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Research article

On stable solutions of the weighted Lane-Emden equation involving Grushin operator

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Abstract: In this article, we study the weighted Lane-Emden equation

$$\mathrm{div}_G(\omega_1(z)|\nabla_G u|^{p-2}\nabla_G u) = \omega_2(z)|u|^{q-1}u,\ z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $N = N_1 + N_2 \geq 2$, $p \geq 2$ and $q > p - 1$, while $\omega_i(z) \in L^1_{\text{loc}}(\mathbb{R}^N) \setminus \{0\}(i = 1, 2)$ are nonnegative functions satisfying $\omega_1(z) \leq C\|z\|_G^\theta$ and $\omega_2(z) \geq C'\|z\|_G^d$ for large $\|z\|_G$ with $d > \theta - p$. Here $\alpha \geq 0$ and $\|z\|_G = (|x|^{2(1+\alpha)} + |y|^2)^{\frac{1}{2(1+\alpha)}}$. div_G (resp., ∇_G) is Grushin divergence (resp., Grushin gradient). We prove that stable weak solutions to the equation must be zero under various assumptions on d, θ, p, q and $N_\alpha = N_1 + (1 + \alpha)N_2$.

Keywords: stable weak solutions; Liouville-type theorem; Grushin operator; Lane-Emden nonlinearity
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1. Introduction

In this work, we examine the nonexistence of stable weak solutions of the problem

$$\mathrm{div}_G(\omega_1(z)|\nabla_G u|^{p-2}\nabla_G u) = \omega_2(z)|u|^{q-1}u,\ z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.1}$$

Here and thereafter, we assume that $p \geq 2$, $q > p-1$ and $\omega_i(z) \in L^1_{\text{loc}}(\mathbb{R}^N)\setminus\{0\}(i = 1, 2)$ are nonnegative functions. For $z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $\alpha \geq 0$, we define the Grushin gradient ∇_G and Grushin divergence div_G as

$$\nabla_G u = (\nabla_x u, (1 + \alpha)|x|^\alpha \nabla_y u),$$

$$\mathrm{div}_G\mathbf{v} = \mathrm{div}_x\mathbf{v} + (1+\alpha)|x|^{\alpha}\mathrm{div}_y\mathbf{v}.$$

The Grushin operator Δ_G is denoted by

$$\Delta_G u = \mathrm{div}_{\mathrm{G}}(\nabla_G u) = \Delta_x u + (1+\alpha)^2|x|^{2\alpha}\Delta_y u,$$

which is just the well-known Laplace operator when $\alpha = 0$.

The anisotropic dilation attached to Δ_G is defined by

$$\tau_\delta(z) = (\delta x, \delta^{1+\alpha}y),\ \delta > 0,\ z = (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

It is easy to check that

$$d\tau_\delta(z) = \delta^{N_\alpha}dxdy = \delta^{N_\alpha}dz,$$

where $N_\alpha = N_1 + (1+\alpha)N_2$ is the homogeneous dimension with respect to the dilation τ_δ and $dxdy$ denotes the Lebesgue measure on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. The norm of z (also known as the Grushin distance) is defined by

$$\|z\|_G = (|x|^{2(1+\alpha)} + |y|^2)^{\frac{1}{2(1+\alpha)}},\ z = (x,y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

The p-Laplace type Grushin operator is given by

$$\Delta_G^p u = \mathrm{div}_G(|\nabla_G u|^{p-2}\nabla_G u).$$

For $x = (x_1, x_2, ..., x_{N_1})$, when x goes to 0 this operator is degenerate if $p > 2$ and is singular as $1 < p < 2$. An significant characteristic of the operator exhibits different scaling behaviors in x and y directions around $x = 0$. In recent years, the degenerate elliptic operators have been attracted the interest of many mathematicians and been studied extensively, we refer the reader to [9, 16, 18, 30].

Let us review some results related to our problem. For problem (1.1) in the case $\alpha = 0$, it becomes the weighted Lane-Emden equation

$$-\mathrm{div}(\omega_1(z)|\nabla u|^{p-2}\nabla u) = \omega_2(z)|u|^{q-1}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Recently, much attention has been focused on studying of the nonexistence and stability of solutions to nonlinear elliptic equations like (1.2). The definition of stability arises in several branches of physical sciences, where a system is called in a stable state if it can recover from small perturbations. More details on physical motivation and recent developments on the topic of stable solutions, we refer to [11].

In the past years, the Liouville property has been refined considerably and emerged as one of the most powerful tools in the study of initial and boundary value problems for nonlinear PDEs. It turn out that one can obtain from Liouville-type theorems a variety of results on qualitative properties of solutions such as universal, pointwise, a priori estimates of local solutions; universal and singularity estimates; decay estimates; blow-up rate of solutions of nonstationary problems, etc., see [25, 26] and references therein.

