Research article

Note on an integral by Anatolii Prudnikov

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Abstract: Closed expressions for the integral

$$\int_0^\infty \frac{x^{m-1} \log^k(ax)}{(x^{2u} + 1)(x^{3u} + 1)} dx$$

are given where the variables a, k, m and u are general complex numbers. Some of these closed expressions are given in [4]. Some special cases of the integral are derived and discussed.

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1. Introduction

In 1986 Prudnikov et al [4] published a famous book containing a significant number of integrals. In this manuscript we focus on the integral

$$\int_0^\infty \frac{x^{m-1} \log^k(ax)}{(x^{2u} + 1)(x^{3u} + 1)} dx$$

which has a closed form in in terms of the Lerch function. We appeal to our method to derive a closed form solution which is continued analytically along with some special cases. A consequence of using our method with this integral is the derivation of the Mellin transform of the product of the logarithmic function and a rational function which is not found in current literature. This work also looks to expand the Table 2.5.1 in [1] where similar and relevant formula are listed. In our case the constants in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [5], [6], [7], [8], [9] and [10]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dy. \tag{1.1}$$

This method involves using a form of Eq (1.1) then multiplies both sides by a function, then takes a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (1.1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. The definite integral of the contour integral

We use the method in [9]. The variable of integration in the contour integral is $\alpha = m + w$. The cut and contour are in the first quadrant of the complex α-plane with $0 < \Re(\alpha) < 5u$. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula, we first replace y by $\log(ax)$ followed by multiplying both sides by $\frac{x^{m-1}}{(x^{2u}+1)(x^{3u}+1)}$, then taking the definite integral with respect $x \in [0, 1]$ to get

$$\int_0^\infty \frac{x^{m-1} \log^k(ax)}{(x^{2u} + 1)(x^{3u} + 1) k!} dx = \frac{1}{2\pi i} \int_0^\infty \int_C \frac{a^w w^{-k-1} x^{\alpha-1}}{(x^{2u} + 1)(x^{3u} + 1)} d\alpha dx$$

$$= \frac{1}{2\pi i} \int_C \int_0^\infty \frac{a^w w^{-k-1} x^{\alpha-1}}{(x^{2u} + 1)(x^{3u} + 1)} dx d\alpha$$

$$= \frac{1}{2\pi i} \int_C \frac{\pi a^w w^{-k-1}}{6u} \left(4 \sin\left(\frac{\pi(u - 4\alpha)}{6u}\right) + 3\sqrt{2} \sin\left(\frac{\pi\alpha}{2u} + \frac{\pi}{4}\right) + 1 \right) \csc\left(\frac{\pi\alpha}{u}\right) d\alpha \tag{2.1}$$

from Eq (2.2.7.7) in [4], where $0 < \Re(\alpha) < 5u$. We are able to switch the order of integration over α and x using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

3. Infinite sum of the contour integrals

3.1. Derivation of the third contour integral $\frac{1}{2\pi i} \int_C \frac{\pi a^w w^{-k-1} \csc\left(\frac{\pi\alpha}{u}\right)}{6u} d\alpha$

Using Eq (1.1) and replacing y by $\log(a) + \frac{i\pi(2y+1)}{u}$, then multiplying both sides by $-\frac{2i\pi}{6u} \exp\left(\frac{(2y+1)i\pi m}{u}\right)$, we get

$$-\frac{2i\pi^{k+1} \left(\frac{i}{u}\right)^k e^{\frac{(2y+1)i\pi m}{u}} \left(-\frac{iu \log(a)}{\pi} + 2y + 1\right)^k}{6uk!} = -\frac{1}{2\pi i} \int_C 2i\pi w^{-k-1} \exp\left(w \log(a) + \frac{i\pi\alpha(2y + 1)}{u}\right) \frac{d\alpha}{6u} \tag{3.1}$$

We then take the infinite sum over $y \in [0, \infty)$ to get

$$-\frac{i(2\pi)^{k+1}\left(\frac{i}{u}\right)^k e^{\frac{i\pi m}{u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{\pi-iu\log(a)}{2\pi}\right)}{6uk!} = -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2i\pi w^{-k-1}\exp\left(w\log(a)+\frac{i\pi\alpha(2y+1)}{u}\right)\frac{d\alpha}{6u}$$

$$=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}2i\pi w^{-k-1}\exp\left(w\log(a)+\frac{i\pi\alpha(2y+1)}{u}\right)\frac{d\alpha}{6u}$$

$$=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}\csc\left(\frac{\pi\alpha}{u}\right)\frac{d\alpha}{6u} \tag{3.2}$$

from (1.232.3) in [3], where $\operatorname{csch}(ix)=-i\csc(x)$ from (4.5.10) in [2] and $\Im(\alpha)>0$.

