial_t\theta_n, P_j) + r(\partial_t P_n, P_j) + h(P_{nx}, P_{jx}) + \xi_2(\partial_t\psi_n, P_{jx}) = 0,
\end{cases}
\tag{3.2}
$$

with initial conditions

$$
\begin{aligned}
&\varphi_n(0) = \varphi_0^n, \partial_t\varphi_n(0) = \varphi_1^n, \partial_{tt}\varphi_n(0) = \varphi_2^n\\
&\partial_{ttt}\varphi_n(0) = \varphi_3^n, \psi_n(0) = \psi_0^n, \partial_t\psi_n(0) = \psi_1^n,\\
&\theta_n(0) = \theta_0^n, P_n(0) = P_0^n,
\end{aligned}
\tag{3.3}
$$

which satisfies

$$
\begin{aligned}
&\varphi_0^n \to \varphi_0,\ \ strongly\ \ in\ \ H_0^1(0, L)\\
&\varphi_1^n \to \varphi_1,\ \ strongly\ \ in\ \ L^2(0, L)\\
&\varphi_2^n \to \varphi_2,\ \ strongly\ \ in\ \ L^2(0, L)\\
&\varphi_3^n \to \varphi_3,\ \ strongly\ \ in\ \ L^2(0, L)\\
&\psi_0^n \to \psi_0,\ \ strongly\ \ in\ \ H_0^1(0, L)\\
&\psi_1^n \to \psi_1,\ \ strongly\ \ in\ \ L^2(0, L)\\
&\theta_0^n \to \theta_0,\ \ strongly\ \ in\ \ L^2(0, L)\\
&P_0^n \to P_0,\ \ strongly\ \ in\ \ L^2(0, L).
\end{aligned}
\tag{3.4}
$$

By using the Caratheodory Theorem for standard ordinary differential equations theory, the problem (3.2) and (3.3) has a solutions $(g_{jn}, \zeta_{jn}, f_{jn}, k_{jn})_{j=1,n} \in (H^3[0, T])^4$ and by using the embedding $H^m[0, T] \to C^m[0, T]$, we deduce that the solution $(g_{jn}, \zeta_{jn}, f_{jn}, k_{jn})_{j=1,n} \in (C^2[0, T])^4$. In turn, this gives a unique $(\varphi_n, \psi_n, \theta_n, P_n)$ defined by (3.1) and satisfying (3.2).

3.2. The first a priori estimate

Multiplying equations of (3.2) by $\partial_t g_{jn}, \partial_t h_{jn}, \partial_t f_{jn}$ and $\partial_t k_{jn}$ respectively and using

$$\kappa\int_0^L \partial_{tt}\varphi\partial_t\psi_x dx = \rho_1\int_0^L \partial_{ttt}\varphi\partial_{tt}\varphi dx - \kappa\int_0^L \partial_t\varphi_{xx}\partial_{tt}\varphi dx + \sigma_1\int_0^L \partial_{tt}\varphi^2 dx,$$

we get

$$\begin{aligned}
&\partial_t\frac{1}{2}\Big[\rho_1\int_0^L \partial_t\varphi_n^2 dx + \frac{\rho_1\rho_2}{\kappa}\int_0^L \partial_{tt}\varphi_n^2 dx + \rho_2\int_0^L \partial_t\varphi_{nx}^2 dx + \kappa\int_0^L (\varphi_{nx}+\psi_n)^2 dx\\
&+\alpha\int_0^L \psi_{nx}^2 dx + \tau_0\int_0^L \theta_n^2 dx + r\int_0^L P_n^2 dx + 2d\int_0^L \theta_n P_n dx\Big]\\
&+\delta\int_0^L \theta_{nx}^2 dx + h\int_0^L P_{nx}^2 dx + \sigma_1\int_0^L \partial_t\varphi_n^2 dx + \frac{\sigma_1\rho_2}{\kappa}\int_0^L \partial_{tt}\varphi_n^2 dx\\
&+\sigma_2\int_0^L \partial_t\psi_n\mathcal{G}(\partial_t\psi_n) dx = 0.
\end{aligned} \tag{3.5}$$

