Mathematics

Research article

A relaxed projection method using a new linesearch for the split feasibility problem

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Abstract: In this work, we propose a new relaxed projection algorithm for the split feasibility problem with a new linesearch. The proposed method does not require the computation on the matrix inverse and the largest eigenvalue of the matrix. We then prove some weak convergence theorems under suitable conditions in the framework of Hilbert spaces. Finally, we give some numerical examples in signal processing to validate the theoretical analysis results. The obtained results improve the corresponding results in the literature.

1. Introduction

In this work, we study the split feasibility problem (shortly, **(SFP)**)) which is formulated as follows:

$$\text{Find a point } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear operator. This problem was first proposed by Censor and Elfving [5] in Euclidean spaces (for recent results on the problem **(SFP)**, see [10, 12, 13, 25]). There are some related topic for splitting problems see [14, 17, 22, 24]. Many applications in real world such

as image reconstruction, signal recovery and intensity-modulated radiation therapy (shortly, (IMRT)) can be seen as solving the problem **(SFP)**.

Byrne [3, 4] proposed the following projection algorithm for solving the problem **(SFP)**:

$$x_{k+1} = P_C(x_k - \alpha A^*(I - P_Q)Ax_k) \tag{1.2}$$

where $\alpha \in (0, 2/\|A^*A\|)$, A^* is the adjoint operator of A, P_C and P_Q are projections onto C and Q, respectively. This method is often called the CQ-algorithm. However, it is observed that, in general, the projections P_C and P_Q are not an easy task to be computed.

To eliminate this difficulty, Fukushima [15] introduced the level sets C and Q which are defined by

$$C = \{x \in H_1 : c(x) \le 0\}, \quad Q = \{y \in H_1 : q(y) \le 0\} \tag{1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and subdifferential functions, and ∂c and ∂q are bounded operators. Fukushima [15] also introduced a relaxed projection algorithm for solving variational inequality problem and gave numerical experiments to support main result.

In 2004, Yang [27] presented a relaxed CQ-algorithm for solving the problem **(SFP)**, by replacing P_C and P_Q by P_{C_k} and P_{Q_k}, respectively. In this case, C_k and Q_k are defined by

$$C_k = \{x \in H_1 : c(x_k) \le \langle \xi_k, x_k - x \rangle\}, \tag{1.4}$$

where $\xi_k \in \partial c(x_k)$, and

$$Q_k = \{y \in H_2 : q(Ax_k) \le \langle \zeta_k, Ax_k - y \rangle\}, \tag{1.5}$$

where $\zeta_k \in \partial q(Ax_k)$.

It is easily seen that $C \subset C_k$ and $Q \subset Q_k$ for all $k \ge 1$. We note that P_{C_k} and P_{Q_k} are easily computed since these sets are half-spaces.

Precisely, Yang [27] proposed the following relaxed CQ-algorithm:

$$x_{k+1} = P_{C_k}(x_k - \alpha_k(A^*(I - P_{Q_k})Ax_k)), \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 2/\|A^*A\|)$, A^* is the adjoint operator of A and C_k and Q_k are given by (1.4) and (1.5), respectively.

Recently, Gibali et al. [16] (see also Qu and Xiu [20]) proposed the modification of the relaxed CQ-algorithm by using the linesearch technique in real Hilbert spaces. For each $k \ge 1$, define a mapping $F_k : H_1 \to H_2$ by

$$F_k(x) = \frac{1}{2}\|Ax - P_{Q_k}Ax\|^2, \quad \nabla F_k(x) = A^*(I - P_{Q_k})Ax. \tag{1.7}$$

They constructed the following algorithm:

Algorithm 1.1. Given constants $\gamma > 0$, $\ell \in (0, 1)$ and $\mu \in (0, 1)$. Let x_1 be arbitrary in H_1. For each $k \ge 1$, calculate

$$y_k = P_{C_k}(x_k - \alpha_k \nabla F_k(x_k)), \tag{1.8}$$

where $\alpha_k = \gamma \ell^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\alpha_k\|\nabla F_k(x_k) - \nabla F_k(y_k)\| \le \mu\|x_k - y_k\|. \tag{1.9}$$

Construct the next iterative step x_{k+1} by

$$x_{k+1} = P_{C_k}(x_k - \alpha_k \nabla F_k(y_k)). \tag{1.10}$$

There have been various methods invented for solving the problem **(SFP)** (see, for example, [5–12, 14–19,21,26]). It is observed that, in each iteration, we have to find the integer m_k such that (1.9) holds. So it takes CPU time and number of iterations in convergence. It is our aim to design new linesearch that takes CPU time less than Algorithm 1.1 and others.

Recently, Tian and Zhang [23] proposed a new regularized CQ algorithm without a priori knowledge of the operator norm to solve the split feasibility problem and provided strong convergence theorem in Hilbert spaces. Later, Dang et. al. [11] introduced inertial relaxed CQ algorithms for the split feasibility problem and proved asymptotical convergence. In 2018, Dong et. al. [13] introduced the projection and contraction methods for solving the split feasibility problem using the inverse strongly monotone property of the underlying operator of the SFP.

Motivated by the previous works, we modify algorithm of Gibali et al. [16] and propose the relaxed CQ-algorithm with the new linesearch in real Hilbert spaces and prove some weak convergence theorems of the proposed method under mild assumptions. Finally, we give some computational results to compare with the algorithms of Gibali et al. [16] and Yang [27]. Numerical experiments show that our algorithm has a better performance than others.

