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Research article

Revisiting of the BT-inverse of matrices

Wanlin Jiang and Kezheng Zuo*

School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China

* **Correspondence:** Email: xiangzuo28@163.com.

Abstract: In this paper, we discuss different characteristics of the BT-inverse of a square matrix introduced by Baksalary and Trenkler [On a generalized core inverse, Appl. Math. Comput., **236** (2014), 450–457]. While the BT-inverse is defined by a expression, we present some necessary and sufficient conditions for a matrix to be the BT-inverse. Then we give a canonical form of BT-inverse and investigate the relationships between BT-inverse and other generalized inverses by Core-EP decomposition. Some properties of BT-inverse concerned with some classes of special matrix are identified by Core-EP decomposition. Furthermore new representations of BT-inverse are given by the maximal classes of matrices.

Keywords: BT-inverse; Core-EP decomposition; Hartwig-Spindelböck decomposition
Mathematics Subject Classification: 15A09

1. Introduction

For many different generalized inverses such as $A^{\dagger}, A^{D}, A^{\#\!\!\!\bigcirc}, A^{\dagger\!\!\!\bigcirc}, A^{D,\dagger}, A^{(B,C)}, A^{Ⓦ}$ below can all be characterized by several equations respectively, while there is no such equations to define $A^{\diamond}$. Our main aim is to develop some necessary and sufficient conditions for a matrix to be the BT-inverse by equations and derive some properties of the BT-inverse.

Throughout this paper, we denote the set of $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. We denote the identity matrix of order n by I_n, the range space, the null space, the conjugate transpose and the rank of the matrix $A \in \mathbb{C}^{m\times n}$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $r(A)$, respectively. The index of $A \in \mathbb{C}^{n\times n}$, denoted by Ind$(A)$, is the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$. $P_{\mathcal{L},\mathcal{M}}$ stands for the projector (idempotent) on the space $\mathcal{L}$ along the $\mathcal{M}$. For $A \in \mathbb{C}^{m\times n}$, P_A represents the orthogonal projection onto $\mathcal{R}(A)$, i.e. $P_A = P_{\mathcal{R}(A)} = AA^{\dagger}$.

For the readers' convenience, we will first recall the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^{\dagger}$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (1) above is called an inner inverse of A and is denoted by $A^{(1)}$. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (2) above is called an outer inverse of A and is denoted by $A^{(2)}$. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (1) and condition (3) above is denoted by $A^{(1,3)}$. The symbol $A\{1\}$, $A\{1,3\}$ stand for the set of all $A^{(1)}$, $A^{(1,3)}$ respectively. Let $A \in \mathbb{C}^{m\times n}$ be of rank r, and $\mathcal{T}$, $\mathcal{S}$ be a subspace of $\mathbb{C}^n, \mathbb{C}^m$ where $\mathcal{T}$, $\mathcal{S}$ is of dimension t ($\leqslant r$), $m - t$, respectively. Then a matrix X satisfies $X = XAX$, $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$ if and only if $A\mathcal{T} \oplus \mathcal{S} = \mathbb{C}^m$, and in this case X denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$ is unique.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, denoted by A^D [2], is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying:

$$XAX = X, \quad AX = XA, \quad XA^{k+1} = A^k.$$

Especially, if $\mathrm{Ind}(A) = 1$, then the Drazin inverse of A is called the group inverse of A and is denoted by $A^{\#}$.

Baksalary and Trenkler [3] introduced the core inverse on the $\mathbb{C}_n^{\mathrm{CM}}$ ($\mathbb{C}_n^{\mathrm{CM}} = \{A|A \in \mathbb{C}^{n\times n}, r(A) = r(A^2)\}$): the core inverse of $A \in \mathbb{C}_n^{\mathrm{CM}}$ is defined to be the unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$AX = P_A, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)$$

and denoted by $A^{\textcircled{\#}}$ (see [3–6]).

Moreover, three kinds of generalizations of the core inverse were given for $n \times n$ complex matrices, called core-EP inverse, DMP-inverse and BT-inverse, respectively.

Firstly, for $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying:

$$XAX = X, \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

is called the Core-EP inverse of A written as $A^{\textcircled{\dagger}}$ (see [7–10]). Moreover, it is seen that $A^{\textcircled{\dagger}} = (A^{k+1}(A^k)^\dagger)^\dagger$ (see [7, Theorem 2.7]).

Secondly, the DMP-inverse of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, written by $A^{D,\dagger}$ [11, 12], is defined as the unique matrix $A \in \mathbb{C}^{n\times n}$ satisfying:

$$XAX = X, \quad XA = A^D A, \quad A^k X = A^k A^\dagger.$$

Moreover, it was proved that $A^{D,\dagger} = A^D AA^\dagger$. Also, the dual DMP inverse of A was introduced in [12], namely $A^{\dagger,D} = A^\dagger AA^D$.