Liouville-type theorems for stable solutions concern about the nonexistence of nontrivial solutions. The pioneering work in this direction is due to Farina [12], where the author established thoroughly the Liouville-type theorem for stable classical solutions of problem (1.2) with $\omega_1(z) \equiv 1 \equiv \omega_2(z)$

and $p = 2$. He showed that the problem possesses no nontrivial stable C^2 solutions if and only if $1 < q < q_c(N)$, where

$$q_c(N) = \begin{cases} +\infty, & \text{if } N \leq 10, \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}, & \text{if } N \geq 11. \end{cases} \tag{1.3}$$

Moreover, this exponent is greater than the classical critical exponent $\frac{N+2}{N-2}$ [15] when $N > 2$. After that, above results were generalized to the weighted case in [5, 8, 13, 17, 29]. In [8], under the restriction that the solutions are locally bounded, the authors presented the nonexistence of nontrivial stable weak solutions of problem (1.2) with $p = 2$, $\omega_1(z) \equiv 1$ and $\omega_2(z) = |z|^d$. In [29], this restriction was withdrawn.

Theorem 1.1. ([29]) *Let u be a stable weak solution of* (1.2) *with* $p = 2$, $\omega_1(z) \equiv 1$ *and* $\omega_2(z) = |z|^d$, *where* $d > -2$. *Then u is a trivial solution provided* $1 < q < q(N, d)$. *Here*

$$q(N,d) = \begin{cases} +\infty, & \text{if } N \leq 10+4d, \\ \frac{(N-2)(N-6-2d)-2(2+d)^2+2(2+d)\sqrt{(2+d)(2N-2+d)}}{(N-2)(N-10-4d)}, & \text{if } N > 10+4d. \end{cases} \tag{1.4}$$

In [5], based on the Farina's approach, Cowan and Fazly established several Liouville-type theorems for stable positive classical solutions of problem (1.2) with $p = 2$ under different assumptions on $\omega_i (i = 1, 2)$. Later, several attempts have been made to extend Farina's results to weighted quasilinear equation (1.2). It is worthy to note that in [6], the authors extended Farina's results to p-Laplace equations for the first time. Paper [3] deals with the problem (1.2) with $\omega_1 \equiv 1$, the author only considered the stable $C^{1,\delta}_{\mathrm{loc}}(\mathbb{R}^N)$ solutions, which are locally bounded. Similar works can be found in [4, 19, 21, 22] and the references therein.

We now consider the case $\alpha > 0$, the problem (1.1) is weighted quasilinear problem involving Grushin operator. It is well-known that the Grushin operator belongs to the wide class of subelliptic operators studied by Franchi et al. in [14](see also [2]). Via Kelvin transform and the method of moving planes, the Liouville-type theorem has been established by Monticelli [24] (resp., Yu [32]) for nonnegative classical (resp., weak) solutions of the problem $-\Delta_G u = u^q$ in $\mathbb{R}^N$, the optimal exponent is $1 < q < \frac{N_\alpha+2}{N_\alpha-2}$. Recently, Duong and Nguyen [10] studied elliptic equations involving Grushin operator and advection

$$-\Delta_G u + \nabla_G w \cdot \nabla_G u = \|z\|_G^s |u|^{q-1} u, \ \text{ in } \mathbb{R}^N, \ s \geq 0.$$

By mean of Farina's approach, the authors obtained several Liouville-type theorems for a class of stable sign-changing weak solutions.

Very recently, Le [20] considered the elliptic problem

$$-\mathrm{div}_{\mathrm{G}}(w_1 \nabla_G u) = w_2 f(u), \ \text{ in } \Omega,$$

with homogeneous Dirichlet boundary condition. Using variable technique, nonexistence of stable weak solutions is proved under various assumptions on Ω, $w_i (i = 1, 2)$ and f. When $\Omega = \mathbb{R}^N$ and f has power or exponential growth, the author also constructed some examples to show the sharpness of his results. For other results of Liouville-type theorems related to Grushin operators or more general subelliptic operators, we refer the reader to [1, 7, 23, 27, 28, 31] and the references therein.

A natural question is whether the analogous Liouville property holds for equation (1.1) with $p \geq 2$, $\alpha > 0$ and $\omega_i \not\equiv 1 (i = 1, 2)$. The present paper is an attempt to answer this interesting question.