2. Preliminaries

In this section, we give some definitions and lemmas which are used in the proof. Throughout this paper, we use the following notations:

- $\rightharpoonup$ denotes the weak convergence;
- $\omega_w(x_k) = \{x : \exists (x_{k_n}) \subset (x_k) \text{ such that } x_{k_n} \rightharpoonup x\}$ denotes the weak ω-limit set of (x_k).

Let H be a real Hilbert space. We recall the following definition.

(1) A mapping $T : H \to H$ is said to be *firmly nonexpansive* if, for all $x, y \in H$,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2. \tag{2.1}$$

(2) A function $f : H \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{2.2}$$

for all $\lambda \in (0, 1)$ and for all $x, y \in H$.

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle \tag{2.3}$$

for all $z \in H$.

(3) An element $g \in H$ is called a *subgradient* of $f : H \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x \rangle \tag{2.4}$$

for all $z \in H$, which is called the *subdifferentiable inequality*. The set of subgradients of f at x is denoted by $\partial f(x)$.

(4) A function $f : H \to \mathbb{R}$ is said to be *subdifferentiable* at x if it has at least one subgradient at x. A function f is said to be *subdifferentiable* if it is subdifferentiable at all $x \in H$.

(5) A function $f : H \to \mathbb{R}$ is said to be *weakly lower semi-continuous* (shortly, *w*-lsc) at x if $x_k \rightharpoonup x$ implies

$$f(x) \le \liminf_{k \to \infty} f(x_k). \tag{2.5}$$

(6) A projection of H onto a nonempty closed convex subset $C \subset H$ is defined by

$$P_C x := \arg \min_{y \in C} \|x - y\|^2 \tag{2.6}$$

for all $x \in H$.

Next, we give some useful lemmas for our proof.

Lemma 2.1. [2] *Let C be a nonempty closed convex subset of a real Hilbert space H. Then, for any $x \in H$, the following assertions hold:*

(1) $\langle x - P_C x, z - P_C x \rangle \le 0$ *for all $z \in C$;*
(2) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle$ *for all $x, y \in H$;*
(3) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2$ *for all $z \in C$.*

From Lemma 2.1, we see that the operator $I - P_C$ is also firmly nonexpansive, where I denotes the identity operator, i.e., for any $x, y \in H$,

$$\langle (I - P_C)x - (I - P_C)y, x - y \rangle \ge \|(I - P_C)x - (I - P_C)y\|^2. \tag{2.7}$$

Lemma 2.2. [1] *Let S be a nonempty closed convex subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H satisfying the following properties:*

(a) $\lim_{k \to \infty} \|x_k - x\|$ *exists for each $x \in S$;*
(b) $\omega_w(x_k) \subset S$.

Then $\{x_k\}$ converges weakly to a point in S.

Now, we give our main results in this paper.

3. Main results

In this section, we introduce a new relaxed projection algorithm using the linesearch technique and prove its weak convergence. We have already defined the mappings F_k and ∇F_k as in (1.7). Our algorithm is generated by the following manner:

Algorithm 3.1. Given constants $\gamma > 0$, $\ell \in (0, 1)$ and $\mu \in (0, \frac{1}{4})$. Let x_1 be arbitrary in H_1. For each $k \ge 1$, calculate

$$y_k = P_{C_k}(x_k - \alpha_k \nabla F_k(x_k)). \tag{3.1}$$

Construct the next iterative step x_{k+1} by

$$x_{k+1} = P_{C_k}(y_k - \alpha_k \nabla F_k(y_k)), \tag{3.2}$$

where $\alpha_k = \gamma \ell^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\alpha_k \max\{\|\nabla F_k(x_{k+1}) - \nabla F_k(y_k)\|, \|\nabla F_k(y_k) - \nabla F_k(x_k)\|\}$$
$$\le \mu(\|x_{k+1} - y_k\| + \|y_k - x_k\|). \tag{3.3}$$

It is remarked that Algorithm 3.1 is based on extra-gradient type method which was firstly introduced by Noor [28, 29].

Following the proof line as in [20], we obtain the following lemma:

Lemma 3.2. *Let $\gamma > 0$, $\ell \in (0, 1)$ and $\mu \in (0, \frac{1}{4})$. Then, the linesearch (3.3) terminates after a finite number of steps. In addition, we have the following:*

$$\frac{\mu\ell}{L} < \alpha_k \le \gamma \tag{3.4}$$

for all $k \ge 1$, where $L = \|A\|^2$.

Proof. From [20], we know that

$$\|\nabla F_k(x_{k+1}) - \nabla F_k(y_k)\| \le L\|x_{k+1} - y_k\| \tag{3.5}$$

and

$$\|\nabla F_k(y_k) - \nabla F_k(x_k)\| \le L\|y_k - x_k\|. \tag{3.6}$$

It follows that

$$\begin{aligned}
\max\{&\|\nabla F_k(x_{k+1}) - \nabla F_k(y_k)\|, \|\nabla F_k(y_k) - \nabla F_k(x_k)\|\} \\
&\le \|\nabla F_k(x_{k+1}) - \nabla F_k(y_k)\| + \|\nabla F_k(y_k) - \nabla F_k(x_k)\| \\
&\le L(\|x_{k+1} - y_k\| + \|y_k - x_k\|).
\end{aligned} \tag{3.7}$$

The second part is obtained by [20]. Hence the linesearch (3.3) is well-defined. This completes the proof. $\square$

In what follows, we denote S by the solution set of the problem **(SFP)** and also assume that S is nonempty.