Thirdly, the BT-inverse of $A \in \mathbb{C}^{n\times n}$, denoted by $A^\diamond$[13], is defined as

$$A^\diamond = (A^2 A^\dagger)^\dagger = (AP_A)^\dagger.$$

In recent years, some new generalized inverses are introduced. The (B,C)-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [14, 15], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying:

$$XAB = B, \quad CAX = C, \quad \mathcal{R}(X) = \mathcal{R}(B), \quad \mathcal{N}(X) = \mathcal{N}(C),$$

where $B, C \in \mathbb{C}^{n\times m}$.

In [16], Wang and Chen introduced a new generalized inverse called the weak group inverse of $A \in \mathbb{C}^{n\times n}$, denoted by $A^{ⓦ}$. It is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying:

$$AX^2 = X, \quad AX = A^{\textcircled{\dagger}}A.$$

Moreover, it is proved that $A^{ⓦ} = (A^{\textcircled{\dagger}})^2 A$.

While the authors in [13] introduced the BT-inverse defined as $A^{\diamond} = (AP_A)^{\dagger}$, the characterizations of how a matrix is $A^{\diamond}$, however, seldom gave. In this paper, we concern more on the necessary and sufficient conditions for a matrix to be $A^{\diamond}$ and characterize the relationships between $A^{\diamond}$ and other generalized inverses. The research is as follows. In Section 2, some indispensable matrix classes and lemmas are given. In Section 3, some characterizations of $A^{\diamond}$ are given too. In Section 4, we first derive a canonical form of $A^{\diamond}$ by Core-EP decomposition and verify the validity of it by Example 1. By the canonical form of $A^{\diamond}$ and Core-EP decomposition, we obtain the relationships between $A^{\diamond}$ and other generalized inverses and some properties of $A^{\diamond}$ when $A^{\diamond}$ or A belongs to some special matrix classes. In Section 5, we extend the representation $A^{\diamond} = (AP_A)^{\dagger}$ to a more general one by the maximal classes of matrices.

2. Preliminaries

For convenience, some matrix classes will be given as follows.

These symbols $\mathbb{C}_n^{\mathrm{CM}}$, $\mathbb{C}_n^{\mathrm{P}}$, $\mathbb{C}_n^{\mathrm{OP}}$ and $\mathbb{C}_n^{\mathrm{EP}}$ will stand for the subsets of $\mathbb{C}^{n\times n}$ consisting of core matrices, projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices) and EP (Range-Hermitian) matrices , respectively, i.e.,

$$\begin{aligned}
&\mathbb{C}_n^{\mathrm{CM}} = \{A|A \in \mathbb{C}^{n\times n}, r(A^2) = r(A)\},\\
&\mathbb{C}_n^{\mathrm{P}} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A\},\\
&\mathbb{C}_n^{\mathrm{OP}} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A = A^{\dagger}\},\\
&\mathbb{C}_n^{\mathrm{EP}} = \{A|A \in \mathbb{C}^{n\times n}, AA^{\dagger} = A^{\dagger}A\} = \{A|A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\}.
\end{aligned}$$

In order to present some characterizations and properties of $A^{\diamond}$, we need to introduce the following lemmas.

Lemma 2.1. [17] *Let $A \in \mathbb{C}^{n\times n}$, $r(A) = r$. Then we have*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2.1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ is the diagonal matrix of singular values of A, $\sigma_i > 0 (i = 1, 2, \cdots, r)$ and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r. \tag{2.2}$$

Moreover, from (2.1), it follows that

$$A^{\dagger} = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*, \quad P_A = AA^{\dagger} = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2.3}$$

By [12, 13], we obtain that

$$A^D = U\begin{bmatrix} (\Sigma K)^D & ((\Sigma K)^D)^2\Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{2.4}$$

$$A^\diamond = U\begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^* \tag{2.5}$$

and

$$A^{\textcircled{\#}} = U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{2.6}$$

The lemma below gives the Core-EP decomposition introduced by Wang which plays an important role in this paper.

Lemma 2.2. [9] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$*. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = A_1 + A_2 = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{2.7}$$

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*,$$

where $T \in \mathbb{C}^{t\times t}$ is nonsingular with $t = r(T) = r(A^k)$ and N is nilpotent of index k.