3.2. Derivation of the contour integral $\frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}}{\sqrt{2}u}\sin\left(\frac{\pi(u+2\alpha)}{4u}\right)\csc\left(\frac{\pi\alpha}{u}\right)d\alpha$

Next using Eq (1.1), we obtain the left-hand side replacing y by $(y+i\pi/(2u))$ and multiplying by $i\exp(i\pi(2m+u)/(4u))$ to get the first equation and by replacing y by $(y-i\pi/(2u))$ and multiplying by $i\exp(-i\pi(2m+u)/(4u))$ to get the second equation and subtracting to get

$$-\frac{e^{-\frac{i\pi(2m+u)}{4u}}\left(y-\frac{i\pi}{2u}\right)^k - e^{\frac{i\pi(2m+u)}{4u}}\left(y+\frac{i\pi}{2u}\right)^k}{k!} = \frac{1}{2\pi i}\int_C 2iw^{-k-1}e^{wy}\sin\left(\frac{\pi(u+2\alpha)}{4u}\right)d\alpha \tag{3.3}$$

Next using Eq (3.3), we replace y by $\log(a)+\frac{i\pi(2y+1)}{u}$ then multiply both sides by $-\frac{1}{2^{1/2}}\frac{\pi}{u}\exp(i\pi m(2y+1)/u)$ take the infinite sum over $y\in[0,\infty)$ to get

$$\sum_{y=0}^{\infty}\frac{\pi\exp\left(\frac{i\pi m(2y+1)}{u}\right)}{\sqrt{2}uk!}\left(e^{-\frac{i\pi(2m+u)}{4u}}\left(\log(a)+\frac{i\pi(2y+1)}{u}-\frac{i\pi}{2u}\right)^k - e^{\frac{i\pi(2m+u)}{4u}}\left(\log(a)+\frac{i\pi(2y+1)}{u}+\frac{i\pi}{2u}\right)^k\right)$$

$$=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{i\sqrt{2}\pi w^{-k-1}}{u}\sin\left(\frac{\pi(u+2\alpha)}{4u}\right)a^w\exp\left(\frac{i\pi w(2y+1)}{u}+\frac{2i\pi my}{u}+\frac{i\pi m}{u}\right)d\alpha \tag{3.4}$$

$$=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{i\sqrt{2}\pi w^{-k-1}}{u}\sin\left(\frac{\pi(u+2\alpha)}{4u}\right)a^w\exp\left(\frac{i\pi w(2y+1)}{u}+\frac{2i\pi my}{u}+\frac{i\pi m}{u}\right)d\alpha$$

Next we simplify to get the Lerch function contour integral representation given by

$$\frac{2^{k-\frac{1}{2}}\pi^{k+1}\left(\frac{i}{u}\right)^k}{uk!}\left(e^{\frac{i\pi(2m-u)}{4u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{\pi-2iu\log(a)}{4\pi}\right) - e^{-\frac{i\pi(6m+u)}{4u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{3}{4}-\frac{iu\log(a)}{2\pi}\right)\right) \tag{3.5}$$

$$=\frac{1}{2\pi i}\int_C\frac{\pi a^w w^{-k-1}}{\sqrt{2}u}\sin\left(\frac{\pi(u+2\alpha)}{4u}\right)\csc\left(\frac{\pi\alpha}{u}\right)d\alpha$$

from (1.232.3) in [3], where $\operatorname{csch}(ix)=-i\csc(x)$ from (4.5.10) in [2] and $\Im(\alpha)>0$.

