Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2636–2652.
DOI:10.3934/math.2021160
Received: 25 October 2020
Accepted: 17 December 2020
Published: 29 December 2020

Research article

Semi-compatible mappings and common fixed point theorems of an implicit relation via inverse $C-$class functions

Mi Zhou[1,*], Mukesh Kumar Jain[2], Mohammad Saeed Khan[3] and Nicolae Adrian Secelean[4]

[1] School of Science and Technology, University of Sanya, Sanya, Hainan, 572022, China
[2] J. N. V. Udalguri Assam, India
[3] College of Science, Department of Mathematics and Statistics, Sultan Qaboos University, Al-Khod, Muscat, Oman
[4] Department of Mathematics and Computer Science, Faculty of Sciences, Lucian Blaga University of Sibiu, Romania

* **Correspondence:** Email: mizhou330@126.com.

Abstract: In this paper, we prove some common fixed point theorems by exploring a new kind of generalized semi-compatibility and an implicit relation via inverse $C-$class functions. The results generalize, extend and improve the main results of [19, 21–23]. Moreover, some examples are given to illustrate the validity of our results.

Keywords: $f-$semi-compatible mapping; $g-$semi-compatible mapping; $f-$compatibility of type (E); semi-compatibility of type (A); inverse $C-$class function; implicit relation; common fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The study of metric fixed point theory is playing an important role in linear and nonlinear analysis. In 1922, Stephen Banach [1] laid the foundation of metric fixed point theory and gave a very fruitful concept of contraction mapping. Since then, many researchers studied that field in many directions. It was indeed a turning point in fixed point theory when Sessa [2] introduced the notion of weak commutativity. Later, this concept was executed by several researchers in considerable amounts. Further, the generalization of weak commutativity came to exist in 1986 when Jungck [3] firstly introduced compatible mappings. Definitely, this research had opened some new directions in fixed point theory for many researchers. Later, in 1996 Jungck generalized his own concept by new class of compatible mappings, named weak compatible mappings [4], and through various examples he had shown that each of these generalizations of commutativity are proper extensions of previous definitions. Abbas eal. [5] pointed out that weakly compatible maps remains a minimal commutativity condition for the existence of unique common fixed of contractive type maps. In the last few decades many generalizations came to exist like compatible mapping of type (A) [6], compatible mapping of type (B) [7], compatible mapping of type (C) [8], compatible mapping of type (P) [9], semi-compatible mappings [10], weak semi-compatible mappings [11], conditional semi-compatible mappings [12], faintly compatible mappings [13], occasionally weakly compatible mappings [14–16] and other types of mappings [17, 18]. In 2011, Singh et al. [19] gave brief discussion of various types of mappings as compatible mappings of type (A), type (B), type (C) and type (P) and compared these mappings with compatible mappings of type (E). He introduced new concepts of $S-$compatible mappings of type (E) and $S-$reciprocal continuous mappings by splitting the concepts of compatible mappings of type (E) and reciprocal continuous mappings [20], and moreover, obtained some common fixed point theorems for non-continuous self-mappings on metric spaces. Recently, Ansari et al. [21] used the concept of compatibility of type (E) and reciprocal continuity and obtained some fixed point results by using an implicit relation via $C-$class functions.

In this paper, we introduce a new concept of semi-compatible mappings and establish some common fixed point results by using an implicit relation introduced by Djoudi [22, 23] via inverse $C-$class functions on metric spaces.

Throughout the paper, we will denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}_0$ the set of natural numbers (positive integers), real numbers, positive real numbers and $\mathbb{N} \cup \{0\}$, respectively.

Definition 1.1. [24] A pair of self-mappings (f, g) on a metric space (X, d) is said to be compatible of type (E), if

$$\lim_{n\to+\infty} ffx_n = \lim_{n\to+\infty} fgx_n = gt \quad \text{and} \quad \lim_{n\to+\infty} ggx_n = \lim_{n\to+\infty} gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

Definition 1.2. [19] A pair of self-mappings (f, g) on a metric space (X, d) is said to be $f-$compatible of type (E), if

$$\lim_{n\to+\infty} ffx_n = \lim_{n\to+\infty} fgx_n = gt,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

Definition 1.3. [19] A pair of self-mappings (f, g) on a metric space (X, d) is said to be $g-$compatible of type (E), if

$$\lim_{n\to+\infty} ggx_n = \lim_{n\to+\infty} gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

It is easy to see that the compatibility of type (E) implies both $f-$ and $g-$ compatibility of type (E), however the $f-$ or $g-$compatibility of type (E) do not imply the compatibility of type (E) (See Example 2.10 [19]).

Definition 1.4. [10] A pair of self-mappings (f, g) on a metric space (X, d) is said to be semi-compatible, if $\lim_{n\to+\infty} fgx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

A simple but genuine question rises: "Does semi-compatibility of (f, g) imply the semi-compatibility of (g, f)?" That is $\lim_{n\to+\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$. Actually they are two different notions. We give the following example to verify it.

Example 1.1. Let $X = [0, +\infty)$ endowed with usual metric d. Define a self-mappings f, g on X as follows:

$$fx = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, +\infty) \end{cases},$$

and $g = I_X$ (the identity mapping). If we consider the sequence $x_n = \frac{1}{2} - \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = \frac{1}{2} \neq f(\frac{1}{2})$. So (g, f) is not semi-compatible. Also, for a sequence $\{x_n\}$ such that $x_n \to x_0$ and $fx_n \to x_0$, we have

$$\lim_{n\to+\infty} fgx_n = \lim_{n\to+\infty} fx_n = x_0 = g(x_0).$$

Then (f, g) is semi-compatible.

In the following we do a modification of the definition of semi-compatibility.

Definition 1.5. A pair (f, g) of self-mappings on a metric space (X, d) is said to be semi-compatible of type (A), if $\lim_{n\to+\infty} fgx_n = gt$ and $\lim_{n\to+\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

We give an example to demonstrate it as follows.

Example 1.2. Let $X = [1, +\infty)$ endowed with the usual metric d and $f, g : X \mapsto X$ be the self-mappings defined by

$$fx = \begin{cases} 2x + 1, & x \in [1, 3) \\ x, & x \in [3, +\infty) \end{cases}, \qquad gx = \begin{cases} 2 + x, & x \in [1, 3) \\ 3, & x \in [3, +\infty) \end{cases}.$$

If we consider the sequence $x_n = 1 + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = 3$. Also $\lim_{n\to+\infty} fgx_n = 3 = g(3)$ and $\lim_{n\to+\infty} gfx_n = 3 = f(3)$. Moreover, if $x_n = 3 + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = 3$. Also $\lim_{n\to+\infty} fgx_n = 3 = g(3)$ and $\lim_{n\to+\infty} gfx_n = 3 = f(3)$.