Motivated by the aforementioned works, we prove the nonexistence of nontrivial stable weak solution to problem (1.1). Since $|\nabla_G u|^{p-2}$ is degenerate when $p > 2$, solutions to (1.1) must be understood in the weak sense. Moreover, solutions to elliptic equations with Hardy potential may possess singularities. Therefore, we need to study weak solutions of (1.1) in a suitable weighted Sobolev space. Based on this reality, we define

$$\|\psi\|_{\omega_1} = \Big(\int_{\mathbb{R}^N} \omega_1(z) |\nabla_G \psi|^p dz \Big)^{1/p}$$

for $\psi \in C_0^\infty(\mathbb{R}^N)$ and denote by $W^{1,p,\alpha}(\mathbb{R}^N; \omega_1)$ the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the $\| \cdot \|_{\omega_1}$-norm. Note that for $\omega_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we have $C_0^1(\mathbb{R}^N) \subset W^{1,p,\alpha}(\mathbb{R}^N; \omega_1)$. Denote also by $W^{1,p,\alpha}_{\mathrm{loc}}(\mathbb{R}^N; \omega_1)$ the space of all functions u such that $u\psi \in W^{1,p,\alpha}(\mathbb{R}^N; \omega_1)$ for all $\psi \in C_0^1(\mathbb{R}^N)$.

Definition 1.2. *Let $X = W^{1,p,\alpha}_{\mathrm{loc}}(\mathbb{R}^N; \omega_1)$, we say that $u \in X$ is a weak solution of* (1.1) *if $\omega_2(z)|u|^q \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and for all $\psi \in C_0^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} \omega_1(z) |\nabla_G u|^{p-2} \nabla_G u \cdot \nabla_G \psi dz = \int_{\mathbb{R}^N} \omega_2(z) |u|^{q-1} u \psi dz. \tag{1.5}$$

Definition 1.3. *A weak solution u of* (1.1) *is stable if $\omega_2(z)|u|^{q-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and for all $\psi \in C_0^1(\mathbb{R}^N)$ we have*

$$q \int_{\mathbb{R}^N} \omega_2(z) |u|^{q-1} \psi^2 dz \leq \int_{\mathbb{R}^N} \omega_1(z) \Big(|\nabla_G u|^{p-2} |\nabla_G \psi|^2 + (p-2) |\nabla_G u|^{p-4} (\nabla_G u \cdot \nabla_G \psi)^2 \Big) dz. \tag{1.6}$$

In other words, the stability condition translates into the fact that the second variation of the energy functional

$$I(u) = \int_{\mathbb{R}^N} \Big(\frac{\omega_1(z) |\nabla_G u|^p}{p} - \frac{\omega_2(z) |u|^{q+1}}{q+1} \Big) dz$$

is nonnegative. Therefore, all the local minima of the functional are stable weak solutions of (1.1).

Remark 1.4. Let u be a stable weak solution of (1.1), by (1.6) and $p \geq 2$, it follows that

$$q \int_{\mathbb{R}^N} \omega_2(z) |u|^{q-1} \psi^2 dz \leq (p-1) \int_{\mathbb{R}^N} \omega_1(z) |\nabla_G u|^{p-2} |\nabla_G \psi|^2 dz. \tag{1.7}$$

It is obvious that (1.5)–(1.7) hold for all $\psi \in W^{1,p,\alpha}(\mathbb{R}^N; \omega_1)$ by density arguments.

Throughout this paper, we assume that the functions $\omega_i(z) (i = 1, 2)$ satisfy the following assumptions

(H) $\omega_i(z) \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \setminus \{0\} (i = 1, 2)$ are nonnegative functions. In addition, there exist $d > \theta - p$, $C, C' > 0$ and $R_0 > 0$ such that

$$\omega_1(z) \leq C \|z\|_G^\theta, \ \omega_2(z) \geq C' \|z\|_G^d, \ \forall \|z\|_G \geq R_0.$$

To facilitate the writing, we denote $\mu_0(p, \theta, d) = \frac{(p-\theta)(p+3)+4d}{p-1}$.

Now, we are ready to give the main result.

Theorem 1.5. *Let $u \in X$ be a stable weak solution of problem* (1.1) *with $p \geq 2$. Assume that* (H) *holds. We further suppose that*

$$\begin{cases} p-1 < q < \infty, & if \ \ N_\alpha \leq \mu_0(p,\theta,d), \\ p-1 < q < q_c(p,N_\alpha,\theta,d), & if \ \ N_\alpha > \mu_0(p,\theta,d) \end{cases}$$

with the critical exponent

$$\begin{aligned} q_c(p,N_\alpha,\theta,d) &= p-1+\Big((p-\theta+d)\big(p(N_\alpha-p+\theta)-2(p-\theta+d)\big) \\ &+2\sqrt{(p-\theta+d)(N_\alpha+d+\frac{N_\alpha+\theta-p}{p-1})}\Big)\Big) \div \Big((N_\alpha-p+\theta)(N_\alpha-\mu_0(p,\theta,d))\Big). \end{aligned} \tag{1.8}$$

Then $u \equiv 0$ in $\mathbb{R}^N$.