Lemma 2.3. [*18, Lemma 6*] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$ *be the form of* (2.7)*. Then*

$$A^\dagger = U\begin{bmatrix} T^*\triangle & -T^*\triangle S N^\dagger \\ (I_{n-t} - N^\dagger N)S^*\triangle & N^\dagger - (I_{n-t} - N^\dagger N)S^*\triangle S N^\dagger \end{bmatrix}U^*, \tag{2.8}$$

where N is not necessary nilpotent, $\triangle = (TT^* + S(I_{n-t} - N^\dagger N)S^*)^{-1}$, $t = r(A^k)$.

From (2.7) and (2.8), a straightforward computation shows that

$$AA^\dagger = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^*, \tag{2.9}$$

$$A^\dagger A = U\begin{bmatrix} T^*\triangle T & T^*\triangle S(I_{n-t} - N^\dagger N) \\ (I_{n-t} - N^\dagger N)S^*\triangle T & N^\dagger N + (I_{n-t} - N^\dagger N)S^*\triangle S(I_{n-t} - N^\dagger N) \end{bmatrix}U^*. \tag{2.10}$$

Lemma 2.4. [*13, Theorem 1*] *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$AA^\diamond = P_{AP_A}, \quad A^\diamond A = P_{\mathcal{R}(P_A A^*),\mathcal{N}((AP_A)^\dagger A)}. \tag{2.11}$$

3. Different characterizations about BT-inverse

It is well-known that some of generalized inverses such as MP-inverse, Drazin inverse, DMP-inverse, etc. can be presented as an outer inverse under the condition of prescribed range and null space. Therefore, we will prove that the same holds in the case of BT-inverse as follows. In the following theorem, we show the other characterizations of BT-inverse by the fact that $A^\diamond AA^\diamond = A^\diamond$.

Theorem 3.1. *Let $A, X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*
(a) $X = A^{\diamond}$;
(b) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(P_AA^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(P_AA^*)$, *i.e.,* $X = A^{(2)}_{\mathcal{R}(P_AA^*),\mathcal{N}(P_AA^*)}$;
(c) $XAX = X$, $AX = A(AP_A)^{\dagger}$ *and* $XA = (AP_A)^{\dagger}A$;
(d) $XAX = X$, $AX = P_{AP_A}$ *and* $XA = (AP_A)^{\dagger}A$.

Proof. (a) $\Rightarrow$ (b). From the definition of BT-inverse and Lemma 2.4, we derive that

$$A(AP_A)^{\dagger} = AA^{\diamond} = P_{AP_A}, \tag{3.1}$$

moreover

$$(AP_A)^{\dagger}A(AP_A)^{\dagger} = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger}. \tag{3.2}$$

From the definition of BT-inverse and (3.2), it follows that

$$A^{\diamond}AA^{\diamond} = (AP_A)^{\dagger}A(AP_A)^{\dagger} = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger} = (AP_A)^{\dagger} = A^{\diamond},$$

$$\mathcal{R}(A^{\diamond}) = \mathcal{R}((AP_A)^{\dagger}) = \mathcal{R}((AP_A)^*) = \mathcal{R}(P_AA^*),$$

$$\mathcal{N}(A^{\diamond}) = \mathcal{N}((AP_A)^{\dagger}) = \mathcal{N}((AP_A)^*) = \mathcal{N}(P_AA^*).$$

(b) $\Rightarrow$ (c). From [19, Remark 3.1], we have that $A^{(2)}_{\mathcal{R}(A^{\diamond}),\mathcal{N}(A^{\diamond})}$ exits. It is easy to check that $A^{\diamond} = A^{(2)}_{\mathcal{R}((AP_A)^{\dagger}),\mathcal{N}((AP_A)^{\dagger})} = A^{(2)}_{\mathcal{R}(P_AA^*),\mathcal{N}(P_AA^*)}$. Since $X = A^{(2)}_{\mathcal{R}(P_AA^*),\mathcal{N}(P_AA^*)}$ and the uniqueness of X, we obtain that $X = A^{\diamond}$. Then the rest of proof is trivial.

(c) $\Rightarrow$ (d). Since $AX = A(AP_A)^{\dagger}$, by (3.1), we obtain that $AX = AP_A(AP_A)^{\dagger} = P_{AP_A}$.