Before proving our main results we introduce some definitions by splitting the concept of semi-compatibility of type (A).

Definition 1.6. A pair (f, g) of self-mappings on a metric space (X, d) is said to be $f-$semi-compatible, if $\lim_{n\to+\infty} fgx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

Definition 1.7. A pair (f, g) of self-mappings on a metric space (X, d) is said to be $g-$semi-compatible, if $\lim_{n\to+\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

It is obvious from the above definitions that semi-compatibility of type (A) implies $f-$semi-compatibility and $g-$semi-compatibility of a pair (f, g), however the converse is not true. Moreover, $f-$semi-compatibility and $g-$semi-compatibility coincide with the semi-compatibility of the pair (f, g) and semi-compatibility of the pair (g, f) introduced by Singh et al. [10], respectively.

Definition 1.8. [20] A pair (f, g) of self-mappings on a metric space (X, d) is said to be reciprocal continuous, if $\lim_{n\to+\infty} fgx_n = ft$ and $\lim_{n\to+\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

We introduce some definitions by splitting the concept of reciprocal continuity of a pair (f, g) of self-mappings as follows.

Definition 1.9. A pair (f, g) of self-mappings on a metric space (X, d) is said to be $f-$reciprocal continuous, if $\lim_{n\to+\infty} fgx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

Definition 1.10. A pair (f, g) of self-mappings on a metric space (X, d) is said to be $g-$reciprocal continuous, if $\lim_{n\to+\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$.

The notion of $f-$reciprocal continuity or $g-$reciprocal continuity coincides with the concept of weak reciprocal continuity introduced by Pant et al. [25].

It is obvious that $f-$semi-compatibility and $g-$semi-compatibility are independent notions with respect to $f-$reciprocal continuity and $g-$reciprocal continuity, respectively. It is noticed that compatibility of type (E) implies semi-compatibility of type (A) but implication is not reversible.

We now provide two examples to verify above discussion and also show the comparison between semi-compatible mappings of type (A) and reciprocal continuous mappings (compatible mappings of type (E)).

Example 1.3. Let us consider $X = [0, +\infty)$ endowed with the usual metric. Define $f, g : X \mapsto X$ by
$fx = \begin{cases} 1, & x \in [0, 1] \\ x, & x \in (1, +\infty) \end{cases}, \quad gx = \begin{cases} 2, & x \in [0, 1) \\ \frac{1}{x}, & x \in [1, +\infty) \end{cases}.$

Then the pair of mappings (f, g) is semi-compatible of type (A) and reciprocal continuous. However, the pair (f, g) is not compatible mapping of type (E).

It is easy to see that $x_n = 1 + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, is the only sequences which satisfy the conditions

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = 1.$$

One has

$$\lim_{n\to+\infty} fgx_n = 1 = g(1),$$
$$\lim_{n\to+\infty} gfx_n = 1 = f(1).$$

Hence (f, g) is semi-compatible of type (A).
The other sentences follow from the following relations

$$\lim_{n\to+\infty} fgx_n = \lim_{n\to+\infty} f(\frac{1}{1+\varepsilon_n}) = 1 = f(1),$$

$$\lim_{n\to+\infty} gfx_n = \lim_{n\to+\infty} g(1+\varepsilon_n) = 1 = g(1),$$

$$\lim_{n\to+\infty} ffx_n = \lim_{n\to+\infty} f(1+\varepsilon_n) = 1 = \lim_{n\to+\infty} fgx_n = g(1),$$

$$\lim_{n\to+\infty} ggx_n = \lim_{n\to+\infty} g(\frac{1}{1+\varepsilon_n}) = 2 \neq \lim_{n\to+\infty} gfx_n = f(1).$$

Example 1.4. Let $X = [0, 1]$ with usual metric. Define $f, g : X \mapsto X$ by

$$fx = \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ 1 - x, & x \in [\frac{1}{2}, 1] \end{cases}, \qquad gx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}) \\ x, & x \in [\frac{1}{2}, 1] \end{cases}.$$

We consider the sequence $x_n = \frac{1}{2} + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = \frac{1}{2}.$$

Also, we have

$$\lim_{n\to+\infty} fgx_n = \lim_{n\to+\infty} f(\frac{1}{2} + \varepsilon_n) = \frac{1}{2} = f(\frac{1}{2}) = g(\frac{1}{2}) \neq \lim_{n\to+\infty} ffx_n = 1.$$

and

$$\lim_{n\to+\infty} gfx_n = \lim_{n\to+\infty} g(\frac{1}{2} - \varepsilon_n) = \frac{1}{2} = f(\frac{1}{2}) = g(\frac{1}{2}) = \lim_{n\to+\infty} ggx_n.$$

Therefore the pair of mappings (f, g) is not only semi-compatible of type (A), but it is also reciprocal continuous, even $g-$compatible of type (E). However, it is not compatible of type (E).

According to the previous examples and Singh [19], we have the following proposition.

Proposition 1.1. Let f and g be self-mappings on a metric space (X, d). Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$. If one of the following conditions is satisfied:

(i) (f, g) is $f-$semi-compatible and $f-$reciprocal continuous,
(ii) (f, g) is $g-$semi-compatible and $g-$reciprocal continuous,

Then
(a) $ft = gt$ and
(b) if there exists $u \in X$ such that $fu = gu = t$, then $fgu = gfu$.

Proof. Follows immediately.

Remark 1.1. By the above, it follows that each of condition of Proposition 1.1 implies the weak compatibility of pair (f, g), introduced by Jungck in [4], however, the inverse is not applicable.

Definition 1.11. [26] A continuous function $F : [0, +\infty)\times[0, +\infty) \mapsto \mathbb{R}$ is called an inverse $C-$class function, if for every $s, t \in [0, +\infty)$, the following conditions hold:

(i) $F(s, t) \geq s$,
(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

We will denote by C_{inv} the class of all inverse $C-$class functions. In the following we will provide some examples (for further details, one should refer [26]).