3.3. Derivation of the contour integral $\frac{1}{2\pi i} \int_C \frac{2\pi a^w w^{-k-1} \sin\left(\frac{\pi(u-4\alpha)}{6u}\right) \csc\left(\frac{\pi\alpha}{u}\right)}{3u} d\alpha$

Here we use Eq (1.1) and replacing y by $y + \frac{2i\pi}{3u}$ and multiply by $\exp\left(-\frac{i\pi(u-4m)}{6u}\right)$ for the first equation and then by $y - \frac{2i\pi}{3u}$ and multiply by $\exp\left(\frac{i\pi(u-4m)}{6u}\right)$ for the second equation. Then we subtract the two to get

$$\frac{e^{\frac{i\pi(u-4m)}{6u}}\left(y - \frac{2i\pi}{3u}\right)^k - e^{\frac{-i\pi(u-4m)}{6u}}\left(y + \frac{2i\pi}{3u}\right)^k}{k!} = \frac{1}{2\pi i} \int_C 2iw^{-k-1}e^{wy}\sin\left(\frac{\pi(u-4\alpha)}{6u}\right)d\alpha \qquad (3.6)$$

Next we replace y by $\log(a) + \frac{i\pi(2y+1)}{u}$ then multiply both sides by $-2i\pi \exp\left(\frac{i\pi m(2y+1)}{u}\right)$ to get

$$-\frac{2i\pi \exp\left(\frac{2i\pi my}{u} + \frac{i\pi m}{u}\right)}{k!}\left(\pi^k \left(\frac{i}{u}\right)^k e^{\frac{i\pi(u-4m)}{6u}}\left(-\frac{iu\log(a)}{\pi} + 2y + \frac{1}{3}\right)^k\right.$$

$$\left. - \pi^k e^{-\frac{i\pi(u-4m)}{6u}}\left(\frac{i}{u}\right)^k \left(-\frac{iu\log(a)}{\pi} + 2y + \frac{5}{3}\right)^k\right) \qquad (3.7)$$

$$= \frac{1}{2\pi i} \int_C 4\pi w^{-k-1}\sin\left(\frac{\pi(u-4\alpha)}{6u}\right)\exp\left(w\left(\log(a) + \frac{i\pi(2y+1)}{u}\right) + \frac{2i\pi my}{u} + \frac{i\pi m}{u}\right)dw$$

Next we take the infinite sum over $y \in [0, \infty)$ and multiply by $1/6u$ to get

$$-\frac{(2\pi)^{k+1}\left(\frac{i}{u}\right)^k}{3uk!}\left(-e^{-\frac{i\pi}{6}}e^{-\frac{i\pi m}{3u}}\Phi\left(e^{\frac{2i m\pi}{u}}, -k, \frac{5}{6} - \frac{iu\log(a)}{2\pi}\right) - e^{\frac{-i\pi}{6}}e^{\frac{i\pi m}{3u}}\Phi\left(e^{\frac{2i m\pi}{u}}, -k, \frac{\pi - 3iu\log(a)}{6\pi}\right)\right)$$

$$= \frac{1}{12u\pi i}\sum_{y=0}^{\infty}\int_C 4\pi w^{-k-1}\sin\left(\frac{\pi(u-4\alpha)}{6u}\right)a^w \exp\left(\frac{i\pi w(2y+1)}{u} + \frac{i\pi m(2y+1)}{u}\right)d\alpha \qquad (3.8)$$

$$= \frac{1}{12u\pi i}\int_C \sum_{y=0}^{\infty} 4\pi w^{-k-1}\sin\left(\frac{\pi(u-4\alpha)}{6u}\right)a^w \exp\left(\frac{i\pi w(2y+1)}{u} + \frac{i\pi m(2y+1)}{u}\right)d\alpha$$

$$= \frac{1}{2\pi}\int_C \frac{2\pi a^w w^{-k-1}}{3u}\sin\left(\frac{\pi(u-4\alpha)}{6u}\right)\csc\left(\frac{\pi\alpha}{u}\right)d\alpha$$

from (1.232.3) in [3], where $\operatorname{csch}(ix) = -i\csc(x)$ from (4.5.10) in [3] and $\Im(\alpha) > 0$.