(d) $\Rightarrow$ (a). By the condition, we conclude that

$$X = XAX = XAP_A(AP_A)^{\dagger} = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger} = (AP_A)^{\dagger} = A^{\diamond}.$$

□

In the following theorem, we present a connection between (B,C)-inverse and BT-inverse showing that a BT-inverse of a matrix $A \in \mathbb{C}^{n\times n}$ is its (P_AA^*, P_AA^*)-inverse.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$. Then $A^{\diamond} = A^{(P_AA^*,P_AA^*)}$.*

Proof. From the definition of BT-inverse and (3.1), it follows that

$$A^{\diamond}AP_AA^* = (AP_A)^{\dagger}AP_A(AP_A)^* = (AP_A)^*,$$

$$P_AA^*AA^{\diamond} = (AP_A)^*A(AP_A)^{\dagger} = (AP_A)^*(AP_A)(AP_A)^{\dagger} = (AP_A)^*,$$

$$\mathcal{R}(A^{\diamond}) = \mathcal{R}(P_AA^*), \mathcal{N}(A^{\diamond}) = \mathcal{N}(P_AA^*).$$

Hence $A^{\diamond} = A^{(P_AA^*,P_AA^*)}$. □

According to the fact that $\mathcal{R}(A^{\diamond}) = \mathcal{R}(P_AA^*)$ and $\mathcal{N}(A^{\diamond}) = \mathcal{N}(P_AA^*)$, there are several different characterizations of BT-inverse as follows.

Theorem 3.3. *Let $A, X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) $X = A^{\diamond}$;

(b) $AX = A(AP_A)^{\dagger}$, $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$;

(c) $AX = P_{AP_A}$, $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$;

(d) $P_A X = (AP_A)^{\dagger}$, $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$;

(e) $A^{\dagger}X = A^{\dagger}(AP_A)^{\dagger}$, $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$;

(f) $XA = (AP_A)^{\dagger}A$, $\mathcal{N}(X) = \mathcal{N}(P_A A^*)$;

(g) $XA = P_{\mathcal{R}(P_A A^*),\mathcal{N}((AP_A)^{\dagger}A)}$, $\mathcal{N}(X) = \mathcal{N}(P_A A^*)$.

Proof. That (a) implies all other items (b), (c), (d), (e), (f) and (g) can be checked directly by Theorem 3.1, the definition of BT-inverse and Lemma 2.4.

$(b) \Rightarrow (a)$. By $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$, we have $X = (AP_A)^{\dagger}T$ for some $T \in \mathbb{C}^{n\times n}$. By (3.2), then

$$X = (AP_A)^{\dagger}T = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger}T = (AP_A)^{\dagger}AX = (AP_A)^{\dagger}A(AP_A)^{\dagger} = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger} = A^{\diamond}.$$

$(c) \Rightarrow (b)$. Since $AX = P_{AP_A}$, by (3.1), we obtain that $AX = P_{AP_A} = AP_A(AP_A)^{\dagger} = A(AP_A)^{\dagger}$.

$(d) \Rightarrow (a)$. By $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$, we get $X = (AP_A)^{\dagger}T$ for some $T \in \mathbb{C}^{n\times n}$. By (3.2), then

$$X = (AP_A)^{\dagger}T = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger}T = (AP_A)^{\dagger}AP_AX = (AP_A)^{\dagger}A(AP_A)^{\dagger} = (AP_A)^{\dagger}AP_A(AP_A)^{\dagger} = A^{\diamond}.$$

$(e) \Rightarrow (d)$. Premultiplying $A^{\dagger}X = A^{\dagger}(AP_A)^{\dagger}$ by A, we obtain that $P_AX = P_A(AP_A)^{\dagger} = (AP_A)^{\dagger}$.

$(f) \Rightarrow (a)$. By $\mathcal{N}(X) = \mathcal{N}(P_A A^*)$, we obtain $X = K(AP_A)^{\dagger}$ for some $K \in \mathbb{C}^{n\times n}$. By (3.2), then

$$X = K(AP_A)^{\dagger} = K(AP_A)^{\dagger}A(AP_A)^{\dagger} = XA(AP_A)^{\dagger} = (AP_A)^{\dagger}A(AP_A)^{\dagger} = (AP_A)^{\dagger} = A^{\diamond}.$$

$(g) \Rightarrow (a)$. Since $XA = P_{\mathcal{R}(P_A A^*),\mathcal{N}((AP_A)^{\dagger}A)} = P_{\mathcal{R}((AP_A)^{\dagger}),\mathcal{N}((AP_A)^{\dagger}A)}$, we get $XA(AP_A)^{\dagger} = (AP_A)^{\dagger}$. By $\mathcal{N}(X) = \mathcal{N}(P_A A^*)$, we have $X = K(AP_A)^{\dagger}$ for some $K \in \mathbb{C}^{n\times n}$. Then

$$X = K(AP_A)^{\dagger} = K(AP_A)^{\dagger}A(AP_A)^{\dagger} = XA(AP_A)^{\dagger} = A^{\diamond}.$$