Example 1.5. The following functions $F : [0, +\infty) \times [0, +\infty) \mapsto \mathbb{R}$ belong to C_{inv}, for all $s, t \in [0, +\infty)$:

1. $F(s, t) = s + t, F(s, t) = s$ implies $t = 0$.
2. $F(s, t) = ms$, for some $m \in (1, + + \infty)$, $F(s, t) = s$ implies $s = 0$.
3. $F(s, t) = s(1 + t)^r$, for some $r \in (0, +\infty)$, $F(s, t) = s$ implies $s = 0$ or $t = 0$.

4. $F(s,t) = \log_a[(t+a^s)(1+t)]$, for some $a > 1$, $F(s,t) = s$ implies $t = 0$.
5. $F(s,t) = \vartheta(s)$, $\vartheta : (0,+\infty) \times (0,+\infty) \mapsto \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function, $F(s,t) = s$ implies $s = 0$.

Definition 1.12. [27] A function $\varphi : [0,+\infty) \mapsto [0,+\infty)$ is called an ultra-altering distance if φ is continuous, and $\varphi(0) = 0$, $\varphi(t) > 0, t > 0$. We denote by Φ_u the set of all ultra-altering distance functions.

An implicit relation, introduced by Djoudi [22,23], is stated as follows.

Let $\mathcal{G}$ be the set of all continuous functions $G(t_1,\ldots,t_6) : \underbrace{\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{6} \mapsto \mathbb{R}$ satisfying the following conditions:

(G_1) : G is non-decreasing in variables t_5 and t_6.
(G_2) : there exists $h \in (1,+\infty)$ such that, for every $u, v \geq 0$ with
$G_a : G(u,v,u,v,u+v,0) \geq 0$, or
$G_b : G(u,v,v,u,0,u+v) \geq 0$,
we have $u \geq hv$.
(G_3) : $G(u,u,0,0,u,u) < 0$, for all $u > 0$.

We now provide some examples of $G \in \mathcal{G}$ (for more details, one can refer Djoudi [22,23]).

Example 1.6. Let $G(t_1,\ldots,t_6) : \underbrace{\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{6} \mapsto \mathbb{R}$ be defined by

$$G(t_1,\ldots,t_6) = \frac{a}{t_2} - \frac{b}{\max\{t_2,t_3,t_4,t_5+t_6\}},$$

where $a, b > 0$ with $b > 2a$.
(G_1): It is clear.
(G_a): Let $u, v \in \mathbb{R}^+$. Suppose that $G(u,v,u,v,u+v,0) = \frac{a}{v} - \frac{b}{\max\{v,u,v,u+v\}} \geq 0$. Then $u \geq (\frac{b-a}{a})v = hv$, where $h = \frac{b-a}{a} \in (1,+\infty)$.
(G_b) : Let $u, v \in \mathbb{R}^+$. Suppose that $G(u,v,v,u,0,u+v) = \frac{a}{v} - \frac{b}{\max\{v,u,v,u+v\}} \geq 0$. Then $u \geq (\frac{b-a}{a})v = hv$, where $h = \frac{b-a}{a} \in (1,+\infty)$. Thus, (G_2) is satisfied when $h = \frac{b-a}{a}$.
(G_3): $G(u,u,0,0,u,u) = \frac{a}{u} - \frac{b}{2u} = \frac{2a-b}{2u} < 0$, for all $u > 0$.

Example 1.7. Let $G(t_1,\ldots,t_6) : \underbrace{\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{6} \mapsto \mathbb{R}$ be defined by

$$G(t_1,\ldots,t_6) = at_1^2 - bt_2^2 + \frac{ct_5t_6}{dt_3^2 + et_4^2 + 1},$$

where $c, d, e \geq 0$, $a > 0$ and $b > a + c$.
(G_1) : It is clear.
(G_a) : Let $u, v \in \mathbb{R}^+$ and suppose that $G(u,v,u,v,u+v,0) = au^2 - bv^2 \geq 0$. Then $u \geq (\frac{b}{a})^{\frac{1}{2}}v = hv$, where $h = (\frac{b}{a})^{\frac{1}{2}}$.
(G_b) : Let $u, v \in \mathbb{R}^+$ and suppose that $G(u,v,v,u,0,u+v) = au^2 - bv^2 \geq 0$. Then $u \geq (\frac{b}{a})^{\frac{1}{2}}v = hv$, where $h = (\frac{b}{a})^{\frac{1}{2}}$. Thus (G_2) is satisfied when $h = (\frac{b}{a})^{\frac{1}{2}}$.
(G_3) : $G(u,u,0,0,u,u) = au^2 - bu^2 + cu^2 = u^2(a-b+c) < 0$, for all $u > 0$.

We generalize the implicit relation of Djoudi [22,23] by using the inverse $C-$class functions. Let $\mathcal{G}_c$ be the set of all continuous functions $G(t_1,\dots,t_6):\underbrace{\mathbb{R}^+\times\mathbb{R}^+\times\cdots\times\mathbb{R}^+}_{6}\mapsto\mathbb{R}$ satisfying the following conditions:

(G_1') : G is non decreasing in variables t_5 and t_6.
(G_2') : there exists $h\in(1,+\infty)$ such that for every $u,v\geq 0$ with
$G_{a'}$: $G(u,v,u,v,u+v,0)\geq 0$, or
$G_{b'}$: $G(u,v,v,u,0,u+v)\geq 0$,
we have $u\geq hF(v,\varphi(v))$, where $F\in C_{\text{inv}}$ and $\varphi\in\Phi_u$.
(G_3') : $G(u,u,0,0,u,u)<0$, for all $u>0$.

It is easy to obtain that $\mathcal{G}\subseteq\mathcal{G}_c$.

In the following we provide some examples of functions $G\in\mathcal{G}_c$.

Example 1.8. Let $G(t_1,\dots,t_6):\underbrace{\mathbb{R}^+\times\mathbb{R}^+\times\cdots\times\mathbb{R}^+}_{6}\mapsto\mathbb{R}$ be given by

$$G(t_1,\dots,t_6)=t_1-m[at_2+\frac{bt_3+ct_4}{t_5t_6+1}],$$

where $a>1, 0<b<mb<1, 0<c<mc<1$.
Define $F\in C_{\text{inv}}$ by $F(s,t)=ms, m>1$ and $\varphi\in\Phi_u$ as $\varphi(t)=2t$, for all $t\geq 0$.
(G_1'): It is clear.
$(G_{a'})$: Let $u,v\in\mathbb{R}^+$. Suppose that $G(u,v,u,v,u+v,0)=u-m[av+bu+cv]\geq 0$. Then $u\geq h_1mv=h_1F(v,\varphi(v))$, where $h_1=\frac{a+c}{1-mb}\in(1,+\infty)$.
$(G_{b'})$: Let $u,v\in\mathbb{R}^+$. Suppose that $G(u,v,v,u,0,u+v)=u-m[av+bv+cu]\geq 0$. Then $u\geq h_2mv=h_2F(v,\varphi(v))$, where $h_2=\frac{a+b}{1-mc}\in(1,+\infty)$. So (G_2') is satisfied.
(G_3'): $G(u,u,0,0,u,u)=u-mau=(1-ma)u<0$, for all $u>0$.