Now integrating (3.5) and by using $(2.3)_1$, we have

$$\begin{aligned}
\mathcal{E}_n(t) \quad + \quad &\delta\int_0^t\int_0^L \theta_{nx}^2(s) dxds + h\int_0^t\int_0^L P_{nx}^2(s) dxds\\
&+\sigma_1\int_0^t\int_0^L \partial_t\varphi_n^2(s) dxds + \frac{\sigma_1\rho_2}{\kappa}\int_0^t\int_0^L \partial_t\varphi_n^2(s) dxds\\
&+\sigma_2\int_0^t\int_0^L \partial_t\psi_n\mathcal{G}(\partial_t\psi_n)(s) dxds \quad = \quad \mathcal{E}_n(0),
\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
\mathcal{E}_n(t) \quad = \quad &\frac{1}{2}\Big[\rho_1\int_0^L \partial_t\varphi_n^2 dx + \frac{\rho_1\rho_2}{\kappa}\int_0^L \partial_t\varphi_n^2 dx + \rho_2\int_0^L \partial_t\varphi_{nx}^2 dx\\
&+\kappa\int_0^L (\varphi_{nx}+\psi_n)^2 dx + \alpha\int_0^L \psi_{nx}^2 dx\\
&+\tau_0\int_0^L \theta_n^2 dx + r\int_0^L P_n^2 dx + 2d\int_0^L \theta_n P_n dx\Big].
\end{aligned} \tag{3.7}$$

Then

$$\mathcal{E}_n(t) \le \mathcal{E}_n(0). \tag{3.8}$$

Thus, there exists a positive constant C independent on n such that

$$\mathcal{E}_n(t) \le C, \quad t \ge 0. \tag{3.9}$$

By (2.1) and (3.9), we have

$$\begin{aligned}
&\int_0^L \partial_t\varphi_n^2 dx + \int_0^L \partial_{tt}\varphi_n^2 dx + \rho_2\int_0^L \partial_t\varphi_{nx}^2 dx + \int_0^L (\varphi_{nx}+\psi_n)^2 dx\\
&+\int_0^L \psi_{nx}^2 dx + \tau_0\int_0^L \theta_n^2 dx + r\int_0^L P_n^2 dx + 2d\int_0^L \theta_n P_n dx \le C.
\end{aligned} \tag{3.10}$$

Then $t_n = T$, for all $T > 0$.

3.3. The second a priori estimate

Differentiating $(3.2)_1$ and multiplying by $\partial_{tt}\varphi_n$, integrating the result over $(0, L)$, we get

$$\frac{\rho_1}{2}\partial_t \int_0^L \partial_{tt}\varphi_n^2 dx + \kappa \int_0^L \partial_t(\varphi_{nx} + \psi_n)\partial_{tt}\varphi_{nx} dx + \sigma_1 \int_0^L \partial_{tt}\varphi_n^2 dx = 0. \tag{3.11}$$

Differentiating $(3.2)_2$, and multiplying by $\partial_{tt}\psi_n$, using the fact that

$$\partial_{tt}\psi_{nx} = \frac{1}{\kappa}\left(\rho_1\partial_{tttt}\varphi_n - \kappa\partial_{tt}\varphi_{nxx} + \sigma_1\partial_{ttt}\varphi_n\right),$$

then integrating the result over $(0, L)$, using $(2.3)_2$, we get

$$\begin{aligned}
&\frac{\rho_1\rho_2}{2\kappa}\partial_t \int_0^L \partial_{ttt}\varphi_n^2 dx + \frac{\rho_2}{2}\partial_t \int_0^L \partial_{tt}\varphi_{nx}^2 dx + \kappa \int_0^L (\partial_t\varphi_{nx} + \partial_t\psi_n)\partial_{tt}\psi_n dx \\
&+\frac{\alpha}{2}\partial_t \int_0^L \partial_t\psi_{nx}^2 dx + \xi_1 \int_0^L \partial_t\theta_n\partial_{tt}\psi_{nx} dx + \xi_2 \int_0^L \partial_t P_n \partial_{tt}\psi_{nx} dx \\
&+\frac{\sigma_1\rho_2}{\kappa} \int_0^L \partial_{tt}\varphi_n^2 dx + \sigma_2 \int_0^L \partial_t\mathcal{G}(\partial_t\psi_n)\partial_{tt}\psi_n^2 dx = 0.
\end{aligned} \tag{3.12}$$