Theorem 3.3. *The sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to a point in S.*

Proof. Let $z \in S$. Since $C \subseteq C_k$ and $Q \subseteq Q_k$, it follows that $z = P_C(z) = P_{C_k}(z)$ and $Az = P_Q(Az) = P_{Q_k}(Az)$. Hence $\nabla F_k(z) = 0$. Using Lemma 2.1 (c), we get

$$\begin{aligned}
\|x_{k+1} - z\|^2 &= \|P_{C_k}(y_k - \alpha_k\nabla F_k(y_k)) - z\|^2 \\
&\le \|y_k - \alpha_k\nabla F_k(y_k) - z\|^2 - \|x_{k+1} - y_k + \alpha_k\nabla F_k(y_k)\|^2 \\
&= \|y_k - z\|^2 + \|\alpha_k\nabla F_k(y_k)\|^2 - 2\alpha_k\langle\nabla F_k(y_k), y_k - z\rangle - \|x_{k+1} - y_k\|^2 \\
&\quad - \|\alpha_k\nabla F_k(y_k)\|^2 - 2\alpha_k\langle\nabla F_k(y_k), x_{k+1} - y_k\rangle \\
&= \|y_k - z\|^2 - 2\alpha_k\langle\nabla F_k(y_k), y_k - z\rangle - \|x_{k+1} - y_k\|^2 \\
&\quad - 2\alpha_k\langle\nabla F_k(y_k), x_{k+1} - y_k\rangle.
\end{aligned} \tag{3.8}$$

Also, we obtain

$$\begin{aligned}
\|y_k - z\|^2 &= \|P_{C_k}(x_k - \alpha_k\nabla F_k(x_k)) - z\|^2 \\
&\le \|x_k - \alpha_k\nabla F_k(x_k) - z\|^2 - \|y_k - x_k + \alpha_k\nabla F_k(x_k)\|^2
\end{aligned}$$

$$
\begin{aligned}
= \quad & \|x_k - z\|^2 + \|\alpha_k \nabla F_k(x_k)\|^2 - 2\alpha_k \langle \nabla F_k(x_k), x_k - z \rangle \\
& - \|y_k - x_k\|^2 - \|\alpha_k \nabla F_k(x_k)\|^2 - 2\alpha_k \langle \nabla F_k(x_k), y_k - x_k \rangle \\
= \quad & \|x_k - z\|^2 - 2\alpha_k \langle \nabla F_k(x_k), x_k - z \rangle - \|y_k - x_k\|^2 \\
& - 2\alpha_k \langle \nabla F_k(x_k), y_k - x_k \rangle.
\end{aligned}
\tag{3.9}
$$

Due to (2.7) and $\nabla F_k(z) = 0$, we have

$$
\begin{aligned}
2\alpha_k \langle \nabla F_k(y_k), y_k - z \rangle &= 2\alpha_k \langle \nabla F_k(y_k) - \nabla F_k(z), y_k - z \rangle \\
&= 2\alpha_k \langle A^T(I - P_{Q_k})Ay_k - A^T(I - P_{Q_k})Az, y_k - z \rangle \\
&= 2\alpha_k \langle (I - P_{Q_k})Ay_k - (I - P_{Q_k})Az, Ay_k - Az \rangle \\
&\geq 2\alpha_k \|(I - P_{Q_k})Ay_k\|^2.
\end{aligned}
\tag{3.10}
$$

It also follows that

$$
2\alpha_k \langle \nabla F_k(x_k), x_k - z \rangle \geq 2\alpha_k \|(I - P_{Q_k})Ax_k\|^2.
\tag{3.11}
$$

From (2.3) and (3.8), we have

$$
\begin{aligned}
& 2\alpha_k \langle \nabla F_k(y_k), x_{k+1} - y_k \rangle \\
= \quad & 2\alpha_k \langle \nabla F_k(y_k) - \nabla F_k(x_{k+1}), x_{k+1} - y_k \rangle + 2\alpha_k \langle \nabla F_k(x_{k+1}), x_{k+1} - y_k \rangle \\
\geq \quad & -2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_{k+1})\| \|x_{k+1} - y_k\| \\
& + 2\alpha_k \left(\frac{1}{2}\|(I - P_{Q_k})Ax_{k+1}\|^2 - \frac{1}{2}\|(I - P_{Q_k})Ay_k\|^2 \right) \\
= \quad & -2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_{k+1})\| \|x_{k+1} - y_k\| \\
& + \alpha_k \|(I - P_{Q_k})Ax_{k+1}\|^2 - \alpha_k \|(I - P_{Q_k})Ay_k\|^2.
\end{aligned}
\tag{3.12}
$$

Also, we have

$$
\begin{aligned}
2\alpha_k \langle \nabla F_k(x_k), y_k - x_k \rangle \geq \quad & -2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_k)\| \|y_k - x_k\| \\
& + \alpha_k \|(I - P_{Q_k})Ay_k\|^2 - \alpha_k \|(I - P_{Q_k})Ax_k\|^2.
\end{aligned}
\tag{3.13}
$$