Example 1.9. Let $G(t_1,\dots,t_6):\underbrace{\mathbb{R}^+\times\mathbb{R}^+\times\cdots\times\mathbb{R}^+}_{6}\mapsto\mathbb{R}$ be given by

$$G(t_1,\dots,t_6)=t_1-[at_2^p+bt_3^p+ct_4^p]^{1/p}+d\sqrt{t_5t_6},$$

where $a>(1+d)^p$, $d\geq 0$ (in particular $a>2^p$ if $d=1$), $0\leq c,\ b<1,\ p\in\mathbb{N}$.
Define $F\in C_{\text{inv}}$ as $F(s,t)=a^{\frac{1}{p}}s$ with $a>(1+d)^p, d\geq 0$ (in particular $a>2^p$ if $d=1$) and $\varphi\in\Phi_u$ as $\varphi(t)=t$, for all $t\geq 0$.
(G_1') : It is clear.
$(G_{a'})$: Let $u,v\in\mathbb{R}^+$. Suppose that $G(u,v,u,v,u+v,0)=u-[av^p+bu^p+cv^p]^{\frac{1}{p}}\geq 0$, then $u\geq(\frac{a+c}{1-b})^{\frac{1}{p}}v=\theta_1F(v,\varphi(v))=\theta_1av$, where $\theta_1=(\frac{a+c}{a(1-b)})^{\frac{1}{p}}>1$.
$(G_{b'})$: Let $u,v\in\mathbb{R}^+$. Suppose that $G(u,v,v,u,0,u+v)=u-[av^p+bv^p+cu^p]^{\frac{1}{p}}\geq 0$; then $u\geq(\frac{a+b}{1-c})^{\frac{1}{p}}v=\theta_2F(v,\varphi(v))=\theta_2av$, where $\theta_2=(\frac{a+b}{a(1-c)})^{\frac{1}{p}}>1$. Hence (G_2') hold for $\theta=\min\{\theta_1,\theta_2\}$. Thus, (G_2') is satisfied.
(G_3') : $G(u,u,0,0,u,u)=u-(au^p)^{\frac{1}{p}}+du=(1-(a)^{\frac{1}{p}}+d)u<0$, for all $u>0$.

Example 1.10. Let $G(t_1,\dots,t_6):\underbrace{\mathbb{R}^+\times\mathbb{R}^+\times\cdots\times\mathbb{R}^+}_{6}\mapsto\mathbb{R}$ be given by

$$G(t_1,\dots,t_6)=at_1-bt_2+c(t_3+t_4)-d\min\{t_3,t_5t_6\},$$

where $\sqrt{\frac{b-c}{a+c}} > 1$ such that $0 < c < \frac{b-a}{2}$ and $a, d > 0$.
Define $F \in C_{\text{inv}}$ by $F(s,t) = hs$ with $h \in (1, +\infty)$ and $\varphi \in \Phi_u$ by $\varphi(t) = t$, for all $t \geq 0$.
(G'_1) : It is clear.
$(G_{a'})$: Let $u, v \in \mathbb{R}^+$. Suppose that $G(u, v, u, v, u+v, 0) = au - bv + c(u+v) - d\min\{u, 0\} = au - bv + cu + cv \geq 0$. Then $u \geq F(v, \varphi(v)) = h^2 v$, where $h = \sqrt{\frac{b-c}{a+c}} > 1$.
$(G_{b'})$: Let $u, v \in \mathbb{R}^+$. Suppose that $G(u, v, v, u, 0, u+v) = au - bv + c(v+u) - d\min\{v, 0\} = au - bv + cu + cv \geq 0$. Then $u \geq F(v, \varphi(v)) = h^2 v$, where $h = \sqrt{\frac{b-c}{a+c}} > 1$. Thus (G'_2) is satisfied.
(G'_3) : $G(u, u, 0, 0, u, u) = au - bu < 0$, for all $u > 0$.

Example 1.11. Let $G(t_1, \ldots, t_6) : \underbrace{\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{6} \mapsto \mathbb{R}$ be given by

$$G(t_1, \ldots, t_6) = 1 - \frac{t_2(h^2+1)}{\max\{t_1, t_2, t_3, t_4, t_5 + t_6\}},$$

where $h \in (1, +\infty)$.
Define $F \in C_{\text{inv}}$ by $F(s,t) = hs$ and $\varphi \in \Phi_u$ by $\varphi(t) = t$, for all $t \geq 0$.
(G'_1) : It is clear.
$(G_{a'})$: Let $u, v \in \mathbb{R}^+$. Suppose that $G(u, v, u, v, u+v, 0) = 1 - \frac{v(h^2+1)}{\max\{u,v,u,v,u+v\}} \geq 0$, that is $1 - \frac{v(h^2+1)}{u+v} \geq 0$.
Then, we have

$$u - h^2 v \geq 0,$$

which implies $u \geq hF(v, \varphi(v)) = h^2 v$.
$(G_{b'})$: Let $u, v \in \mathbb{R}^+$. Suppose that $G(u, v, v, u, 0, u+v) = 1 - \frac{v(h^2+1)}{\max\{u,v,v,u,u+v\}} \geq 0$, that is $1 - \frac{v(h^2+1)}{u+v} \geq 0$.
Then, we have

$$u - h^2 v \geq 0,$$

which implies $u \geq hF(v, \varphi(v)) = h^2 v$. Thus (G'_2) is satisfied.
(G'_3) : $G(u, u, 0, 0, u, u) = 1 - \frac{u(h^2+1)}{\max\{u,u,0,0,2u\}} = 1 - \frac{h^2+1}{2} < 0$, for all $u > 0$.

2. Main results

In this section we prove some common fixed point theorems of a kind of implicit relation via inverse $C-$class functions.