Remark 1.6. Indeed, the assumption on q in Theorem 1.5 is equivalent to

$$N_\alpha < p-\theta+\frac{(p-\theta+d)((p-1)(p-2)+2q+2\sqrt{q(q-p+1)})}{(p-1)(q-p+1)}. \tag{1.9}$$

The critical exponent $q_c(p,N_\alpha,\theta,d)$ can be calculated directly from the above quadratic inequality for q. Moreover, our result recovers the known result for weighted elliptic problem in [5, Theorem 3] when $\alpha = 0$ and $p = 2$, and the previous result in [13, Theorem 2.3] with $\alpha = \theta = 0$ and $p = 2$.

Remark 1.7. If $\alpha = 0$, we obtain

$$\begin{aligned} &q_c(p,N,\theta,d) = \\ &p-1+\frac{(p-\theta+d)(p(N-p+\theta)-2(p-\theta+d)+2\sqrt{(p-\theta+d)(N+d+\frac{N+\theta-p}{p-1})})}{(N-p+\theta)(N-\mu_0(p,\theta,d))}. \end{aligned}$$

It is the critical exponent q_c in [4,22]. If $\alpha = \theta = 0$, we have

$$q_c(p,N,0,d) = p-1+\frac{(p+d)(p(N-p)-2(p+d)+2\sqrt{(p+d)(N+d+\frac{N-p}{p-1})})}{(N-p)(N-\mu_0(p,0,d))},$$

which is the critical exponent q_c in [3]. If $\alpha = \theta = d = 0$, then

$$q_c(p,N,0,0) = p-1+\frac{p^2(N-p)-2p^2+2p^2\sqrt{\frac{N-1}{p-1}}}{(N-p)(N-\mu_0(p,0,0))},$$

which equals the critical exponent p_c in [6]. If $p = 2$ and $\alpha = \theta = 0$, we conclude

$$q_c(2,N,0,d) = 1+\frac{2(2+d)(N-4-d+\sqrt{(2+d)(2N-2+d)})}{(N-2)(N-10-4d)}.$$

It is the critical exponent $\bar{p}(d)$ in [8]. If $p = 2$ and $\alpha = \theta = d = 0$, we get

$$q_c(2,N,0,0) = 1+\frac{4(N-4+2\sqrt{N-1})}{(N-2)(N-10)},$$

which is the critical exponent $p_c(N)$ in [12]. Finally, if $p = 2$, Theorem 1.5 recovers the known result for the Grushin operator in [20, Proposition 3], and if $\alpha = 0$, Theorem 1.5 recovers [22, Theorem 1.5]. Therefore, our conclusion of Theorem 1.5 can be viewed as an expansion of the previous works, which is therefore interesting and meaningful.

The rest of the paper is devoted to the proof of Theorem 1.5. In the following , C stands for a generic positive constant which may vary from line to line even in the same line. If this constant depends on an arbitrary small number ε, then we denote it by C_ε.

2. Proofs

We begin with the following proposition.

Proposition 2.1. *Let $u \in X$ be a stable weak solution of* (1.1) *with $q > p-1 \geq 1$. Then for every $s \in (1, h(p))$, where*

$$h(t) = -1 + \frac{2(t + \sqrt{t(t-p+1)})}{p-1}, \quad t > p-1 \tag{2.1}$$

and for any constant $m \geq \frac{q+s}{q-p+1}$, there exists a constant $C > 0$ depending only on p, q, s and m such that

$$\int_{\mathbb{R}^N} \left(\omega_2(z)|u|^{q+s} + \omega_1(z)|\nabla_G u|^p |u|^{s-1}\right)\varphi^{pm} dz \leq C \int_{\mathbb{R}^N} \omega_1(z)^{\frac{q+s}{q-p+1}} \omega_2(z)^{-\frac{p-1+s}{q-p+1}} |\nabla_G \varphi|^{\frac{p(q+s)}{q-p+1}} dz \tag{2.2}$$

holds for all functions $\varphi \in C_0^1(\mathbb{R}^N)$ satisfying $0 \leq \varphi \leq 1$ and $\nabla_G \varphi = 0$ in a neighborhood of $\{z \in \mathbb{R}^N : \omega_2(z) = 0\}$.