4. The definite integral in terms of the Lerch function

Since the right-hand side of Eq (2.1) is equal to the sum of Eqs (3.2), (3.5) and (3.8), we may equate the left hand sides to get

$$\int_0^\infty \frac{x^{m-1}\log^k(ax)}{(x^{2u}+1)(x^{3u}+1)}dx = \frac{2^{k-\frac{1}{2}}\pi^{k+1}\left(\frac{i}{u}\right)^k e^{\frac{i\pi(2m-u)}{4u}}}{u}\left(\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{\pi-2iu\log(a)}{4\pi}\right)\right.$$

$$\left.- ie^{\frac{i\pi m}{u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{3}{4}-\frac{iu\log(a)}{2\pi}\right)\right)$$

$$+ \frac{e^{-i\pi/6}(2\pi)^{k+1}\left(\frac{i}{u}\right)^k e^{\frac{i\pi m}{3u}}}{3u}\left(e^{\frac{4i\pi m}{3u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{5}{6}-\frac{iu\log(a)}{2\pi}\right)\right.$$

$$\left.- e^{\pi i/3}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{\pi-3iu\log(a)}{6\pi}\right)\right)$$

$$- \frac{1}{3}2^k\pi^{k+1}\left(\frac{i}{u}\right)^{k+1}e^{\frac{i\pi m}{u}}\Phi\left(e^{\frac{2im\pi}{u}},-k,\frac{\pi-iu\log(a)}{2\pi}\right) \tag{4.1}$$

The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^\infty (v+n)^{-s}z^n \tag{4.2}$$

where $|z|< 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{4.3}$$

where $Re(v) > 0$, or $|z|\leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Derivation of entry 2.2.7.7 in [4]

Using Eq (4.1) and setting $k = 0$ and simplifying, we get

$$\int_0^\infty \frac{x^{m-1}}{(x^{2u}+1)(x^{3u}+1)}dx = \frac{\pi\csc\left(\frac{\pi m}{u}\right)}{6u}\left(-2\sqrt{3}\sin\left(\frac{2\pi m}{3u}\right)+3\sqrt{2}\sin\left(\frac{\pi(2m+u)}{4u}\right)+2\cos\left(\frac{2\pi m}{3u}\right)+1\right) \tag{5.1}$$

6. Derivation of Prudnikov type integral in Table 2.6.19

Using Eq (4.1), then taking the first partial derivative with respect to m, then set $m = 1$, $k = 1$ and $a = 1/a$ on the right-hand side followed by using L'Hopital's rule on the right-hand side as $u \to 1$, we get

$$\int_0^\infty \frac{\log(x)\log\left(\frac{x}{a}\right)}{(x^2+1)(x^3+1)}dx = \frac{1}{432}\pi^2\left(27\pi - 37\log\left(\frac{1}{a}\right)\right) \tag{6.1}$$

7. Derivation of an integral of product of logarithms

Using Eq (4.1), then taking the first partial derivative with respect to m, then set $m = 2$, $k = 1$ and $a = 1/a$ followed by using L'Hopital's rule on the right-hand side as $u \to 1$, we get

$$\int_0^\infty \frac{x \log(x) \log\left(\frac{x}{a}\right)}{(x^2 + 1)(x^3 + 1)} dx = \frac{\pi^2 \left(\left(160\sqrt{3} - 243\right)\pi - 45\log\left(\frac{1}{a}\right)\right)}{3888} \tag{7.1}$$

8. Derivation of a special case

Using Eq (4.1) setting $k = a = 1$ and $m = 1/2$, we get

$$\int_0^\infty \frac{\log(x)}{\sqrt{x}\,(x^{2u} + 1)(x^{3u} + 1)} dx = -\frac{\pi^2 \csc^2\left(\frac{\pi}{2u}\right)}{72u^2} \left(20\cos\left(\frac{\pi}{6u}\right) + 27\cos\left(\frac{\pi}{4u}\right) + 12\cos\left(\frac{\pi}{2u}\right) + 9\cos\left(\frac{3\pi}{4u}\right)\right.$$
$$\left. + 4\left(-9\sin^3\left(\frac{\pi}{4u}\right) + 5\sqrt{3}\sin\left(\frac{\pi}{6u}\right) - \sqrt{3}\sin\left(\frac{5\pi}{6u}\right) + \cos\left(\frac{5\pi}{6u}\right)\right)\right) \tag{8.1}$$

8.1. When u = 1/2

Using L'Hopital's rule as $u \to 1/2$ to the right-hand side of equation (8.1), we get

$$\int_0^\infty \frac{\log(x)}{x^3 + x^2 + x^{3/2} + \sqrt{x}} dx = -\frac{37\pi^2}{108} \tag{8.2}$$