Differentiating the equations of (3.2), multiplying by $\partial_t\theta_n, \partial_t P_n$, and then integrating the result over $(0, L)$, we get

$$\begin{aligned}
&\frac{\tau_0}{2}\partial_t \int_0^L \partial_t\theta_n^2 dx + \frac{r}{2}\partial_t \int_0^L \partial_t P_n^2 dx + d\partial_t \int_0^L \partial_t\theta_n\partial_t P_n dx \\
&+\xi_1 \int_0^L \partial_t\theta_{nx}\partial_{tt}\psi_n dx + \xi_2 \int_0^L \partial_t P_{nx}\partial_{tt}\psi_n dx \\
&+\delta \int_0^L \partial_t\theta_{nx}^2 dx + h \int_0^L \partial_t P_{nx}^2 dx = 0.
\end{aligned} \tag{3.13}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned}
\mathcal{R}_n(t) \quad + \quad & \delta \int_0^t \int_0^L \partial_t\theta_{nx}^2 dxds + h \int_0^t \int_0^L \partial_t P_{nx}^2 dxds + \sigma_1 \int_0^t \int_0^L \partial_t\varphi_n^2 dxds \\
& +\frac{\sigma_1\rho_2}{\kappa} \int_0^t \int_0^L \partial_{tt}\varphi_n^2 dxds + \sigma_2 \int_0^t \int_0^L \mathcal{G}(\partial_t\psi_n)\partial_{tt}\psi_n dxds \\
= \quad & \mathcal{R}_n(0),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_n(t) \quad = \quad & \frac{1}{2}\Big[\rho_1 \int_0^L \partial_{tt}\varphi_n^2 dx + \frac{\rho_1\rho_2}{\kappa} \int_0^L \partial_{ttt}\varphi_n^2 dx + \rho_2 \int_0^L \partial_{tt}\varphi_{nx}^2 dx \\
& +\kappa \int_0^L (\partial_t\varphi_{nx} + \partial_t\psi_n)^2 dx + \alpha \int_0^L \partial_t\psi_{nx}^2 dx \\
& +\tau_0 \int_0^L \partial_t\theta_n^2 dx + r \int_0^L \partial_t P_n^2 dx + 2d \int_0^L \partial_t\theta_n\partial_t P_n dx\Big].
\end{aligned} \tag{3.14}$$

As in the fist a priori estimate, there exists $C > 0$ independent on n such that

$$\mathcal{R}_n(t) \leq C, \quad t \geq 0. \tag{3.15}$$

Passage to limit

From (3.10) and (3.14), we conclude that for any $n \in \mathbb{N}$,

$$\begin{array}{rl} \varphi_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, H_0^1(0,L)) \\ \partial_t\varphi_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ \partial_{tt}\varphi_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ \psi_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, H_0^1(0,L)) \\ \partial_t\psi_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ \theta_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ \partial_t\theta_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ P_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)) \\ \partial_t P_n & \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2(0,L)). \end{array} \tag{3.16}$$

Thus we get

$$\begin{array}{rl} \varphi_n & \textit{weakly star in } L^2(\mathbb{R}_+, H_0^1(0,L)) \\ \partial_t\varphi_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ \partial_{tt}\varphi_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ \psi_n & \textit{weakly star in } L^2(\mathbb{R}_+, H_0^1(0,L)) \\ \partial_t\psi_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ \theta_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ \partial_t\theta_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ P_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)) \\ \partial_t P_n & \textit{weakly star in } L^2(\mathbb{R}_+, L^2(0,L)). \end{array} \tag{3.17}$$

By (3.17), we deduce that φ_n, ψ_n is bounded in $L^2(\mathbb{R}_+, H_0^1(0,L))$ and $\partial_t\varphi_n, \partial_{tt}\varphi_n$ are bounded in $L^2(\mathbb{R}_+, L^2(0,L))$, and $\partial_t\theta_n, \partial_t P_n$ are bounded in $L^2(\mathbb{R}_+, L^2(0,L))$. Then from Aubin-Lions theorem [18], we infer that for and, $T > 0$,

$$\begin{array}{rl} \varphi_n & \textit{strongly in } L^\infty(0,T, H_0^1(0,L)) \\ \psi_n & \textit{strongly in } L^\infty(0,T, H_0^1(0,L)) \\ \theta_n & \textit{strongly in } L^\infty(0,T, L^2(0,L)) \\ P_n & \textit{strongly in } L^\infty(0,T, L^2(0,L)). \end{array} \tag{3.18}$$

We also obtain by Lemma 1.4 in Kim [15] that

$$
\begin{array}{lll}
\varphi_n & \textit{strongly in} & C(0,T,H_0^1(0,L)) \\
\psi_n & \textit{strongly in} & C(0,T,H_0^1(0,L)) \\
\theta_n & \textit{strongly in} & C(0,T,L^2(0,L)) \\
P_n & \textit{strongly in} & C(0,T,L^2(0,L)).
\end{array} \tag{3.19}
$$

Then we can pass to limit the approximate problem (3.2) and (3.3) in order to get a weak solution of problem (1.2)–(1.4).