Combining (3.8)–(3.13), by Lemma 3.2, we obtain

$$
\begin{aligned}
& \|x_{k+1} - z\|^2 \\
\leq \quad & \|x_k - z\|^2 - 2\alpha_k \langle \nabla F_k(x_k), x_k - z \rangle - \|y_k - x_k\|^2 - 2\alpha_k \langle \nabla F_k(x_k), y_k - x_k \rangle \\
& - 2\alpha_k \langle \nabla F_k(y_k), y_k - z \rangle - \|x_{k+1} - y_k\|^2 - 2\alpha_k \langle \nabla F_k(y_k), x_{k+1} - y_k \rangle \\
\leq \quad & \|x_k - z\|^2 - 2\alpha_k \|(I - P_{Q_k})Ax_k\|^2 - \|y_k - x_k\|^2 \\
& + 2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_k)\| \|y_k - x_k\| - \alpha_k \|(I - P_{Q_k})Ay_k\|^2 \\
& + \alpha_k \|(I - P_{Q_k})Ax_k\|^2 - 2\alpha_k \|(I - P_{Q_k})Ay_k\|^2 - \|x_{k+1} - y_k\|^2 \\
& + 2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_{k+1})\| \|x_{k+1} - y_k\| + \alpha_k \|(I - P_{Q_k})Ay_k\|^2 \\
\leq \quad & \|x_k - z\|^2 - \alpha_k \|(I - P_{Q_k})Ax_k\|^2 - \|y_k - x_k\|^2 \\
& + 2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_k)\| \|y_k - x_k\| - 2\alpha_k \|(I - P_{Q_k})Ay_k\|^2 - \|x_{k+1} - y_k\|^2 \\
& + 2\alpha_k \|\nabla F_k(y_k) - \nabla F_k(x_{k+1})\| \|x_{k+1} - y_k\|.
\end{aligned}
\tag{3.14}
$$

Using (3.3) and (3.4), we have

$$
\begin{aligned}
& \|x_{k+1} - z\|^2 \\
\leq \quad & \|x_k - z\|^2 - \frac{\mu\ell}{L}\|(I - P_{Q_k})Ax_k\|^2 - \|y_k - x_k\|^2 \\
& + 2\mu(\|x_{k+1} - y_k\| + \|y_k - x_k\|)\|y_k - x_k\| - 2\frac{\mu\ell}{L}\|(I - P_{Q_k})Ay_k\|^2 \\
& - \|x_{k+1} - y_k\|^2 + 2\mu(\|x_{k+1} - y_k\| + \|y_k - x_k\|)\|x_{k+1} - y_k\| \\
= \quad & \|x_k - z\|^2 - \frac{\mu\ell}{L}\|(I - P_{Q_k})Ax_k\|^2 - \|y_k - x_k\|^2 + 2\mu\|x_{k+1} - y_k\|\|y_k - x_k\| \\
& + 2\mu\|y_k - x_k\|^2 - 2\frac{\mu\ell}{L}\|(I - P_{Q_k})Ay_k\|^2 - \|x_{k+1} - y_k\|^2 + 2\mu\|x_{k+1} - y_k\|^2 \\
& + 2\mu\|y_k - x_k\|\|x_{k+1} - y_k\| \\
\leq \quad & \|x_k - z\|^2 - \frac{\mu\ell}{L}\|(I - P_{Q_k})Ax_k\|^2 - \|y_k - x_k\|^2 + \mu\|x_{k+1} - y_k\|^2 \\
& + \mu\|y_k - x_k\|^2 + 2\mu\|y_k - x_k\|^2 - 2\frac{\mu\ell}{L}\|(I - P_{Q_k})Ay_k\|^2 - \|x_{k+1} - y_k\|^2 \\
& + 2\mu\|x_{k+1} - y_k\|^2 + \mu\|y_k - x_k\|^2 + \mu\|x_{k+1} - y_k\|^2 \\
= \quad & \|x_k - z\|^2 - \frac{\mu\ell}{L}\|(I - P_{Q_k})Ax_k\|^2 - (1 - 4\mu)\|y_k - x_k\|^2 - (1 - 4\mu)\|x_{k+1} - y_k\|^2 \\
& - 2\frac{\mu\ell}{L}\|(I - P_{Q_k})Ay_k\|^2.
\end{aligned}
\tag{3.15}
$$

This implies that

$$
\|x_{k+1} - z\| \leq \|x_k - z\| \tag{3.16}
$$

and hence $\lim_{k\to\infty} \|x_k - z\|$ exists. Moreover, $\{x_k\}$ is bounded. From (3.15), we also have

$$
\lim_{k\to\infty} \|y_k - x_k\| = 0 \tag{3.17}
$$

and

$$
\lim_{k\to\infty} \|x_{k+1} - y_k\| = 0. \tag{3.18}
$$

Moreover, we have

$$
\lim_{k\to\infty} \|(I - P_{Q_k})Ax_k\| = 0. \tag{3.19}
$$

So, we obtain

$$
\begin{aligned}
\|x_{k+1} - x_k\| & = \|x_{k+1} - y_k + y_k - x_k\| \\
& \leq \|x_{k+1} - y_k\| + \|y_k - x_k\| \to 0
\end{aligned}
\tag{3.20}
$$

as $k \to \infty$. Since $\{x_k\}$ is bounded, assume that $\omega_w(x_k)$ is nonempty. Let $x^* \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x^*$.