□

Remark 3.4. *Notice that the condition $\mathcal{R}(X) = \mathcal{R}(P_A A^*)$ in items (b), (c), (d) and (e) of Theorem 3.3 can be replaced by $\mathcal{R}(X) \subseteq \mathcal{R}(P_A A^*)$. Also the condition $\mathcal{N}(X) = \mathcal{N}(P_A A^*)$ in items (f), (g) of Theorem 3.3 can be replaced by $\mathcal{N}(P_A A^*) \subseteq \mathcal{N}(X)$.*

Theorem 3.5. *Let $A, X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) $X = A^{\diamond}$;

(b) $r(X) = r(A^2)$, $XA(AP_A)^* = (AP_A)^*$ *and* $AX = A(AP_A)^{\dagger}$;

(c) $r(X) = r(A^2)$, $(AP_A)^*AX = (AP_A)^*$ *and* $XA = A(AP_A)^{\dagger}A$;

(d) $r(X) = r(A^2)$, $XA(AP_A)^* = (AP_A)^*$ *and* $AX = P_{AP_A}$;

(e) $r(X) = r(A^2)$, $XA(AP_A)^* = (AP_A)^*$ *and* $P_AX = (AP_A)^{\dagger}$;

(f) $r(X) = r(A^2)$, $XA(AP_A)^* = (AP_A)^*$ *and* $A^{\dagger}X = A^{\dagger}(AP_A)^{\dagger}$;

(g) $r(X) = r(A^2)$, $(AP_A)^*AX = (AP_A)^*$ *and* $XA = P_{\mathcal{R}(P_A A^*),\mathcal{N}(P_A A^*)}$.

Proof. $(a) \Rightarrow (b)$. For $X = A^{\diamond}$, we get that $r(A^{\diamond}) = r(AP_A)$. For $\mathcal{R}(A^2) = \mathcal{R}(AP_AA) \subseteq \mathcal{R}(AP_A) \subseteq \mathcal{R}(A^2)$, then we get that $\mathcal{R}(AP_A) = \mathcal{R}(A^2)$, hence $r(A^{\diamond}) = r(AP_A) = r(A^2)$. From the definition of BT-inverse and the latter half of (2.11), we derive that $A^{\diamond}A(AP_A)^* = (AP_A)^*$ and $AA^{\diamond} = A(AP_A)^{\dagger}$.

That (a) implies all other items (c), (d), (e), (f) and (g) can be similarly proved.

$(b) \Rightarrow (a)$. Combining $r(X) = r(A^2) = r(AP_A)$ with $XA(AP_A)^* = (AP_A)^*$, we obtain $\mathcal{R}(X) = \mathcal{R}(P_AA^*)$. Hence it follows from (b) of Theorem 3.3 that $X = A^{\diamond}$.

$(c) \Rightarrow (a)$. From $r(X) = r(A^2) = r(AP_A)$ and $(AP_A)^*AX = (AP_A)^*$, we get $\mathcal{N}(X) = \mathcal{N}(P_AA^*)$. Hence we get $X = A^{\diamond}$ by (f) of Theorem 3.3.

The proofs of $(d) \Rightarrow (a)$, $(e) \Rightarrow (a)$ and $(f) \Rightarrow (a)$ are analogous to that of $(b) \Rightarrow (a)$. Also $(g) \Rightarrow (a)$ follows similarly as in the part $(c) \Rightarrow (a)$. □

4. Canonical form and some properties of BT-inverse

In this section, we first give the canonical form of BT-inverse by using Core-EP decomposition. Then some properties of BT-inverse will be given by utilizing the definition and the canonical form of BT-inverse.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (2.7). *Then*

$$A^{\diamond} = U\begin{bmatrix} T^*\triangle & -T^*\triangle SN^{\diamond} \\ (P_N - P_{N^{\diamond}})S^*\triangle & N^{\diamond} - (P_N - P_{N^{\diamond}})S^*\triangle SN^{\diamond} \end{bmatrix}U^*, \tag{4.1}$$

where $\triangle = [TT^* + S(P_N - P_{N^{\diamond}})S^*]^{-1}$.

Proof. By (2.9) of Lemma 2.3, we get that

$$A^{\diamond} = (AP_A)^{\dagger} = \left(U\begin{bmatrix} T & SP_N \\ 0 & NP_N \end{bmatrix}U^*\right)^{\dagger} = U\begin{bmatrix} T & SP_N \\ 0 & NP_N \end{bmatrix}^{\dagger}U^*.$$

From (2.8) of Lemma 2.3, we have that

$$A^{\diamond} = U\begin{bmatrix} T^*\triangle & -T^*\triangle SP_NN^{\diamond} \\ (P_N - P_{N^{\diamond}})S^*\triangle & N^{\diamond} - (P_N - P_{N^{\diamond}})S^*\triangle SP_NN^{\diamond} \end{bmatrix}U^*,$$

where $\triangle = [TT^* + S(P_N - P_NP_{N^{\diamond}})S^*]^{-1}$.