Proof. Some ideas in this proof are inspired by [22, 31]. Since $\omega_i(z)(i = 1, 2)$ are not necessarily locally bounded, the solutions of (1.1) are not necessarily locally bounded. To overcome this difficulty, we shall construct a sequence of suitable cut-off functions. Let n be a positive integer, we denote

$$\delta_n(t) = \begin{cases} |t|^{\frac{s-1}{2}} t, & |t| \leq n, \\ n^{\frac{s-1}{2}} t, & |t| > n, \end{cases} \qquad \nu_n(t) = \begin{cases} |t|^{s-1} t, & |t| \leq n, \\ n^{s-1} t, & |t| > n. \end{cases}$$

By a direct computation, we obtain that for any $t \in \mathbb{R}$, there exists a positive constant C depending only on s such that

$$\begin{gathered} \delta_n^2(t) = t\nu_n(t), \quad \delta_n'(t)^2 \leq \frac{(1+s)^2}{4s} \nu_n'(t), \\ |\delta_n(t)|^p \delta_n'(t)^{2-p} + |\nu_n(t)|^p \nu_n'(t)^{1-p} \leq C|t|^{p-1+s}. \end{gathered} \tag{2.3}$$

Moreover, since $u \in X$ we deduce that $\delta_n(u)$, $\nu_n(u) \in X$ for any $n \in \mathbb{Z}^+$.

For any nonnegative function $\phi \in C_0^1(\mathbb{R}^N)$ satisfying $0 \leq \phi \leq 1$, set $\psi = \nu_n(u)\phi^p$ as a test function in (1.5). Then we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \nu_n'(u)\phi^p dz + p\int_{\mathbb{R}^N} \omega_1(z)\nu_n(u)\phi^{p-1}|\nabla_G u|^{p-2}\nabla_G u \cdot \nabla_G \phi dz \\ &= \int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1} u \nu_n(u)\phi^p dz. \end{aligned}$$

Applying Young's inequality, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \nu_n'(u)\phi^p dz \\
&\le p \int_{\mathbb{R}^N} \omega_1(z)|\nu_n(u)||\nabla_G u|^{p-1}|\nabla_G \phi|\phi^{p-1} dz + \int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz \\
&\le \varepsilon \int_{\mathbb{R}^N} \left(\omega_1(z)^{(p-1)/p}|\nabla_G u|^{p-1}\nu_n'(u)^{(p-1)/p}\phi^{p-1}\right)^{p/(p-1)} dz \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} \left(\omega_1(z)^{1/p}|\nu_n(u)|\nu_n'(u)^{-(p-1)/p}|\nabla_G \phi|\right)^p dz + \int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz \\
&= \varepsilon \int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \nu_n'(u)\phi^p dz + C_\varepsilon \int_{\mathbb{R}^N} \omega_1(z)|\nu_n(u)|^p \nu_n'(u)^{1-p}|\nabla_G \phi|^p dz \\
&\quad + \int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(1-\varepsilon)\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \nu_n'(u)\phi^p dz \\
&\le C_\varepsilon \int_{\mathbb{R}^N} \omega_1(z)|\nu_n(u)|^p \nu_n'(u)^{1-p}|\nabla_G \phi|^p dz + \int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz.
\end{aligned}
\tag{2.4}
$$

On the other hand, by virtue of the stability definition, we take $\psi = \delta_n(u)\phi^{p/2}$ in (1.7) and yield

$$
\begin{aligned}
&q\int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz \\
&\le (p-1)\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \delta_n'(u)^2\phi^p dz \\
&\quad + p(p-1)\int_{\mathbb{R}^N} \omega_1(z)\delta_n'(u)|\delta_n(u)||\nabla_G u|^{p-1}|\nabla_G \phi|\phi^{p-1} dz \\
&\quad + \frac{p^2(p-1)}{4}\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^{p-2}\delta_n^2(u)|\nabla_G \phi|^2\phi^{p-2} dz.
\end{aligned}
\tag{2.5}
$$

We use Young's inequality to estimate the last two terms of the right-hand side of (2.5)

$$
\begin{aligned}
&p(p-1)\int_{\mathbb{R}^N} \omega_1(z)\delta_n'(u)|\delta_n(u)||\nabla_G u|^{p-1}|\nabla_G \phi|\phi^{p-1} dz \\
&\le \frac{\varepsilon(p-1)}{2}\int_{\mathbb{R}^N} \left(\omega_1(z)^{(p-1)/p}|\nabla_G u|^{p-1}\delta_n'(u)^{2(p-1)/p}\phi^{p-1}\right)^{p/(p-1)} dz \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} \left(\omega_1(z)^{1/p}|\delta_n(u)|\delta_n'(u)^{(2-p)/p}|\nabla_G \phi|\right)^p dz \\
&= \frac{\varepsilon(p-1)}{2}\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \delta_n'(u)^2\phi^p dz + C_\varepsilon \int_{\mathbb{R}^N} \omega_1(z)|\delta_n(u)|^p \delta_n'(u)^{2-p}|\nabla_G \phi|^p dz
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{p^2(p-1)}{4}\int_{\mathbb{R}^N}\omega_1(z)|\nabla_G u|^{p-2}\delta_n^2(u)|\nabla_G\phi|^2\phi^{p-2}dz\\
&\le \frac{\varepsilon(p-1)}{2}\int_{\mathbb{R}^N}\left(\omega_1(z)^{(p-2)/p}|\nabla_G u|^{p-2}\delta_n'(u)^{2(p-2)/p}\phi^{p-2}\right)^{p/(p-2)}dz\\
&\quad+C_\varepsilon\int_{\mathbb{R}^N}\left(\omega_1(z)^{2/p}\delta_n^2(u)\delta_n'(u)^{2(2-p)/p}|\nabla_G\phi|^2\right)^{p/2}dz\\
&=\frac{\varepsilon(p-1)}{2}\int_{\mathbb{R}^N}\omega_1(z)|\nabla_G u|^p\delta_n'(u)^2\phi^p dz+C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)|\delta_n(u)|^p\delta_n'(u)^{2-p}|\nabla_G\phi|^p dz.
\end{aligned}
$$