Now, we show that $x^* \in S$. Since $x_{k_n+1} \in C_{k_n}$ by the definition of C_{k_n}, we get

$$
c(x_{k_n}) \leq \langle \xi_{k_n}, x_{k_n} - x_{k_n+1} \rangle, \tag{3.21}
$$

where $\xi_{k_n} \in \partial c(x_{k_n})$. By the boundedness of ∂c and (3.20), we have

$$c(x_{k_n}) \leq \|\xi_{k_n}\|\|x_{k_n} - x_{k_n+1}\| \to 0 \tag{3.22}$$

as $n \to \infty$. By the w-lsc of c, $x_{k_n} \rightharpoonup x^*$ and (3.22), it follows that

$$c(x^*) \leq \liminf_{n\to\infty} c(x_{k_n}) \leq 0. \tag{3.23}$$

Hence $x^* \in C$.

Next, we show that $Ax^* \in Q$. Since $P_{Q_{k_n}}(Ax_{k_n}) \in Q_{k_n}$, we have

$$q(Ax_{k_n}) \leq \langle \eta_{k_n}, Ax_{k_n} - P_{Q_{k_n}}(Ax_{k_n})\rangle, \tag{3.24}$$

where $\eta_{k_n} \in \partial q(Ax_{k_n})$. So, we have

$$q(Ax_{k_n}) \leq \|\eta_{k_n}\|\|Ax_{k_n} - P_{Q_{k_n}}(Ax_{k_n})\| \to 0 \tag{3.25}$$

as $n \to \infty$. Since $x_{k_n} \rightharpoonup x^*$, $Ax_{k_n} \rightharpoonup Ax^*$. The w-lsc of q and (3.25) imply that

$$q(Ax^*) \leq \liminf_{n\to\infty} q(Ax_{k_n}) \leq 0. \tag{3.26}$$

Hence $Ax^* \in Q$. Using Lemma 2.2, we conclude that the sequence $\{x_k\}$ converges weakly to a point in S. This completes the proof. $\qquad\square$

Remark 3.4. *Algorithm 3.1 in Theorem 3.3 is quite different from that of Gibali et al. [16]. To be more precise, the linesearch (3.3) is defined by a new way of two step method.*

Remark 3.5. *Theorem 3.3 is more convenient than the results of Yang [27] in practice. In fact, we do not require the information of the operator norm which is not easy in computation.*

4. Numerical experiments

In this section, we provide some numerical experiments in signal recovery to compare our algorithm with those of Yang [27] and Gibali et al. [16]. In signal processing, the compressed sensing can be formulated as inverting the equation system:

$$y = Ax + \varepsilon, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε and $A : \mathbb{R}^N \to \mathbb{R}^M$ ($M < N$) is a bounded linear observation operator. Here A is sparse and the range of it is not closed in most inverse problems and thus A is often ill-condition and the problem is also ill-posed.

When x is a sparse expansion, to find solutions of the problem (4.1) can be seen as solving the following LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Ax\|^2 \text{ such that } \|x\|_1 \leq t, \tag{4.2}$$

where $t > 0$ is a given constant. The sparse vector $x \in \mathbb{R}^N$ is generated from the uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $A \in \mathbb{R}^{M\times N}$ is generated from a normal

distribution with mean zero and variance one. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR=40. The process is started with $t = m$ and initial point x_1 is picked randomly.

Let $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$. Then the minimization problem (4.2) can be seen as the problem **(SFP)** and, since the projection onto the closed convex C does not have a closed form solution, we make use of the subgradient projection. Define a convex function $c(x) = \|x\|_1 - t$ and denote the level set C_k by:

$$C_k = \{x : c(x_k) + \langle \xi_k, x - x_k \rangle \leq 0\}, \tag{4.3}$$

where $\xi_k \in \partial c(x_k)$. Then the orthogonal projection onto C_k can be calculated by the following:

$$P_{C_k}(x) = \begin{cases} x, & \text{if } c(x_k) + \langle \xi_k, x - x_k \rangle \leq 0, \\ x - \frac{c(x_k) + \langle \xi_k, x - x_k \rangle}{\|\xi_k\|^2} \xi_k, & \text{otherwise.} \end{cases} \tag{4.4}$$

The subdifferential ∂c at x_k is

$$\partial c(x_k) = \begin{cases} 1, & x_k > 0, \\ [-1, 1], & x_k = 0, \\ -1, & x_k < 0. \end{cases} \tag{4.5}$$

In our experiment, we consider two cases as follows:

Case 1: $N = 512$, $M = 256$ and $m = 10$;
Case 2: $N = 4096$, $M = 2048$ and $m = 100$.

Next, we give some numerical results by using the relaxed CQ-algorithms defined by Yang [27], Gibali et al. [16] and our algorithms (Algorithm 3.1).

The iteration is stopped when the following criteria is satisfied:

$$\text{MSE} = \frac{1}{N} \|x^* - x\|^2 < 10^{-5}, \tag{4.6}$$

where x^* is an estimated signal of x.

In what follows, let $\alpha_k = \dfrac{1}{\|A\|^2}$ in the CQ-algorithm by Yang [27]. Define $\gamma = 2$, $\ell = 0.5$ and $\mu = 0.2$ in that of Gibali et al. [16]. Then the numerical results are reported as follows:

Remark 4.1. (1) In Figures 1 and 4, we see that the signal recovered by Algorithm 3.1 in both **Case 1** and **Case 2** has number of iterations and cpu time less than algorithm of Yang [27] and Gibali et al. [16]. In the algorithm of Yang [27], the stepsize α_k depends on the operator norm $\|A\|$ whenever the matrix has a large dimension, it may be compute very hard and has a costly cpu time. It is observed that our new linesearch has a less cpu time than that of Gibali et al. [16].