It is easy to check that $P_NN^{\diamond} = N^{\diamond}$ by (2.3) and (2.5). Hence

$$A^{\diamond} = U\begin{bmatrix} T^*\triangle & -T^*\triangle SN^{\diamond} \\ (P_N - P_{N^{\diamond}})S^*\triangle & N^{\diamond} - (P_N - P_{N^{\diamond}})S^*\triangle SN^{\diamond} \end{bmatrix}U^*,$$

where $\triangle = [TT^* + S(P_N - P_NP_{N^{\diamond}})S^*]^{-1} = [TT^* + S(P_N - P_{N^{\diamond}})S^*]^{-1}$. □

Next, we will verify the correctness of the expression (4.1) as follows.

Example 1. Given matrix

$$A = \begin{bmatrix}
0.5191 & 0.5922 & 0.8096 & 0.3341 & 0.7491 & 0.0801 & 0.3664 & 0.6988 & 0.1834 & 0.1987 \\
0.3897 & 0.2828 & 0.5073 & 0.6534 & 1.1533 & 0.1098 & 0.5847 & 0.7325 & 0.9618 & -0.1729 \\
1.1683 & 0.3983 & 0.5191 & 0.3454 & 0.5072 & 0.3863 & -0.0372 & 1.0568 & 0.5583 & 0.3311 \\
0.8177 & 0.3113 & 1.0133 & 0.7451 & 0.6738 & 0.5783 & 0.0714 & 0.1584 & 0.0524 & 0.1195 \\
0.8294 & 0.3371 & 0.8222 & 0.9830 & 1.4529 & -0.1282 & -0.0299 & 0.3507 & 0.7032 & 0.5101 \\
0.7189 & 0.0200 & 0.8032 & 0.5823 & 0.5989 & 0.5793 & 0.4254 & 0.0908 & 0.4943 & 0.9090 \\
0.5923 & 0.6193 & 0.5685 & 0.4965 & 0.4073 & 0.3121 & 0.1642 & 0.2414 & 0.3979 & 0.3385 \\
1.1399 & -0.0433 & 0.0694 & 0.6084 & 0.7149 & 0.8039 & 0.2417 & 0.3485 & 0.4629 & 0.3436 \\
0.3883 & 0.3624 & 0.9590 & 0.4811 & 0.5895 & 0.2980 & 0.3599 & 0.4059 & 0.3457 & 0.4983 \\
0.4063 & 0.3763 & 0.2283 & 0.7486 & 1.0007 & 0.8114 & 0.4796 & 0.3602 & -0.1058 & 0.5583
\end{bmatrix}.$$

By the definition of BT-inverse, it turns out that

$$r1 = (AP_A)^{\dagger} = \begin{bmatrix} 1.2507 & -0.0226 & -0.0663 & -0.6058 & 0.2154 & 0.2790 & 0.0448 & 1.1114 & -0.8224 & -1.1597 \\ 0.2073 & 0.1052 & -0.0244 & -0.6952 & 0.0287 & -0.4345 & 1.7112 & -0.1370 & -0.2799 & -0.0348 \\ 0.0140 & 0.0540 & 0.0636 & 0.7072 & -0.1473 & 0.2573 & -0.5076 & -0.4322 & 0.5430 & -0.3668 \\ -1.3634 & 0.0740 & 0.1260 & 0.7041 & 0.1529 & -0.4219 & 0.4008 & -0.5087 & 0.3199 & 0.6953 \\ 0.5619 & 0.1288 & -0.2313 & -0.2723 & 0.5275 & -0.0635 & -0.6696 & 0.2278 & -0.2854 & 0.1209 \\ -1.1576 & 0.1623 & 0.3274 & 0.7051 & -0.6261 & -0.1563 & 0.1071 & -0.2272 & 0.4554 & 0.7088 \\ 1.5361 & 0.6173 & -0.7601 & -0.9706 & -0.4480 & 0.6544 & 0.0055 & 0.8060 & -0.3946 & -0.7210 \\ -0.2661 & -0.2618 & 0.7675 & 0.4571 & -0.2999 & -0.3593 & -0.7283 & -0.5581 & 0.5152 & 0.8686 \\ -0.7639 & 0.1845 & -0.2022 & 0.2158 & -0.1960 & 0.1204 & 0.9133 & -0.0600 & 0.1465 & -0.0297 \\ -0.0314 & -0.7461 & 0.1217 & -0.5991 & 0.2760 & 0.5232 & 0.0696 & -0.2471 & 0.2561 & 0.4946 \end{bmatrix}.$$