Substituting the above two inequalities into (2.5), one has

$$
\begin{aligned}
q\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz\ &\le (1+\varepsilon)(p-1)\int_{\mathbb{R}^N}\omega_1(z)|\nabla_G u|^p\delta_n'(u)^2\phi^p dz\\
&\quad+C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)|\delta_n(u)|^p\delta_n'(u)^{2-p}|\nabla_G\phi|^p dz.
\end{aligned}
\tag{2.6}
$$

With the help of (2.3), it follows from (2.4) and (2.6) that

$$
\begin{aligned}
q\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz\ &\le \frac{(1+\varepsilon)(1+s)^2(p-1)}{4s}\int_{\mathbb{R}^N}\omega_1(z)|\nabla_G u|^p\nu_n'(u)\phi^p dz\\
&\quad+C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)|\delta_n(u)|^p\delta_n'(u)^{2-p}|\nabla_G\phi|^p dz\\
&\le \frac{(1+\varepsilon)(1+s)^2(p-1)}{4s(1-\varepsilon)}\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz\\
&\quad+C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)(|\delta_n(u)|^p\delta_n'(u)^{2-p}+|\nu_n(u)|^p\nu_n'(u)^{1-p})|\nabla_G\phi|^p dz\\
&\le \frac{(1+\varepsilon)(1+s)^2(p-1)}{4s(1-\varepsilon)}\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz\\
&\quad+C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)|u|^{s+p-1}|\nabla_G\phi|^p dz.
\end{aligned}
$$

Consequently,

$$
q_\varepsilon\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz\le C_\varepsilon\int_{\mathbb{R}^N}\omega_1(z)|u|^{p-1+s}|\nabla_G\phi|^p dz,\tag{2.7}
$$

where $q_\varepsilon=q-\frac{(1+\varepsilon)(1+s)^2(p-1)}{4s(1-\varepsilon)}$. Since $\lim_{\varepsilon\to0^+}q_\varepsilon=q_0=q-\dfrac{(1+s)^2(p-1)}{4s}$, we have $q_0>0$ under assumption on $s\in(1,h(p))$, we can fix some $\varepsilon>0$ sufficiently small such that $q_\varepsilon>0$. Therefore,

$$
\int_{\mathbb{R}^N}\omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz\le C\int_{\mathbb{R}^N}\omega_1(z)|u|^{p-1+s}|\nabla_G\phi|^p dz,\tag{2.8}
$$

where positive constant C depends only on $q,\ p$ and s.

From (2.8) and Fatou's Lemma, we derive as $n \to \infty$,

$$\int_{\mathbb{R}^N} \omega_2(z)|u|^{q+s}\phi^p dz \le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}|\nabla_G \phi|^p dz. \tag{2.9}$$

On the other hand, choosing $\varepsilon = 1/2$ in (2.4) and combining (2.3) with (2.8), we can find

$$\begin{aligned}
&\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p \nu_n'(u)\phi^p dz \\
&\le C \int_{\mathbb{R}^N} \omega_1(z)|\nu_n(u)|^p \nu_n'(u)^{1-p}|\nabla_G \phi|^p dz + 2\int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}u\nu_n(u)\phi^p dz \\
&\le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}|\nabla_G \phi|^p dz + 2\int_{\mathbb{R}^N} \omega_2(z)|u|^{q-1}\delta_n^2(u)\phi^p dz \\
&\le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}|\nabla_G \phi|^p dz.
\end{aligned}$$

Letting $n \to \infty$ in above inequality, we have from Fatou's Lemma that

$$\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p|u|^{s-1}\phi^p dz \le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}|\nabla_G \phi|^p dz. \tag{2.10}$$