3.4. Continuous dependence and uniqueness

We prove the continuous dependence of unique solution of (1.2)–(1.4).
Let $(\varphi, \partial_t\varphi, \partial_{tt}\varphi, \psi, \Upsilon, \Psi)$, and $(\Gamma, \partial_t\Gamma, \partial_{tt}\Gamma, \Xi, \Pi, \Omega)$ be two global solutions of (1.2)–(1.4) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \Theta_0, \Psi_0)$, and $(\Gamma_0, \Gamma_1, \Gamma_2, \Xi_0, \Phi_0, \Omega_0)$.
Let

$$
\begin{aligned}
\Lambda(t) &= \varphi - \Gamma \\
\Sigma(t) &= \psi - \Xi \\
\chi(t) &= \Pi - \Phi \\
M(t) &= \Psi - \Omega.
\end{aligned} \tag{3.20}
$$

Then $(\Lambda, \Sigma, \chi, M)$ verifies (1.2)–(1.4), and we have

$$
\begin{cases}
\rho_1\partial_{tt}\Lambda - \kappa(\Lambda_x + \Sigma)_x + \sigma_1\partial_t\Lambda = 0 \\
-\rho_2\partial_{tt}\Lambda_x - \alpha\Sigma_{xx} + \kappa(\Lambda_x + \Sigma) - \xi_1\chi_x - \xi_2 M_x + \sigma_2\mathcal{G}(\partial_t\Sigma) = 0 \\
\tau_0\partial_t\chi + d\partial_t M - \delta\chi_{xx} - \xi_1\partial_t\Sigma_x = 0 \\
d\partial_t\chi + r\partial_t M - hM_{xx} - \xi_2\partial_t\Sigma_x = 0.
\end{cases} \tag{3.21}
$$

Multiplying $(3.21)_1$ by $\partial_t\Lambda$, $(3.21)_2$ by $\partial_t\Sigma$, integrating over $(0, L)$, and since

$$
\begin{aligned}
\kappa\int_0^L \partial_{tt}\Lambda\partial_t\Sigma_x dx &= \rho_1\int_0^L \partial_{ttt}\Lambda\partial_{tt}\Lambda dx - \kappa\int_0^L \partial_t\Lambda_{xx}\partial_{tt}\Lambda dx \\
&+ \sigma_1\int_0^L \partial_{tt}\Lambda^2 dx,
\end{aligned} \tag{3.22}
$$

we get

$$
\begin{aligned}
&\partial_t\frac{1}{2}\Big[\rho_1\int_0^L \partial_t\Lambda^2 dx + \frac{\rho_1\rho_2}{\kappa}\int_0^L \partial_{tt}\Lambda^2 dx + \rho_2\int_0^L \partial_t\Lambda_x^2 dx + \kappa\int_0^L (\Lambda_x + \Sigma)^2 dx \\
&\quad + \alpha\int_0^L \Sigma_x^2 dx + \tau_0\int_0^L \chi^2 dx + r\int_0^L M^2 dx + 2d\int_0^L \chi M dx\Big] \\
&+ \delta\int_0^L \chi_x^2 dx + h\int_0^L M_x^2 dx + \sigma_1\int_0^L \partial_t\Lambda^2 dx + \frac{\sigma_1\rho_2}{\kappa}\int_0^L \partial_{tt}\Lambda^2 dx \\
&+ \sigma_2\int_0^L \partial_t\Sigma\mathcal{G}(\partial_t\Sigma) dx = 0.
\end{aligned} \tag{3.23}
$$

Then

$$
\begin{aligned}
\partial_t\mathcal{E}(t) &\leq 0\\
&\leq c\Big(\int_0^L \partial_t\Lambda^2 dx + \int_0^L \partial_{tt}\Lambda^2 dx + \int_0^L \partial_t\Lambda_x^2 dx + \int_0^L \Sigma_x^2 dx + \int_0^L (\Lambda_x+\Sigma)^2 dx\\
&\quad + \int_0^L \chi^2 dx + \int_0^L M^2 dx\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\Big[\rho_1\int_0^L \partial_t\Lambda^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^L \partial_{tt}\Lambda^2 dx + \rho_2\int_0^L \partial_t\Lambda_x^2 dx + \beta\int_0^L (\Lambda_x+\Sigma)^2 dx\\
&\quad +\alpha\int_0^L \Sigma_x^2 dx + \tau_0\int_0^L \chi^2 dx + r\int_0^L M^2 dx + 2d\int_0^L \chi M dx\Big]. \qquad (3.24)
\end{aligned}
$$