Assume that A is of the form (2.7), we obtain that

$$U = \begin{bmatrix} 0.2922 & 0.3567 & 0.2593 & 0.3427 & 0.0253 & 0.2289 & 0.6603 & -0.0103 & -0.3353 & -0.0323 \\ 0.3330 & -0.4801 & 0.1381 & 0.4201 & 0.2087 & 0.3648 & -0.0541 & 0.1485 & 0.4849 & -0.1622 \\ 0.3316 & 0.2241 & 0.3288 & -0.4195 & 0.1860 & -0.2229 & 0.0996 & -0.4765 & 0.4440 & -0.1934 \\ 0.2955 & -0.1610 & -0.2112 & 0.0892 & 0.2513 & 0.1539 & -0.3538 & -0.5646 & -0.5254 & -0.1655 \\ 0.3824 & 0.1840 & -0.2339 & -0.1527 & -0.7245 & 0.4114 & -0.1367 & -0.0663 & 0.1455 & 0.0590 \\ 0.3327 & 0.1261 & -0.6005 & 0.1360 & 0.0405 & -0.4635 & 0.1275 & 0.2584 & 0.0703 & -0.4358 \\ 0.2649 & 0.2488 & 0.0699 & -0.4504 & 0.4074 & 0.3044 & -0.2470 & 0.5568 & -0.1739 & -0.0187 \\ 0.2975 & -0.5624 & -0.1892 & -0.3928 & 0.0290 & -0.0960 & 0.4427 & 0.0202 & -0.1203 & 0.4293 \\ 0.3012 & 0.2960 & -0.0514 & 0.3464 & 0.1693 & -0.2581 & -0.2645 & -0.0215 & 0.1389 & 0.7170 \\ 0.3164 & -0.2249 & 0.5499 & 0.0559 & -0.3689 & -0.4342 & -0.2532 & 0.2237 & -0.2986 & -0.1258 \end{bmatrix},$$

$$T = \begin{bmatrix} 4.9695 & 0.5955 & 0.0256 & -0.1136 & -0.5071 & 0.4929 & 0.5074 & -1.0539 \\ 0 & -0.3745 & 0.7615 & -0.1175 & 0.0914 & -0.1466 & 0.0771 & -0.2335 \\ 0 & -0.6028 & -0.3745 & -0.1623 & 0.0536 & 0.3600 & 0.2317 & 0.3098 \\ 0 & 0 & 0 & -0.6836 & 0.1055 & 0.1977 & -0.5123 & -0.0501 \\ 0 & 0 & 0 & 0 & 0.6185 & -0.2633 & 0.4003 & 0.1953 \\ 0 & 0 & 0 & 0 & 0.0392 & 0.6185 & -0.0897 & -0.5558 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3705 & 0.2909 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.6230 & 0.3705 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.3973 & 0.0962 \\ 0.4349 & -0.0431 \\ 0.1727 & 0.1383 \\ 0.4068 & 0.0375 \\ 0.2437 & 0.0132 \\ 0.4205 & 0.5983 \\ -0.1454 & 0.3339 \\ -0.0429 & -0.0343 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

According to (4.1), a straightforward computation shows that

$$A^{\diamond} = \begin{bmatrix} 1.2507 & -0.0226 & -0.0663 & -0.6058 & 0.2154 & 0.2790 & 0.0448 & 1.1114 & -0.8224 & -1.1597 \\ 0.2073 & 0.1052 & -0.0244 & -0.6952 & 0.0287 & -0.4345 & 1.7112 & -0.1370 & -0.2799 & -0.0348 \\ 0.0140 & 0.0540 & 0.0636 & 0.7072 & -0.1473 & 0.2573 & -0.5076 & -0.4322 & 0.5430 & -0.3668 \\ -1.3634 & 0.0740 & 0.1260 & 0.7041 & 0.1529 & -0.4219 & 0.4008 & -0.5087 & 0.3199 & 0.6953 \\ 0.5619 & 0.1288 & -0.2313 & -0.2723 & 0.5275 & -0.0635 & -0.6696 & 0.2278 & -0.2854 & 0.1209 \\ -1.1576 & 0.1623 & 0.3274 & 0.7051 & -0.6261 & -0.1563 & 0.1071 & -0.2272 & 0.4554 & 0.7088 \\ 1.5361 & 0.6173 & -0.7601 & -0.9706 & -0.4480 & 0.6544 & 0.0055 & 0.8060 & -0.3946 & -0.7210 \\ -0.2661 & -0.2618 & 0.7675 & 0.4571 & -0.2999 & -0.3593 & -0.7283 & -0.5581 & 0.5152 & 0.8686 \\ -0.7639 & 0.1845 & -0.2022 & 0.2158 & -0.1960 & 0.1204 & 0.9133 & -0.0600 & 0.1465 & -0.0297 \\ -0.0314 & -0.7461 & 0.1217 & -0.5991 & 0.2760 & 0.5232 & 0.0696 & -0.2471 & 0.2561 & 0.4946 \end{bmatrix}.$$