Theorem 2.1. Let $G \in \mathcal{G}_c$ and let A, B, S and T be four self-mappings on a complete metric space (X, d) satisfying the following conditions:

(i) S and T are surjective,
(ii) for every $x, y \in X$,

$$G(d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) \geq 0,$$

(iii) the pair (A, S) is A-semi-compatible and A-reciprocal continuous and the pair (B, T) is B-semi-compatible and B-reciprocal continuous.

Then A, B, S and T have a unique common fixed point.

Proof. Let any $x_0 \in X$. Condition (i) assures that one can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, where $n \in \mathbb{N}_0$. Using condition (ii) with $x = x_{2n}, y = x_{2n+1}$, we obtain

$$\begin{aligned}
&G(d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}),\\
&\quad d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n}), d(Ax_{2n}, Tx_{2n+1}))\\
&= G(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n}))\\
&= G(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n-1}), 0)\\
&\geq 0.
\end{aligned}$$

By (G_1'), we have

$$G(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), 0) \geq 0.$$

From (G_a'), we deduce

$$\begin{aligned}
d(y_{2n-1}, y_{2n}) &\geq hF(d(y_{2n}, y_{2n+1})), \varphi((d(y_{2n}, y_{2n+1})))\\
&\geq hd(y_{2n}, y_{2n+1}),
\end{aligned}$$

that is

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{h} d(y_{2n-1}, y_{2n}).$$

Repeating the previous argument and using (ii) and (G_b') leads to

$$d(y_{2n-1}, y_{2n}) \leq \frac{1}{h} d(y_{2n-2}, y_{2n-1}).$$

Consequently, we have

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{h^{2n}} d(y_0, y_1) \quad \text{and} \quad d(y_{2n-1}, y_{2n}) \leq \frac{1}{h^{2n-1}} d(y_0, y_1),$$

where $h \in (1, +\infty)$. Hence $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$.

We will now show that $\{y_n\}$ is a Cauchy sequence. For any integer $p > 0$, we get

$$\begin{aligned}
d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p})\\
&\leq \frac{1}{h^n} d(y_0, y_1) + \frac{1}{h^{n+1}} d(y_0, y_1) + \cdots + \frac{1}{h^{n+p-1}} d(y_0, y_1)\\
&= \frac{1}{h^n}[1 + \frac{1}{h} + \frac{1}{h^2} + \cdots \frac{1}{h^{p-1}}] d(y_0, y_1)\\
&\leq k^n \frac{1}{1-k} d(y_0, y_1),
\end{aligned}$$

where $k = \frac{1}{h} \in (0, 1)$. Letting $n, p \to +\infty$, we obtain $d(y_n, y_{n+p}) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence. Using the completeness of X one can find $t \in X$ such that $\lim_{n\to+\infty} y_n = t$. Consequently all subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to t, i.e.,

$$\lim_{n\to+\infty} Ax_{2n} = t, \lim_{n\to+\infty} Bx_{2n+1} = t, \lim_{n\to+\infty} Sx_{2n} = t, \lim_{n\to+\infty} Tx_{2n+1} = t.$$

Condition (i) implies that $t = Su = Tv$ for some $u, v \in X$.

We now prove that $Au = t$. From (ii), we have

$$G(d(Su, Tx_{2n+1}), d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Bx_{2n+1}, Su), d(Au, Tx_{2n+1})) \geq 0.$$

Taking limit in the above inequality as $n \to +\infty$, one obtains

$$G(0, d(Au, t), d(Au, t), 0, 0, d(Au, t)) \geq 0.$$

Using $(G_{b'})$, we get $Au = t$ and $Su = Au = t$.

Next, we also claim that $Bv = t$. To see this, note that by condition (ii) with $x = x_{2n}, y = v$, we have

$$\begin{aligned} &G(d(Sx_{2n}, Tv), d(Ax_{2n}, Bv), d(Ax_{2n}, Sx_{2n}), d(Bv, Tv), d(Bv, Sx_{2n}), d(Ax_{2n}, Tv)) \\ &= G(0, d(t, Bv), 0, d(Bv, t), d(Bv, t), 0) \\ &\geq 0. \end{aligned}$$

From $(G_{a'})$, we deduce $Bv = t$, that is $Bv = Tv = t$. Hence $Au = Su = Bv = Tv = t$.

From (iii), $A-$semi-compatibility of the pair (A, S) yields $\lim_{n\to+\infty} ASx_{2n} = St$ and $A-$reciprocally continuity of the pair (A, S) yields $\lim_{n\to+\infty} ASx_{2n} = At$. These both yield $At = St$.

Again, $B-$semi-compatibility of and $B-$reciprocal continuity pair (B, T) yield $BTv = TBv$ or $Bt = Tt$.

In the following, we prove $Bt = At$.

From (ii), we have

$$\begin{aligned} &G(d(St, Tt), d(At, Bt), d(At, St), d(Bt, Tt), d(Bt, St), d(At, Tt)) \\ &= G(d(At, Bt), d(At, Bt), 0, 0, d(At, Bt), d(At, Bt)) \\ &\geq 0, \end{aligned}$$

which is a contradiction to (G_3'), so $Bt = At$. Therefore $At = Bt = St = Tt$.

Finally, we show that t is a fixed point of A. Indeed, on the contrary, condition (ii) implies

$$\begin{aligned} &G(d(St, Tv), d(At, Bv), d(At, St), d(Bv, Tv), d(Bv, St), d(At, Tv)) \\ &= G(d(At, t), d(At, t), 0, 0, d(At, t), d(At, t)) \\ &\geq 0, \end{aligned}$$

which contradicts (G_3'). Hence $At = t$. Accordingly $At = Bt = St = Tt = t$, that is t is a common fixed point of A, B, S and T.

To prove uniqueness, we suppose that t' is another common fixed point of A, B, S and T. Then $At' = Bt' = St' = Tt' = t'$. Now, from condition (ii) with $x = t, y = t'$, we have

$$\begin{aligned} &G(d(St, Tt'), d(At, Bt'), d(At, St), d(Bt', Tt'), d(Bt', St), d(At, Tt')) \\ &= G(d(t, t'), d(t, t'), 0, 0, d(t, t'), d(t, t')) \\ &\geq 0, \end{aligned}$$

which contradicts (G_3'). Consequently $t = t'$.

Remark 2.1. Theorem 2.1 holds true if we suppose that the pair (A, S) is $S-$semi-compatible and $S-$reciprocal continuous and the pair (B, T) is $T-$semi-compatible and $T-$reciprocal continuous.