8.2. When u = 1

We get

$$\int_0^\infty \frac{\log(x)}{\sqrt{x}\,(x^2 + 1)(x^3 + 1)} dx = -\frac{2\pi^2}{3\sqrt{3}} \tag{8.3}$$

9. Derivation of a Mellin transform with an example

Using Eq (4.1), we first set $a = 1$ and replace u by $2m$ and m by $m/2$, then take the first partial derivative with respect to k, then set $k = 0$ simplify to get

$$\int_0^\infty \frac{x^{\frac{m}{2} - 1} \log(\log(x))}{(x^{2m} + 1)(x^{3m} + 1)} dx = -\frac{\pi}{6m}\left(3\sqrt{2}\log\left(-\frac{35im\Gamma\left(-\frac{7}{8}\right)\Gamma\left(-\frac{5}{8}\right)}{12\pi\Gamma\left(-\frac{3}{8}\right)\Gamma\left(-\frac{1}{8}\right)}\right)\right.$$
$$\left. + \log\left(\frac{900i\pi\Gamma\left(-\frac{3}{4}\right)^2\Gamma\left(-\frac{5}{12}\right)^2\Gamma\left(-\frac{1}{12}\right)^2}{5929m\Gamma\left(-\frac{11}{12}\right)^2\Gamma\left(-\frac{7}{12}\right)^2\Gamma\left(-\frac{1}{4}\right)^2}\right) - 4i\sqrt{3}\coth^{-1}\left(\sqrt{3}\right)\right) \tag{9.1}$$

Next we set $m = 1$ in Eq (9.1) simplify to get

$$\int_0^\infty \frac{\log(\log(x))}{\sqrt{x}\,(x^2+1)\,(x^3+1)}dx = -\frac{1}{6}\pi\left(-4i\sqrt{3}\coth^{-1}\left(\sqrt{3}\right) + 3\sqrt{2}\log\left(-\frac{35i\Gamma\left(-\frac{7}{8}\right)\Gamma\left(-\frac{5}{8}\right)}{12\pi\Gamma\left(-\frac{3}{8}\right)\Gamma\left(-\frac{1}{8}\right)}\right)\right.$$
$$\left. + \log\left(\frac{900i\pi\Gamma\left(-\frac{3}{4}\right)^2\Gamma\left(-\frac{5}{12}\right)^2\Gamma\left(-\frac{1}{12}\right)^2}{5929\Gamma\left(-\frac{11}{12}\right)^2\Gamma\left(-\frac{7}{12}\right)^2\Gamma\left(-\frac{1}{4}\right)^2}\right)\right) \tag{9.2}$$

Note there is a singularity at $x = 1$ which is removable using the principal value of the integral.

10. Derivation of special cases

In this section, we will list some interesting examples by evaluating Eq (4.1) for various values of the parameters k, a, m and u in terms of fundamental constants and trigonometric functions.

10.1. Example 1

Using Eq (4.1) and setting $k = a = -1$, $u = 1$ and $m = 1/2$ and rationalizing the real and imaginary parts, we get

$$\int_0^\infty \frac{\log(x)}{\sqrt{x}\,(x^2+1)\,(x^3+1)\left(\log^2(x)+\pi^2\right)}dx = \frac{\sqrt{3}}{2} - \frac{\pi}{3} \tag{10.1}$$

and

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x^2+1)\,(x^3+1)\left(\log^2(x)+\pi^2\right)}dx = \frac{-1 + 2\sqrt{2} + \log\left(6 - 4\sqrt{2}\right)}{2\pi} \tag{10.2}$$

10.2. Example 2

Using Eq (4.1) and setting $k = -1$, $a = i$, $u = 1$ and $m = 1/2$ and rationalizing the real and imaginary parts, we get

$$\int_0^\infty \frac{\log(x)}{\sqrt{x}\,(x^2+1)\,(x^3+1)\left(4\log^2(x)+\pi^2\right)}dx = \frac{1}{8}\left(4\sqrt{3} - \sqrt{2}\pi + \sqrt{2}\log\left(5 - 2\sqrt{6}\right)\right) \tag{10.3}$$

and

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x^2+1)\,(x^3+1)\left(4\log^2(x)+\pi^2\right)}dx = \frac{-48 + 9\sqrt{2}\pi + 2\sqrt{2}\log\left(8\left(99 + 70\sqrt{2}\right)\right)}{48\pi} \tag{10.4}$$