Now, we assert that (2.2) holds true. In fact, we can select some positive constant $m \gg 1$ such that

$$\frac{(m-1)(q+s)}{p-1+s} \ge m, \quad \text{or} \quad m \ge \frac{q+s}{q-p+1}.$$

Recalling $0 \le \phi(z) \le 1$ in $\mathbb{R}^N$, we obtain

$$(\phi(z))^{\frac{p(m-1)(q+s)}{p-1+s}} \le (\phi(z))^{pm}, \quad \forall z \in \mathbb{R}^N.$$

Then, by (2.9) with $\phi = \varphi^m$ and Hölder's inequality, one sees

$$\begin{aligned}
&\int_{\mathbb{R}^N} \omega_2(z)|u|^{q+s}\varphi^{pm} dz \le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}\varphi^{p(m-1)}|\nabla_G \varphi|^p dz \\
&\le C\Big(\int_{\mathbb{R}^N} \big(\omega_2(z)^{\frac{p-1+s}{q+s}}|u|^{p-1+s}\varphi^{p(m-1)}\big)^{\frac{q+s}{p-1+s}} dz\Big)^{\frac{p-1+s}{q+s}} \Big(\int_{\mathbb{R}^N} \big(\omega_1(z)\omega_2(z)^{-\frac{p-1+s}{q+s}}|\nabla_G \varphi|^p\big)^{\frac{q+s}{q-p+1}} dz\Big)^{\frac{q-p+1}{q+s}} \\
&= C\Big(\int_{\mathbb{R}^N} \omega_2(z)|u|^{q+s}\varphi^{\frac{p(m-1)(q+s)}{p-1+s}} dz\Big)^{\frac{p-1+s}{q+s}} \Big(\int_{\mathbb{R}^N} \omega_1(z)^{\frac{q+s}{q-p+1}}\omega_2(z)^{-\frac{p-1+s}{q-p+1}}|\nabla_G \varphi|^{\frac{p(q+s)}{q-p+1}} dz\Big)^{\frac{q-p+1}{q+s}} \\
&\le C\Big(\int_{\mathbb{R}^N} \omega_2(z)|u|^{q+s}\varphi^{pm} dz\Big)^{\frac{p-1+s}{q+s}} \Big(\int_{\mathbb{R}^N} \omega_1(z)^{\frac{q+s}{q-p+1}}\omega_2(z)^{-\frac{p-1+s}{q-p+1}}|\nabla_G \varphi|^{\frac{p(q+s)}{q-p+1}} dz\Big)^{\frac{q-p+1}{q+s}}.
\end{aligned} \tag{2.11}$$

Hence,

$$\int_{\mathbb{R}^N} \omega_2(z)|u|^{q+s}\varphi^{pm} dz \le C \int_{\mathbb{R}^N} \omega_1(z)^{\frac{q+s}{q-p+1}}\omega_2(z)^{-\frac{p-1+s}{q-p+1}}|\nabla_G \varphi|^{\frac{p(q+s)}{q-p+1}} dz. \tag{2.12}$$

Analogously, take $\phi = \varphi^m$ in (2.10) and combining (2.11) with (2.12), one can achieve

$$\begin{aligned}
\int_{\mathbb{R}^N} \omega_1(z)|\nabla_G u|^p|u|^{s-1}\varphi^{pm} dz &\le C \int_{\mathbb{R}^N} \omega_1(z)|u|^{p-1+s}\varphi^{p(m-1)}|\nabla_G \varphi|^p dz \\
&\le C \int_{\mathbb{R}^N} \omega_1(z)^{\frac{q+s}{q-p+1}}\omega_2(z)^{-\frac{p-1+s}{q-p+1}}|\nabla_G \varphi|^{\frac{p(q+s)}{q-p+1}} dz.
\end{aligned}$$

Therefore, combining this with (2.12), (2.2) is obtained immediately. This completes the proof. □

Let $R > 0$, $\Omega_{2R} = B_1(0, 2R) \times B_2(0, 2R^{1+\alpha})$, where $B_i \subset \mathbb{R}^{N_i}$, with $i = 1, 2$, are open ball centered at 0, the radii are $2R$ and $2R^{1+\alpha}$, respectively. We consider a cut-off function $\kappa(t) \in C_0^\infty([0, +\infty); [0, 1])$ satisfying

$$\kappa(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t \ge 2. \end{cases}$$

Moreover, we define

$$\varphi_{1,R}(x) = \kappa(\frac{|x|}{R}), \quad x \in \mathbb{R}^{N_1}, \quad \varphi_{2,R}(y) = \kappa(\frac{|y|}{R^{1+\alpha}}), \quad y \in \mathbb{R}^{N_2}$$

and

$$\varphi_R(x, y) = \varphi_{1,R}(x)\varphi_{2,R}(y), \quad (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{2.13}$$