Corollary 2.1. Let $G \in \mathcal{G}_c$ and let A, B, S and T be four self-mappings on a complete metric space (X, d) satisfying the following conditions:

(i) S and T are surjective,

(ii) for every $x, y \in X$,

$$G(d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) \geq 0,$$

(iii) the pairs (A, S) and (B, T) are semi-compatible of type (A)(or compatible of type (E)) and reciprocal continuous.

Then A, B, S and T have a unique common fixed point.

Proof. Since the semi-compatibility of type (A) (or compatibility of type (E)) and the reciprocally continuity of the pairs (A, S) and (B, T) imply the $A-(B-)$ semi-compatibility and the $A-(B-)$ reciprocally continuity of the pairs (A, S) and (B, T), respectively, then the conclusion follows from the proof of Theorem 2.1.

We now present some examples which verify the validity of Theorem 2.1.

Example 2.1. Let $X = [1, +\infty)$ and $x, y \in X(y \geq x)$ with usual metric d. We define the maps $A, B, S, T : X \to X$ by,

$$A(x) = B(x) = 1, \quad \forall x \in [1, +\infty).$$

$$S(x) = T(x) = x, \quad \forall x \in [1, +\infty).$$

It is obvious that S, T are surjective. Taking the sequence $x_n = \{1 + \varepsilon_n\}$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, one can verify that the pair (A, S) is $A-$semi-compatible and $A-$reciprocal continuous. Also (B, T) is $B-$semi-compatible and $B-$reciprocal continuous. Define now an implicit function $G \in \mathcal{G}_c$, $G(t_1, \ldots, t_6) : \mathbb{R}^{+6} \mapsto \mathbb{R}$ as in Example 1.10:

$$G(t_1, \ldots, t_6) = at_1 - bt_2 + c(t_3 + t_4) - d\min\{t_3, t_5 t_6\},$$

where $\sqrt{\frac{b-c}{a+c}} > 1$ such that $0 < c < \frac{b-a}{2}$ and $a, d > 0$.
Define $F \in C_{\text{inv}}$ as $F(s, t) = hs$ with $h \in (1, +\infty)$ and $\varphi \in \Phi_u$.
For all $x, y \in [1, +\infty)$, we have

$$\begin{aligned} &a|x-y| - b \cdot 0 + c(|1-x| + |1-y|) - d\min\{|1-x|, |1-y| \times |1-x|\} \\ &= a|y-x| + c(x+y-2) - d(x-1). \end{aligned}$$

Choosing $a = 1, b = 4, c = 1$ and $d = 2$, one has $0 < c < \frac{b-a}{2}$ and

$$a|y-x| + c(x+y-2) - d(x-1) = \begin{cases} 2(y-x) & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x > y \end{cases}.$$

Therefore all the conditions in Theorem 2.1 are satisfied and 1 is the common fixed point of A, B, S and T.

Example 2.2. Let $X = [1, +\infty)$ and $x, y \in X$ with usual metric d. We define maps $A, B, S, T : X \mapsto X$ such that,

$$Sx = x, \quad x \in [1, +\infty),$$

$$Tx = x^2 \quad x \in [1, +\infty),$$

$$Ax = \begin{cases} 1, & x \in [1, 2) \\ \frac{x-1}{3}, & x \in [2, +\infty) \end{cases},$$

$$Bx = \begin{cases} 1, & x \in [1, 2) \\ \frac{x^2-1}{3}, & x \in [2, +\infty) \end{cases}.$$

It is obvious from the example that mappings S and T are surjective. On taking sequence $\{x_n\} = 1 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to +\infty$, then it is easy to show that pair (A, S) is $A-$Semi compatible and $A-$reciprocal continuous. Also pair (B, T) is $B-$semi compatible and $B-$reciprocal continuous.
Now we define $G(t_1, \dots, t_6) : \underbrace{\mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{6} \mapsto \mathbb{R}$ as in Example 1.11:

$$G(t_1, \dots, t_6) = 1 - \frac{t_2(h^2 + 1)}{\max\{t_1, t_2, t_3, t_4, t_5 + t_6\}},$$

where $h \in (1, +\infty)$.
Define $F \in C_{\text{inv}}$ and $F(s, t) = hs$ for all $h \in (1, +\infty)$ and $\varphi \in \Phi_u$.
Now for all $x, y \in [1, 2)$, we have

$$\begin{aligned} &(h^2 + 1)d(Ax, By) \\ &= (h^2 + 1)|Ax - By| \\ &\leq d(Sx, Ty), \end{aligned}$$

which shows that

$$(h^2 + 1)d(Ax, By) \leq \max\{d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx) + d(Ax, Ty)\}.$$

Thus,

$$1 - \frac{(h^2 + 1)d(Ax, By)}{\max\{d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx) + d(Ax, Ty)\}} \geq 0.$$

Now for all $x, y \in [2, +\infty)$, we have

$$\begin{aligned} &(h^2 + 1)d(Ax, By) \\ &= (h^2 + 1)|Ax - By| \\ &= (h^2 + 1)|\frac{x-1}{3} - \frac{y^2-1}{3}|. \end{aligned}$$

Taking $h = \sqrt{2}$, we get

$$(h^2 + 1)d(Ax, By)$$

$$= |x - y^2|$$
$$= d(Sx, Ty)$$
$$\leq \max\{d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx) + d(Ax, Ty)\},$$

which implies that

$$1 - \frac{(h^2 + 1)d(Ax, By)}{\max\{d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx) + d(Ax, Ty)\}} \geq 0.$$

Therefore all the conditions of Theorem 2.1 hold and 1 is the unique common fixed point of A, B, S and T.

Setting $A = A_{2n}, B = A_{2n+1}, S = B_{2n}$ and $T = B_{2n+1}$, for $n \in \mathbb{N}_0$, in Theorem 2.1, we obtain the following result for two infinite families of self-mappings.

Theorem 2.2. Let $G \in \mathcal{G}_c$ and let $\{A_i\}_{i\in\mathbb{N}_0}$ and $\{B_i\}_{i\in\mathbb{N}_0}$ be two sequences of self-mappings on a complete metric space (X, d) satisfying condition (ii) of Theorem 2.1. Assume that, for every $n \in \mathbb{N}_0$, the following properties are satisfied:

(i) $A_{2n}(X) \subseteq B_{2n+1}(X)$ and $A_{2n+1}(X) \subseteq B_{2n}(X)$,
(ii) (A_{2n}, B_{2n}) is $A_{2n}-$semi-compatible and $A_{2n}-$reciprocal continuous,
(iii) (A_{2n+1}, B_{2n+1}) is $A_{2n+1}-$semi-compatible and $A_{2n+1}-$reciprocal continuous.