(2) In Figures 2 and 5, we plot the error value of iterations. We see that the errors of Algorithm 3.1 decrease faster than those of other algorithms. Also, in Figures 3 and 6, the objective function values obtained by Algorithm 3.1 have a better convergence behaviour than other algorithms.
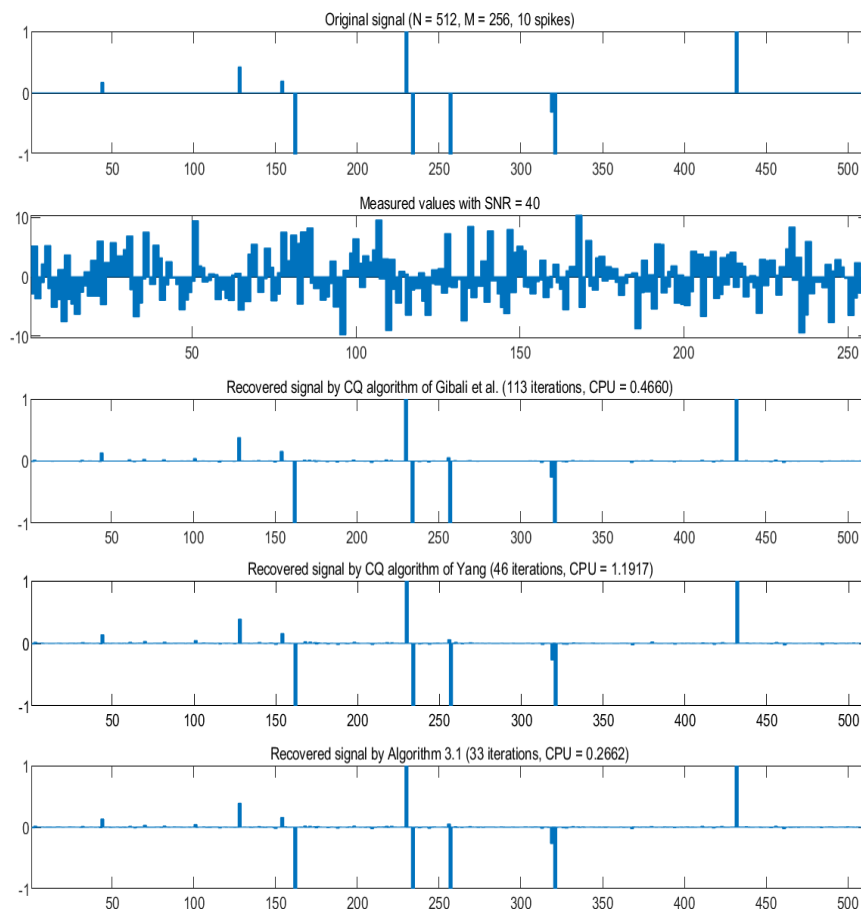
Figure 1. From top to bottom: original signal, observation data, recovered signal by Gibali et al., Yang and Algorithm 3.1 with $N = 512$ and $M = 256$, respectively.
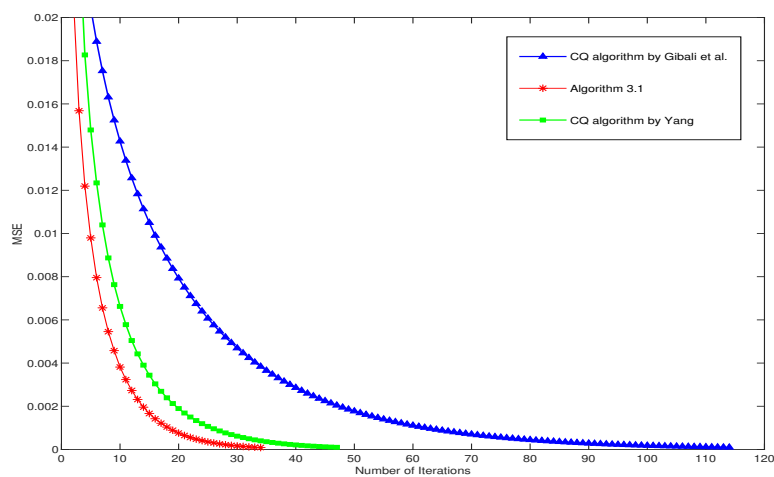