The direct calculations yield

$$\begin{aligned} |\nabla_x \varphi_{1,R}| \le CR^{-1}, \ |\nabla_y \varphi_{2,R}| \le CR^{-(1+\alpha)}, \\ |\Delta_x \varphi_{1,R}| \le CR^{-2}, \ |\Delta_y \varphi_{2,R}| \le CR^{-2(1+\alpha)}, \\ |\nabla_G \varphi_R|^2 + |\Delta_G \varphi_R| \le CR^{-2}, \quad \forall x \in \mathbb{R}^{N_1}, \ y \in \mathbb{R}^{N_2}, \\ R \le ||z||_G \le CR, \quad \forall z = (x, y) \in \Omega_{2R} \setminus \Omega_R, \end{aligned} \tag{2.14}$$

where positive constant C is independent of R.

Proof of Theorem 1.5. By contradiction, we assume that (1.1) admits a nontrivial stable weak solution u. Applying (2.2) with a test function $\varphi_R(x, y)$ which is given by (2.13), we derive that for all $R \ge R_0$ (R_0 comes from a set of assumptions denoted by (H)), there exists a constant $C > 0$ independent of R such that

$$\int_{\Omega_R} (\omega_2(z)|u|^{q+s} + \omega_1(z)|\nabla_G u|^p |u|^{s-1}) dz \le CR^{-\frac{p(q+s)}{q-p+1}} \int_{\Omega_{2R} \setminus \Omega_R} ||z||_G^{\frac{(q+s)\theta - (p-1+s)d}{q-p+1}} dz \le CR^\mu \tag{2.15}$$

with

$$\mu = N_\alpha - \frac{(p - \theta)(q + s) + (p - 1 + s)d}{q - p + 1}.$$

Here, we have utilized (H) and (2.14).

Clearly, if $\mu < 0$ for some certain $s \in (1, h(p))$, it implies from (2.15) that

$$\int_{\mathbb{R}^N} (\omega_2(z)|u|^{q+s} + \omega_1(z)|\nabla_G u|^2 |u|^{s-1}) dz = 0$$

as $R \to +\infty$, i.e., $u \equiv 0$ in $\mathbb{R}^N$, which contradicts the assumption about u. Therefore, we obtain the desired conclusion.

Now, we consider the cases in which $\mu < 0$. Set

$$g(t) = \frac{(p - \theta)(t + h(t)) + (p - 1 + h(t))d}{t - p + 1}, \ t > p - 1,$$

where $h(t)$ is given by (2.1). Elementary calculations lead to

$$\lim_{t\to(p-1)^+} h(t) = 1,\ \lim_{t\to+\infty} h(t) = +\infty,\ h'(t) > 0,\ t > p-1$$

and

$$\lim_{t\to(p-1)^+} g(t) = +\infty,\ \lim_{t\to+\infty} g(t) = \mu_0(p,\theta,d).$$

Since

$$g'(t) = \frac{(p-\theta+d)}{(t-p+1)^2}\Big(-p - \frac{t-p+1}{\sqrt{t(t-p+1)}}\Big) < 0,\ t > p-1,$$

the function $g(t)$ is decreasing on $(p-1,+\infty)$.

Therefore, if $N_\alpha \le \mu_0(p,\theta,d)$, then $N_\alpha < g(t)$ for $t > p-1$. Thus if we fix $s \in (1, h(p))$ sufficiently near to $h(p)$, we see that

$$\mu = N_\alpha - \frac{(p-\theta)(q+s)+(p-1+s)d}{q-p+1} < 0,\ q > p-1,$$

which implies the nonexistence of nontrivial stable weak solutions of (1.1).

Assume now $N_\alpha > \mu_0(p,\theta,d)$. According to the monotonicity of $g(t)$, there is a unique critical value $q_c(p,N,\theta,d) > p-1$ such that $N_\alpha < g(t)$ for $p-1 < t < q_c(p,N,\theta,d)$. So if we choose $s \in (1,h(p))$ sufficiently near to $h(p)$, we get

$$\mu = N_\alpha - \frac{(p-\theta)(q+s)+(p-1+s)d}{q-p+1} < 0,\ p-1 < q < q_c(p,N,\theta,d),$$

which implies the nonexistence of nontrivial stable weak solutions of (1.1). Moreover, $q_c(p,N,\theta,d)$ can be derived from the equation $N_\alpha = h(p)$, which are given by (1.8). The proof is finished. □

3. Conclusion

We consider a class of weighted Lane-Emden equation involving Grushin operator. Based on the approaches by Farina [12] and Le [22], we establish a Liouville-type theorem for the class of stable sign-changing weak solution under various assumptions.

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Conflict of interest

The authors declare no conflict of interest.

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