By integrating (3.23), we get

$$
\begin{aligned}
\mathcal{E}(t) &\leq \mathcal{E}(0) + C_1\int_0^t \Big(||\partial_t\Lambda||^2 + ||\partial_{tt}\Lambda||^2 + ||\partial_t\Lambda_x||^2 + ||\Sigma_x||^2 + ||(\Lambda_x+\Sigma)||^2\\
&\quad +||\chi||^2 + ||M||^2\Big)ds. \qquad (3.25)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathcal{E}(t) &\geq c_0\Big(||\partial_t\Lambda||^2 + ||\partial_{tt}\Lambda||^2 + ||\partial_t\Lambda_x||^2 + ||\Sigma_x||^2 + ||(\Lambda_x+\Sigma)||^2\\
&\quad +||\chi||^2 + ||M||^2\Big). \qquad (3.26)
\end{aligned}
$$

Owing to Gronwall's inequality to (3.27), we have

$$
\begin{aligned}
&(||\partial_t\Lambda||^2 + ||\partial_{tt}\Lambda||^2 + ||\partial_t\Lambda_x||^2 + ||\Sigma_x||^2 + ||(\Lambda_x+\Sigma)||^2\\
&+||\chi||^2 + ||M||^2) \leq e^{C_2 t}\mathcal{E}(0), \qquad (3.27)
\end{aligned}
$$

which implies that solution of (1.2)–(1.4) depends continuously on the initial data. □

4. Asymptotic behavior

Using the multiplied techniques, we prove the stability result.

Theorem 4.1.

- *Assume that (2.1), (2.2) and* $(2.3)_1$ *hold. Then, there exist positive constants* λ_1 *and* λ_2 *such that (2.4) satisfies*
$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \forall t \geq 0. \qquad (4.1)$$
- *Assume that (2.1), (2.2) and* $(2.3)_2$ *hold. Then, there exist positive constants* β *and* γ *such that (2.4) satisfies*
$$\mathcal{E}(t) \leq \beta\mathcal{H}_0^{-1}(\frac{\gamma}{t}). \qquad (4.2)$$

where $\mathcal{H}_0(t) = t\partial_t\mathcal{H}(\varepsilon_0 t),\ \ \forall\varepsilon_0 > 0.$

First, we need to introduce an auxiliary Lemmas.
Let

$$F_1(t) = -\frac{\sigma_1}{2}\int_0^L \partial_t\varphi^2 dx - \kappa\int_0^L \partial_t\varphi_x\varphi_x dx, \tag{4.3}$$

$$\begin{aligned} F_2(t) &= \rho_1\int_0^L \varphi\partial_t\varphi dx + \frac{\sigma_1}{2}\int_0^L \varphi^2 dx \\ &+ \frac{\sigma_1\rho_2}{2\kappa}\int_0^L \partial_t\varphi^2 dx + \rho_2\int_0^L \partial_t\varphi_x\varphi_x dx. \end{aligned} \tag{4.4}$$

Lemma 4.2. *The functional $F_1(t)$ satisfies*

$$F_1'(t) \le -\kappa\int_0^L \partial_t\varphi_x^2 dx + \varepsilon_1\int_0^L \psi_x^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^L \partial_{tt}\varphi^2 dx. \tag{4.5}$$

Proof. Direct computation using integration by parts, we get

$$F_1'(t) = \rho_1\int_0^L \partial_{tt}\varphi^2 dx - \kappa\int_0^L \psi_x\partial_{tt}\varphi dx - \kappa\int_0^L \partial_t\varphi_x^2 dx.$$

Owing to Young and Poincare's inequalities, we obtain (4.5). □

Lemma 4.3. *The functional $F_2(t)$ satisfies,*

$$\begin{aligned} F_2(t) \le& -\frac{\kappa}{2}\int_0^L (\varphi_x+\psi)^2 dx - \frac{\alpha}{2}\int_0^L \psi_x^2 dx + c\int_0^L \partial_{tt}\varphi^2 dx \\ &+\rho_2\int_0^L \partial_t\varphi_x^2 dx + \rho_1\int_0^L \partial_t\varphi^2 dx + c\int_0^L \theta_x^2 dx \\ &+c\int_0^L P_x^2 dx + c\int_0^L P_x^2 dx + c\int_0^L \mathcal{G}^2(\partial_t\psi)dx. \end{aligned} \tag{4.6}$$