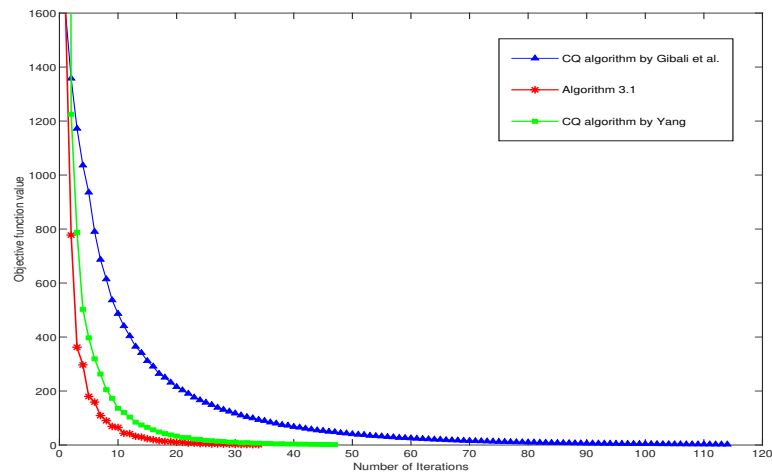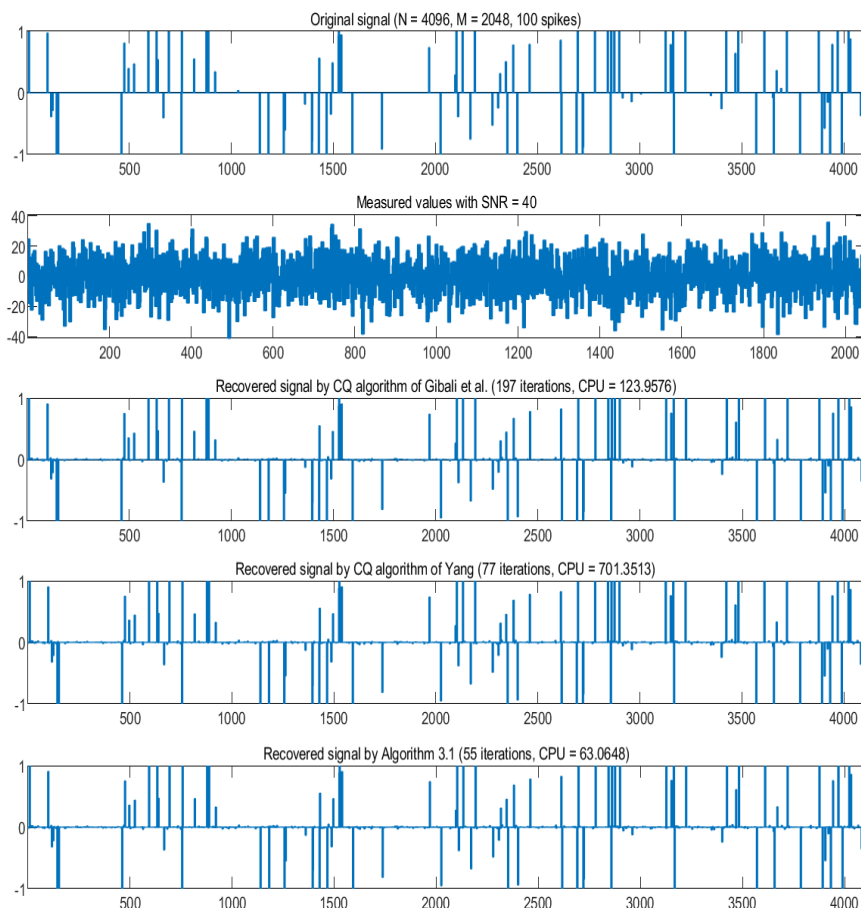**Figure 2.** MSE versus number of iterations in case N=512, M=256.

Figure 3. The objective function value versus number of iterations in case N=512, M=256.

Figure 4. From top to bottom: original signal, observation data, recovered signal by Gibali et al., Yang and Algorithm 3.1 with $N = 4096$ and $M = 2048$, respectively.
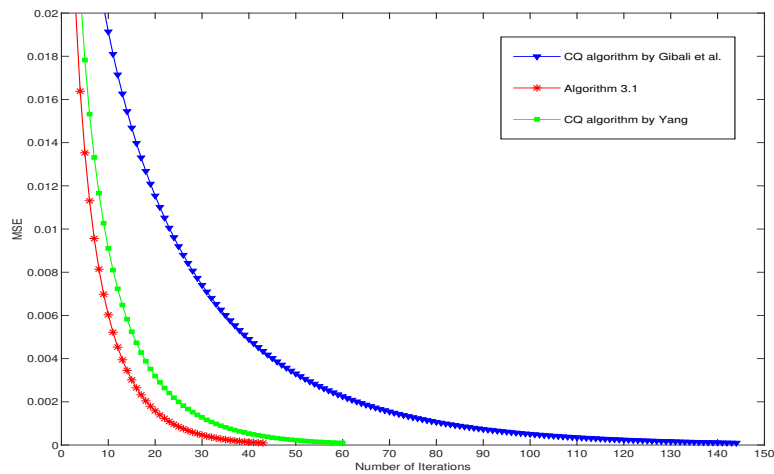
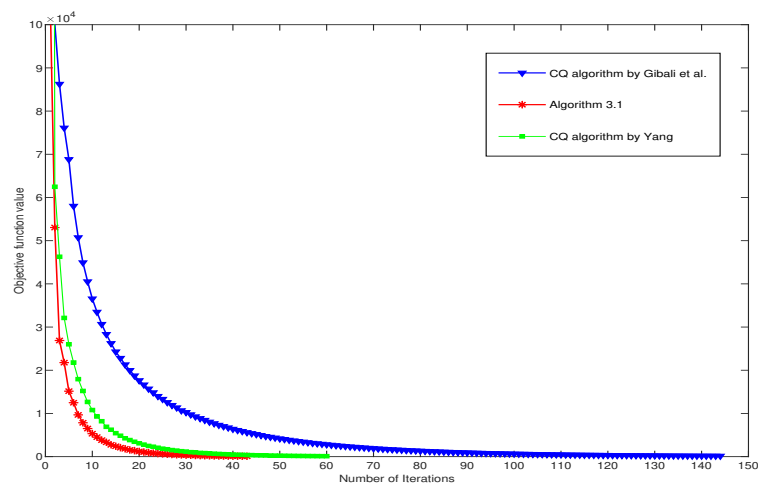**Figure 5.** MSE versus number of iterations in case N=4096, M=2048.