Then $\{A_i\}_{i\in\mathbb{N}_0}$ and $\{B_i\}_{i\in\mathbb{N}_0}$ have a unique common fixed point.

Proof. Fix $k \in \mathbb{N}_0$. From hypothesis, we deduce that A_{2k}, A_{2k+1}, B_{2k} and B_{2k+1} satisfy the inequality

$$G(d(B_{2k}x, B_{2k+1}y), d(A_{2k}x, A_{2k+1}y), d(B_{2k}x, A_{2k}x), d(B_{2k+1}y, A_{2k+1}y),$$
$$d(B_{2k+1}y, A_{2k}x), d(B_{2k}x, A_{2k+1}y)) \geq 0.$$

for all $x, y \in X$, where $G \in \mathcal{G}_c$. Therefore all the conditions of Theorem 2.1 are satisfied. So A_{2n}, A_{2n+1}, B_{2n} and B_{2n+1} have a common fixed point in X.

For the uniqueness, suppose that, for some $t, t' \in X$, $t \neq t'$, one has $A_{2n}t = B_{2n}t = A_{2n+1}t = B_{2n+1}t = t$ and $A_{2m}t' = B_{2m}t' = A_{2m+1}t' = B_{2m+1}t' = t'$, for all $n, m \in \mathbb{N}_0$. Using condition (ii) of Theorem 2, we obtain

$$G(d(B_{2n}t, B_{2m+1}t'), d(A_{2n}t, A_{2m+1}t'), d(B_{2n}t, A_{2n}t), d(B_{2m+1}t', A_{2m+1}t'),$$
$$d(B_{2m+1}t', A_{2n}t), d(B_{2n}t, A_{2m+1}t')) \geq 0,$$

that is

$$G(d(t, t'), d(t, t'), 0, 0, d(t, t'), d(t, t')) \geq 0.$$

which contradicts (G_3'). Therefore $t = t'$ and so the sequences of maps $\{A_i\}_{i\in\mathbb{N}_0}$ and $\{B_i\}_{i\in\mathbb{N}_0}$ have a unique fixed point.

Remark 2.2. According to [19, Remark 3.9], one can verify the followings:

(i) Theorems 2.1 improves Theorems 3.7, 3.8 in [19] by exploring new kind of semi-compatibility in lieu of responding compatibility of type (E).

(*ii*) Under the assumptions of Theorems 2.1, the main results of Theorems 3.1, 3.2 in [21] are still validate, respectively.

Theorem 2.3. Let $G \in \mathcal{G}_c$ and let A, B and S be three self-mappings on a complete metric space (X, d) satisfying the following conditions:

(*i*) S is surjective,
(*ii*) for all $x, y \in X$,

$$G(d(Sx, Sy), d(Ax, By), d(Ax, Sx), d(By, Sy), d(By, Sx), d(Ax, Sy)) \geq 0,$$

(*iii*) the pair (A, S) is $A-$semi-compatible and $A-$reciprocal continuous and the pair (B, S) is $B-$semi-compatible and $B-$reciprocal continuous.

Then A, B and S have a unique common fixed point.

Proof. Follows immediately in a similar way to that in the proof of Theorem 2.1.

Theorem 2.4. Let $G \in \mathcal{G}_c$ and let A, B and $\{g_i\}_{i\in\mathbb{N}}$ be self-mappings on a complete metric space (X, d) satisfying the following conditions:

(*i*) $\{g_i\}_{i\in\mathbb{N}}$ are surjective,
(*ii*) for all $x, y \in X$ and $i \in \mathbb{N}$,

$$G(d(g_i x, g_{i+1} y), d(Ax, By), d(Ax, g_{i+1} x), d(By, g_i y), d(By, g_i x), d(Ax, g_i y)) \geq 0,$$

(*iii*) the pair (A, g_i) is $A-$semi-compatible and $A-$reciprocal continuous and the pair (B, g_{i+1}) is $B-$semi-compatible and $B-$reciprocal continuous, for all $i \in \mathbb{N}$.

Then A, B and $\{g_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Letting $i = 1$ in the inequality of condition (*ii*), we get exactly the hypothesis of Theorem 2.1 for the mappings A, B, g_1 and g_2 and so they have a unique common fixed point t. Indeed, if t' is another fixed point for A, B, g_1, g_2 with $t' \neq t$, then using (*ii*) for $i = 1$, we have

$$\begin{aligned} &G(d(g_1 t', g_2 t), d(At', Bt), d(At', g_2 t'), d(Bt, g_1 t), d(Bt, g_1 t'), d(At', g_1 t)) \\ &= G(d(t', t), d(t', t), 0, 0, d(t', t), d(t, t')) \geq 0, \end{aligned}$$

which contradicts (G_3'), hence $t' = t$.

By letting now $i = 2$, we get the hypothesis of Theorem 2.1 for the mappings A, B, g_2 and g_3, and consequently they have a unique common fixed point t''. Thus $t = t''$. In this way, we clearly see that t is the required point.

Remark 2.3. Theorem 2.4 generalizes Theorem 4.3 in [23] by reducing weak compatibility to certain semi-compatibility and reciprocal continuity.

Corollary 2.2. Let A, B, S and T be four self-mappings on a complete metric space (X, d) satisfying conditions (*i*), (*iii*) of Theorem 2.1. Suppose that, for all $x, y \in X$, we have the following inequality:

$$d^p(Sx, Ty) \geq ad^p(Ax, By) + bd^p(Ax, Sx) + cd^p(By, Ty),$$

where $a > 1$, $0 \leq c, b < 1$, $p \in \mathbb{N}$. Then A, B, S and T have a unique common fixed point.

Proof. Take a function $G \in \mathcal{G}_c$ as in Example 1.9 with $d = 0$. We have

$$G(d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty))$$
$$= d(Sx, Ty) - [ad^p(Ax, By) + bd^p(Ax, Sx) + cd^p(By, Ty)]^{\frac{1}{p}} \geq 0.$$

The conclusion follows from Theorem 2.1.