Proof. Differentiating F_2, by (1.4) and integration by parts, we have

$$F_2'(t) = \rho_1\int_0^L \partial_t\varphi^2 dx - \kappa\int_0^L (\varphi_x+\psi)^2 dx - \alpha\int_0^L \psi_x^2 dx \tag{4.7}$$

$$\begin{aligned} &-\frac{\rho_1\rho_2}{\kappa}\int_0^L \partial_{tt}\varphi^2 dx + \xi_1\int_0^L \theta_x\psi dx - \rho_2\int_0^L \partial_{tt}\varphi\psi_x dx \\ &+\rho_2\int_0^L \partial_t\varphi_x^2 dx + \xi_2\int_0^L P_x\psi dx + \sigma_2\int_0^L \psi\mathcal{G}(\partial_t\psi)dx. \end{aligned} \tag{4.8}$$

Owing to Young and Poincaré's inequalities, we get (4.6). □

Proof. (Of Theorem 4.1). We define an appropriate Lyapunov functional as

$$\mathcal{L}(t) = N\mathcal{E}(t) + N_1F_1(t) + F_2(t), \tag{4.9}$$

where $N, N_1 > 0$. By differentiating (4.9) and using (2.5), (4.5) and (4.6) we have

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -[N\sigma_1-\rho_2]\int_0^L \partial_t\varphi^2 dx - \left[\frac{\alpha}{2}-\varepsilon_1 N_1\right]\int_0^L \psi_x^2 dx \\
& -\left[\frac{\sigma_1\rho_2}{\kappa}N+\frac{\rho_1\rho_2}{\kappa}-\frac{2\rho_2^2}{\alpha}-cN_1(1+\frac{1}{\varepsilon_1})\right]\int_0^L \partial_{tt}\varphi^2 dx \\
& -\kappa\int_0^L(\varphi_x+\psi)^2dx-[\kappa N_1-\rho_2]\int_0^L \partial_t\varphi_x^2 dx + c\int_0^L \mathcal{G}^2(\partial_t\psi)dx \\
& -[Nh-c]\int_0^L P_x^2 dx-[N\delta-c]\int_0^L \theta_x^2 dx - N\sigma_2\int_0^L \partial_t\psi\mathcal{G}(\partial_t\psi)dx.
\end{aligned}$$

By setting $\varepsilon_1=\frac{\alpha}{4N_1}$, and we choose N_1 large enough so that

$$\alpha_1=\kappa N_1-\rho_2>0,$$

thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -[N\sigma_1-\rho_1]\int_0^L \partial_t\varphi^2 dx - \left[\frac{\sigma_1\rho_2}{\kappa}N+\alpha_2-c\right]\int_0^L \partial_{tt}\varphi^2 dx \\
& -\alpha_3\int_0^L \psi_x^2 dx-\alpha_4\int_0^L(\varphi_x+\psi)^2dx-\alpha_1\int_0^L \partial_t\varphi_x^2 dx \\
& -[Nh-c]\int_0^L P_x^2 dx-[N\delta-c]\int_0^L \theta_x^2 dx \\
& -N\sigma_2\int_0^L \partial_t\psi\mathcal{G}(\partial_t\psi)dx + c\int_0^L \mathcal{G}^2(\partial_t\psi)dx, \qquad (4.10)
\end{aligned}$$

where $\alpha_2=\frac{\rho_1\rho_2}{\kappa}-\frac{2\rho_2^2}{\alpha}, \alpha_3=\frac{\alpha}{4}, \alpha_4=\kappa$.
On the other hand, if we let

$$\mathcal{K}(t)=N_1F_1(t)+F_2(t),$$

then

$$\begin{aligned}
|\mathcal{K}(t)| \quad \leq \quad & \frac{\sigma_1}{2}N_1\int_0^L \partial_t\varphi^2 dx+\kappa N_1\int_0^L|\partial_t\varphi_x\varphi_x|dx+\rho_1\int_0^L|\varphi\partial_t\varphi|dx \\
& +\frac{\sigma_1}{2}\int_0^L\varphi^2dx+\frac{\sigma_1\rho_2}{2\kappa}\int_0^L\partial_t\varphi^2dx+\rho_2\int_0^L|\partial_t\varphi_x\varphi_x|dx.
\end{aligned}$$