Figure 6. The objective function value versus number of iterations in case N=4096, M=2048.

5. Conclusions

In this paper, we proposed the relaxed *CQ*-algorithm with new two steps by using a new linesearch in real Hilbert spaces. The computation of matrix inverse and norm of operators is not required in our algorithm. Also, we gave in a simple and novel way how the sequence generated by the method weakly converges to a solution of the problem **(SFP)**. All the results are compared to the relaxed *CQ*-algorithms of Yang [27] and Gibali et al. [16]. Finally, we found that the proposed algorithm is effective and outruns other known methods in the literature.

Acknowledgments

Conflict of interest

The authors declare no conflict of interest.

References

1. H. H. Bauschke, P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248–264.

2. H. H. Bauschke, P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, London, 2011.

3. C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Probl.*, **18** (2002), 441–453.

4. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120.

5. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phys. Med. Biol.*, **51** (2006), 2353–2365.

6. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *The split feasibility model leading to a unified approach for inversion problems in intensity-modulated radiation therapy*, Technical Report, Department of Mathematics, University of Haifa, Israel, 2005.

7. Y. Censor, T. Elfving, A multiprojection algorithms using Bregman projection in a product space, *Numer. Algorithms*, **8** (1994), 221–239.

8. Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problem, *Inverse Prob.*, **21** (2005), 2071–2084.

9. Y. Censor, M. Jiang, G. Wang, *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems*, Medical Physics Publishing, Madison, Wisconsin, 2010.

10. P. Cholamjiak, S. Kesornprom, N. Pholasa, Weak and strong convergence theorems for the inclusion problem and the fixed-point problem of nonexpansive mappings, *Mathematics*, **7** (2019), 1–19.

11. Y. Z. Dang, J. Sun, H. L. Xu, Inertial accelerated algorithms for solving a split feasibility problem, *J. Ind. Manage. Optim.*, **13** (2017), 1383–1394.

12. Q. L. Dong, X. H. Li, D. Kitkuan, Y. J. Cho, P. Kumam, Some algorithms for classes of split feasibility problems involving paramonotone equilibria and convex optimization, *J. Inequal. Appl.*, **2019** (2019), 1–23.

13. Q. L. Dong, Y. C. Tang, Y. J. Cho, Th. M. Rassias, "Optimal" choice of the step length of the projection and contraction methods for solving the split feasibility problem, *J. Global Optim.*, **71** (2018), 341–360.

14. M. Farid, The subgradient extragradient method for solving mixed equilibrium problems and fixed point problems in Hilbert spaces, *J. Appl. Numer. Optim.*, **1** (2019), 335–345.

15. M. Fukushima, A relaxed projection method for variational inequalities, *Math. Program.*, **35** (1986), 58–70.

16. A. Gibali, L. W. Liu, Y. C. Tang, Note on the modified relaxation *CQ*-algorithm for the split feasibility problem, *Optim. Lett.*, **12** (2017), 1–14.

17. S. Husain, N. Singh, A hybrid iterative algorithm for a split mixed equilibrium problem and a hierarchical fixed point problem, *Appl. Set-Valued Anal. Optim.*, **1** (2019), 149–169.

18. G. López, V. Martín-Márquez, F. H. Wang, H. K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Prob.*, **28** (2012), 085004. Available from: `http://iopscience.iop.org/0266-5611/28/8/085004`.

19. G. López, V. Martín-Márquez, H. K. Xu, Perturbation techniques for nonexpansive mappings with applications, *Nonlinear Anal.: Real Word Appl.*, **10** (2009), 2369–2383.

20. B. Qu, N. H. Xiu, A note on the *CQ*-algorithm for the split feasibility problem, *Inverse Prob.*, **21** (2005), 1655–1665.

21. S. Suantai, S. Kesornprom, P. Cholamjiak, A new hybrid CQ algorithm for the split feasibility problem in Hilbert spaces and its applications to compressed sensing, *Mathematics*, **7** (2019), 789.

22. W. Takahahsi, J. C. Yao, The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces, *J. Nonlinear Convex Anal.*, **20** (2019), 173–195.

23. M. Tian, H. F. Zhang, The regularized CQ algorithm without a priori knowledge of operator norm for solving the split feasibility problem, *J. Inequalities Appl.*, **2017** (2017), 1–10.

24. S. Wang, Y. Zhang, W. Wang, H. Guo, Extragradient algorithms for split pseudomonotone equilibrium problems and fixed point problems in Hilbert spaces, *J. Nonlinear Funct. Anal.*, **2019** (2019), 1–19.

25. U. Witthayarat, Y. J. Cho, P. Cholamjiak, On solving proximal split feasibility problems and applications, *Ann. Funct. Anal.*, **9** (2018), 111–122.

26. H. K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, *Inverse Prob.*, **26** (2010), 105018.

27. Q. Z. Yang, The relaxed *CQ*-algorithm for solving the split feasibility problem, *Inverse Prob.*, **20** (2004), 1261–1266.

28. M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.*, **152** (2004), 199–277.

29. M. A. Noor, K. I. Noor, M. Th. Rassias, New trends in general variational inequalities, *Acta Applicandae Math.*, **170** (2020), 981–1064.