Remark 2.4. Corollary 2.2 is an improved result of Corollary 1 in [22] and Corollary 4.1 in [23] in the following aspects:

(i) compatibility of type (B) in [22] and weak compatibility in [23] are replaced by semi-compatibility and reciprocal continuity,
(ii) the requirement of continuity of mappings in [22] are relaxed.

Theorem 2.5. Let $G \in \mathcal{G}_c$ and let A and S be two self-mappings on a complete metric space (X, d) satisfying the following conditions:

(i) $A(X) \subseteq S(X)$ or A and S are surjective,
(ii) for every $x, y \in X$,

$$G(d(Sx, Sy), d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)) \geq 0,$$

(iii) the pair (A, S) is A-semi-compatible and A-reciprocal continuous or the pair (A, S) is S-semi-compatible and S-reciprocal continuous.

Then A and T have unique common fixed point.

Proof. Follows immediately using similar arguments as in the proof of Theorem 2.1.

3. Conclusion

Based on the notions of semi-compatibility and reciprocal continuity of a pair of self-mappings (f, g), we introduce some new types of a pair of self-mappings (f, g), called semi-compatibility of type (A), $f-$semi-compatibility of type (A), $g-$semi-compatibility of type (A), $f-$reciprocal continuity and $g-$reciprocal continuity, which are extensions of the corresponding notions. Some valid examples are set up to demonstrate the comparisons between these conceptions. Moreover, by using the inverse $C-$class functions, we provide a new kind of implicit relations $\mathcal{G}_c$ which is a generalization of the implicit relations $\mathcal{G}$ introduced by Djoudi. The achievement of this paper is to extend and improve the results of [19, 21–23] by using general implicit relations, weakening compatibility and dropping continuity.

Acknowledgments

We would like to thank Hainan Provincial Natural Science Foundation of China (Grant No.118MS081) and Science and Technology Cooperation Project of Sanya City(Grant Nos.2018YD13, 2018YD08) for providing a financial support.

Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux quations intgrales, *Fund. Math.*, **3** (1922), 133–183.
2. S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.*, **32** (1982), 149–153.
3. G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, **9** (1986), 771–779.
4. G. Jungck, Common fixed points for non-continuous non-self maps on non-metric spaces, *Far East J. Math. Sci.*, **4** (1996), 199–215.
5. M. Abbas, D. Gopal, S. Radenović, A note on recent introduced commutative conditions, *Indian J. Math.*, **55** (2012), 195–202.
6. G. Jungck, P. P. Murthy, Y. J. Cho, Compatible mappings of type (A) and common fixed points, *Math. Japonica*, **36** (1993), 381–390.
7. H. K. Pathak, M. S. Khan, Compatible mappings of type (B) and common fixed point theorems of Greguš type, *Czechoslovak Math. J.*, **45** (1995), 685–698.
8. H. K. Pathak, Y. J. Cho, S. M. Kang, B. Madharia, Compatible mappings of type (C) and common fixed point theorem of Greguš type, *Demonstr. Math.*, **31** (1998), 499–517.
9. H. K. Pathak, S. S. Chang, Y. J. Cho, Fixed point theorems for compatible mappings of type (P), *Indian J. Math.*, **36** (1994), 151–166.
10. B. Singh, S. Jain, Semi-compatibility, compatibility and fixed point theorems in fuzzy metric spaces, *J. Chungcheong Math. Soc.*, **18** (2005), 1–22.
11. A. S. Saluja, M. K. Jain, P. K. Jhade, Weak semi-compatibility and fixed point theorems, *Bull. Int. Math. Virt. Inst.*, **2** (2012), 205–217.
12. A. S. Saluja, M. K. Jain, Fixed point theorems under conditional semicompatibility with control function, *Adv. Fixed Point Theory*, **3** (2013), 648–666.
13. R. K. Bisht, N. Shahzad, Faintly compatibel mappings and common fixed points, *Fixed Point Theory Appl.*, (2013), Article ID: 156.
14. M. A. Al-Thagafi, N. Shahzad, Generalized I-nonexpansive self maps and invariant proximations, *Acta Math. Sinica*, **24** (2008), 867–876.
15. D. Dorić, Z. Kadelburg, S. Radenović, A note on occasionally weakly compatible mappings and common fixed points, *Fixed Point Theory*, **13** (2012), 475–480.
16. N. Hussain, S. M. Hussain, S. Radenović, Fixed points of weakly contractions through occasionally weak compatibility, *J. Comput. Anal. Appl.*, **13** (2011), 532–543.
17. S. Ivković, On Various Generalizations of Semi- $\mathcal{A}$-Fredholm Operators, *Complex Anal. Oper. Theory*, **14** (2020), Article No. 41.

18. M. A. Alghamdi, S. Radenović, N. Shahzad, On Some Generalizations of Commuting Mappings, *Abstr. Appl. Anal.*, **2012** (2012), Article ID: 952052.
19. M. R. Singh, Y. M. Singh, On various type of compatible mappings and common fixed point theorems for non-continuous mappings, *Hacet. J. Math. Stat.*, **40** (2011), 503–513.
20. R. P. Pant, A common fixed point theorem under a new condition, *Indian J. Pure Appl. Math.*, **30** (1999), 147–152.
21. A. H. Ansari, V. Popa, Y. M. Singh, M. S. Khan, Fixed point theorems of an implicit relation via $C-$class function in metric spaces, *J. Adv. Math. Stud.*, **13** (2020), 1–10.
22. A. Djoudi, A unique common fixed point for compatible mappings of type (B) satisfying an implicit relation, *Demonstratio Math.*, **36** (2003), 763–770.
23. A. Djoudi, General fixed point theorems for weakly compatible maps, *Demonstratio Math.*, **38** (2005), 197–206.
24. M. R. Singh, Y. M. Singh, Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type, *Int. J. Math. Sci. Eng. Appl.*, **1** (2007), 299–315.
25. R. P. Pant, R. K. Bisht, D. Arora, Weak reciprocal continuity and fixed point theorems, *Ann. Univ. Ferrara*, **57** (2011), 181–190.
26. N. Saleem, A. H. Ansari, M. K. Jain, Some fixed point theorems of inverse $C-$class function under weak semi-compatibility, *J. Fixed Point Theory*, **2018** (2018), 9.
27. A. H. Ansari, Note on "$\varphi - \psi -$ contractive type mappings and related fixed point", The 2nd regional conference on Mathematics and Applications, Payame Noor University, 2014, 377–380.

AIMS Press