By using Young, Poincaré's inequalities, and the fact that

$$\int_0^L\varphi^2dx\leq 2c\int_0^L(\varphi_x+\psi)^2dx+2c\int_0^L\psi_x^2dx,$$

we get

$$
\begin{aligned}
|\mathcal{K}(t)| &\le c\int_0^L \left(\partial_t\varphi^2+\partial_t\varphi_x^2+\psi_x^2+\partial_{tt}\varphi^2+(\varphi_x+\psi)^2\right)dx \\
&\le c\mathcal{E}(t)\,.
\end{aligned}
$$

Consequently,

$$
|\mathcal{H}(t)| = |\mathcal{L}(t)-N\mathcal{E}(t)| \le c\mathcal{E}(t)\,,
$$

which yield

$$
(N-c)\,\mathcal{E}(t) \le \mathcal{L}(t) \le (N+c)\,\mathcal{E}(t). \tag{4.11}
$$

By choosing N large enough such that

$$
\frac{\sigma_1\rho_2}{\kappa}N+\alpha_2-c>0, N\sigma_1-\rho_1>0, N-c>0, N\delta-c>0, Nh-c>0,
$$

we obtain

$$
c_1\mathcal{E}(t) \le \mathcal{L}(t) \le c_2\mathcal{E}(t), \forall t\ge 0. \tag{4.12}
$$

Using (2.5), estimates (4.10), (4.11), respectively, we get

$$
\mathcal{L}'(t) \le -k_1\mathcal{E}(t)+k_2\int_0^L \mathcal{G}^2(\partial_t\psi)dx, \forall t\ge t_0, \tag{4.13}
$$

for some $k_1, k_2, c_1, c_2 > 0$.
At this point, we distinguish two cases:

- If $\mathcal{H}$ is linear on $[0,\varepsilon]$, In this case, using the assumption $(2.3)_1$, we can write

$$
\begin{aligned}
k_2\int_0^L \mathcal{G}^2(\partial_t\psi)dx &\le k_2\int_0^L (\partial_t\psi^2+\mathcal{G}^2(\partial_t\psi))dx \\
&\le k_2\int_0^L \partial_t\psi\mathcal{G}(\partial_t\psi))dx \le -k_3\partial_t\mathcal{E}(t),
\end{aligned} \tag{4.14}
$$

  where $k_3 = \frac{k_2}{\sigma_2}$.
  Inserting (4.14) in (4.13). Then, we have

$$
\mathcal{L}_1'(t) \le -k_1\mathcal{E}(t)\,, \tag{4.15}
$$

  and

$$
m_1\mathcal{E}(t) \le \mathcal{L}_1(t) \le m_2\mathcal{E}(t), \tag{4.16}
$$

  with

$$
m_1 = c_1, \quad m_2 = c_2+k_3\mathcal{E}(0),
$$

  where

$$
\mathcal{L}_1(t) = \mathcal{L}(t)+k_3\mathcal{E}(t) \sim \mathcal{E}(t). \tag{4.17}
$$

A combination (4.15) with (4.17), gives

$$\mathcal{L}_1'(t) \le -\lambda_1 \mathcal{L}_1(t), \tag{4.18}$$

where $\lambda_1 = \frac{k_1}{m_2}$.
A simple integration of (4.18), we obtain (4.1).

- If $\mathcal{H}$ is nonlinear on $[0, \varepsilon]$, we choose $0 \le \varepsilon_1 \le \varepsilon$ and let us consider

$$I_1(t) = \{x \in (0, L),\ \ |\partial_t \psi| \le \varepsilon_1\},\ \ I_2 = \{x \in (0, L),\ \ |\partial_t \psi| > \varepsilon_1\},$$

we define

$$I = \int_{I_1} \partial_t \psi \mathcal{G}(\partial_t \psi) dt.$$

Using Jensen's inequality and the assumption $(2.3)_2$, we have

$$\begin{aligned} k_2 \int_0^L (\partial_t \psi^2 + \mathcal{G}^2(\psi_t)) dx &\le k_2 \int_0^L \partial_t \psi \mathcal{G}(\partial_t \psi)) dx \\ &\le k_4 \mathcal{H}^{-1}(I(t)) - k_4 \partial_t \mathcal{E}(t). \end{aligned} \tag{4.19}$$

Inserting (4.19) in (4.13), $\partial_t \mathcal{E}(t) \le 0$, we obtain

$$\mathcal{L}_2'(t) \le -k_1 \mathcal{E}(t) + k_4 \mathcal{H}^{-1}(I(t)), \tag{4.20}$$

and

$$m_3 \mathcal{E}(t) \le \mathcal{L}_2(t) \le m_4 \mathcal{E}(t), \tag{4.21}$$

with

$$m_3 = c_1, \quad m_4 = c_2 + k_4 \mathcal{E}(0),$$

where

$$\mathcal{L}_2(t) = \mathcal{L}(t) + k_4 \mathcal{E}(t) \sim \mathcal{E}(t). \tag{4.22}$$