10.3. Example 3

Using Eq (4.1) and setting $k = -1$, $a = -1$, and $u = 1/2$ followed by taking the first partial derivative with respect to m, then apply L'Hopital's rule as $m \to 1/2$ simplify the real and imaginary parts, we get

$$\int_0^\infty \frac{\log(x)}{\left(x^{3/2} + x^3 + x^2 + \sqrt{x}\right)\left(\log^2(x) + \pi^2\right)}\,dx = 4 - \frac{3\pi}{4} - \log(6) \tag{10.5}$$

and

$$\int_0^\infty \frac{\log^2(x)}{\left(x^{3/2} + x^3 + x^2 + \sqrt{x}\right)\left(\log^2(x) + \pi^2\right)}\,dx = \frac{\pi}{2} - \frac{1}{2}\pi\log(2) \tag{10.6}$$

10.4. Example 4

Using Eq (4.1) and setting $k = -1$, $a = -1$, and $u = 1$ and $m = 1/4$ for the first equation, then replace $m = 1/2$ for the second equation, followed by subtracting the two and rationalizing the real and imaginary parts, we get

$$\int_0^\infty \frac{\left(\sqrt{x} - \sqrt[4]{x}\right)\log(x)}{(x^2 + 1)(x^3 + 1)\left(\log^2(x) + \pi^2\right)}\,dx = \frac{1}{48\sqrt{2}}\Bigg[2\left(4\sqrt{2} - 5\right)\pi + 4\bigg(-30 - 6\sqrt{3} + 6\sqrt{6} + 12\sqrt{2 + \sqrt{2}}$$
$$+ \log(8) + \cosh^{-1}(1351) - 12\tanh^{-1}\left(\sqrt{1 - \frac{1}{\sqrt{2}}}\right)\bigg)\Bigg] \tag{10.7}$$

and

$$\int_0^\infty \frac{\sqrt{x} - \sqrt[4]{x}}{(x^2 + 1)(x^3 + 1)\left(\log^2(x) + \pi^2\right)}\,dx = \frac{5\sqrt{2}\pi - 2}{48\pi}\Bigg(\log(4096)$$
$$+ \sqrt{2}\log\left[\frac{\left(\sqrt{3} - 2\right)^4\left(1 + \sqrt{2 + \sqrt{2}}\right)^6}{8\left(2 + \sqrt{3}\right)^2\left(\sqrt{2 + \sqrt{2}} - 1\right)^6}\right]$$
$$+ 6\left(-2 + \sqrt{6} + 4\sqrt{2}\sin\left(\frac{\pi}{8}\right) - 2\sqrt{13}\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{5}{12}\right)\right)\right)\Bigg) \tag{10.8}$$

10.5. Example 5

Using Eq (4.1) and setting $k = -1$, $a = 1$, and $u = 1/2$ simplifying, we get

$$\int_0^\infty \frac{x^{m-1}}{(x + 1)(x^{3/2} + 1)\log(x)}\,dx = \frac{1}{3}\left(-3i\tan^{-1}\left(e^{i\pi m}\right) - 2\tanh^{-1}\left(e^{\frac{2i\pi m}{3}}\right)\right.$$
$$\left. + 2\tanh^{-1}\left(\sqrt[3]{-1}e^{\frac{2i\pi m}{3}}\right) + 4\tanh^{-1}\left((-1)^{2/3}e^{\frac{2i\pi m}{3}}\right) - 3\tanh^{-1}\left(e^{i\pi m}\right) - \tanh^{-1}\left(e^{2i\pi m}\right)\right) \tag{10.9}$$

Then we form a second equation by replacing m by n then taking the difference. Then we set $n = 1/4$ followed by using L'Hopital's rule as $m \to 1/2$ simplify to get

$$\int_0^\infty \frac{\sqrt[4]{x} - 1}{x^{3/4}(x+1)\left(x^{3/2}+1\right)\log(x)}dx = \frac{1}{2}\log\left(\frac{2}{9}\left(7 + 4\sqrt{3}\right)\right) \tag{10.10}$$

11. Summary

In this article, we derived some interesting definite integrals in [4]. We found that we are able to achieve a wider range of computation using one formula as opposed to previous works. We will be looking at other integrals using this contour integral method for future work. The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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