Now, for $\varepsilon_0 < \varepsilon_1$ and by using $\partial_t \mathcal{E}(t) \le 0, \partial_t \mathcal{H} > 0$ and $\partial_{tt} \mathcal{H} > 0$ on $(0, \varepsilon]$, we define the functional $\mathcal{L}_3(t)$ by

$$\mathcal{L}_3(t) = \partial_t \mathcal{H}(\varepsilon_0 \mathcal{E}(t)) \mathcal{L}_2(t) \sim \mathcal{E}(t),$$

satisfies

$$\begin{aligned} \partial_t \mathcal{L}_3(t) &= \partial_t \mathcal{E}(t)(\varepsilon_0 \partial_t \mathcal{H}(\varepsilon_0 \mathcal{E}(t)) \mathcal{L}_2(t)) + \partial_t \mathcal{L}_2(t) \partial_t \mathcal{H}(\varepsilon_0 \mathcal{E}(t)) \\ &\le -k_1 \mathcal{E}(t) \partial_t \mathcal{H}(\varepsilon_0 \mathcal{E}(t)) + k_4 \partial_t \mathcal{H}(\varepsilon_0 \mathcal{E}(t)) \mathcal{H}^{-1}(I(t)). \end{aligned} \tag{4.23}$$

To estimate the last term of (4.20), using the general Young's inequality

$$AB \le \mathcal{H}^*(A) + \mathcal{H}(B), \quad if\ \ A \in (0, \partial_t \mathcal{H}(\varepsilon)),\ \ B \in (0, \varepsilon),$$

where

$$\mathcal{H}^*(A) = s(\partial_t\mathcal{H})^{-1}(s) - \mathcal{H}((\partial_t\mathcal{H})^{-1}(s)), \quad if \;\; s \in (0, \partial_t\mathcal{H}(\varepsilon)),$$

satisfies

$$k_4\partial_t\mathcal{H}(\varepsilon_0\mathcal{E}(t))\mathcal{H}^{-1}(I(t)) \;\; \leq \;\; k_1\varepsilon_0\mathcal{H}_0(E(t)) - k_4\partial_t\mathcal{E}(t). \tag{4.24}$$

Inserting (4.24) in (4.20) and letting $\varepsilon_0 = \dfrac{k_1}{2k_4}$, we get

$$\partial_t\mathcal{L}_3(t) + k_4\partial_t\mathcal{E}(t) \;\; \leq \;\; -k_1\mathcal{H}_0(\mathcal{E}(t)). \tag{4.25}$$

Then

$$\partial_t\mathcal{L}_4(t) \;\; \leq \;\; -k_1\mathcal{H}_0(\mathcal{E}(t)), \tag{4.26}$$

where

$$\mathcal{L}_4(t) \;\; = \;\; \mathcal{L}_3(t) + k_4\mathcal{E}(t) \sim \mathcal{E}(t).$$

Since $\mathcal{H}_0(\mathcal{E}(t)), \partial_t\mathcal{H}(\varepsilon_0\mathcal{E}(t))$ are non-increasing functions. Then, by integrating (4.26) for any $T > 0$, we get

$$k_1\mathcal{H}_0(\mathcal{E}(T)) \;\; \leq \;\; \mathcal{L}_4(0),$$

which gives (4.2). The proof is completed.

□

5. Conclusions

Our research falls within the scope of the modern interests, it is considered among the issues that have wide applications in modern science and engineering related to the energy systems. The importance of this research, although it is theoretical, lies in the following:

1. There are several generalizations and contributions that are very important in terms of the system itself. We proposed a system related to a large number of Bresse-Timoshenko type with the presence of three different types of damping, each one has functionality and physical properties, and we look at the overlapping of these three terms.
2. The great importance lies in the presence of a non-linear sources, which makes the problem have a very wide applications and importan in terms of applications in modern science.
3. Qualitatively, we proposed a new tools to study the asymptotic behavior of solutions commensurate with the existence of nonlinear term after proving the existence of the solution using a usual method. We found a new decay rate of system's energ, although the system's energy decreased according to a very general rate that includes all previous results and more than that, so, to our knowledge, there is no generalization more than this.

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Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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