Let $\| \cdot \|$ be the Frobenius norm, then it follows that

$$\| A^{\diamond} - r1 \| = 3.5313 \times 10^{-14}$$

which implies the validity of the representation (4.1).

Lemma 4.2. [*20*] *Let* $A \in \mathbb{C}^{n\times n}$ *written as in* (2.7). *Then*

$$A^D = U \begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\tilde{T} \\ 0 & 0 \end{bmatrix} U^*, \tag{4.2}$$

where $\tilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

In [13], the necessary and sufficient conditions for $A^{\diamond} = A^{\dagger}, A^{\#}$ were given by using the Hartwig-Spindelböck decomposition in Lemma 2.1. We will prove the conditions that $A^{\diamond} = A^{D}$, $A^{\diamond} = A^{\dagger,D}$ and $A^{\diamond} = A^{Ⓦ}$ are equivalent by utilizing Core-EP decomposition as follows.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ be decomposed by* (2.7). *Then the following statements are equivalent:*
(a) $S = 0$ *and* $N^2 = 0$;
(b) $A^{\diamond} = A^{D}$;
(c) $A^2 \in \mathbb{C}_n^{\mathrm{EP}}$;
(d) $A^{\diamond} = A^{\dagger,D}$;
(e) $A^{\diamond} = A^{Ⓦ}$.

Proof. $(a) \Longleftrightarrow (b)$. It follows from the definition of $A^{\diamond}$, Lemma 2.3 and (4.2).

$$\begin{aligned} A^{\diamond} = A^{D} &\Longleftrightarrow A^2A^{\dagger} = (A^{D})^{\dagger} \\ &\Longleftrightarrow U\begin{bmatrix} T & SP_N \\ 0 & NP_N \end{bmatrix}U^* = \left(U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\tilde{T} \\ 0 & 0 \end{bmatrix}U^*\right)^{\dagger} \\ &\Longleftrightarrow \tilde{T} = 0,\ SP_N = 0,\ NP_N = 0 \\ &\Longleftrightarrow S = 0,\ N^2 = 0. \end{aligned}$$

$(a) \Longleftrightarrow (c)$. From (2.7) and (2.8), we can calculate that

$$A^2 = U\begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix}U^*,$$

$$(A^2)^{\dagger} = U\begin{bmatrix} (T^2)^*\triangle' & -(T^2)^*\triangle'(TS+SN)(N^2)^{\dagger} \\ (I_{n-t}-(N^2)^{\dagger}N^2)(TS+SN)^*\triangle' & (N^2)^{\dagger} - (I_{n-t}-(N^2)^{\dagger}N^2)(TS+SN)^*\triangle'(TS+SN)(N^2)^{\dagger} \end{bmatrix}U^*,$$

where $\triangle' = (T^2(T^2)^* + (TS+SN)(I_{n-t}-(N^2)^{\dagger}N^2)(TS+SN)^*)^{-1}$.

Then it follows that

$$\begin{aligned} A^2 \in \mathbb{C}_n^{\mathrm{EP}} &\Longleftrightarrow A^2(A^2)^{\dagger} = (A^2)^{\dagger}A^2 \\ &\Longleftrightarrow (TS+SN) = (TS+SN)(N^2)^{\dagger}N^2,\ (N^2)^{\dagger}N^2 = N^2(N^2)^{\dagger} \\ &\Longleftrightarrow N^2 = 0,\ TS + SN = 0 \\ &\Longleftrightarrow S = 0,\ N^2 = 0. \end{aligned}$$

$(d) \Longrightarrow (a)$. We can get $AA^{\diamond} = AA^{D}$ by $A^{\diamond} = A^{\dagger,D}$. From (2.1), (2.4) and (2.5), $AA^{\diamond} = AA^{D}$ is equivalent to

$$U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Sigma K)^{D} & ((\Sigma K)^{D})^2\Sigma L \\ 0 & 0 \end{bmatrix}U^*.$$

Thus $\Sigma K(\Sigma K)^{\dagger} = \Sigma K(\Sigma K)^{D}$. Then we have $\Sigma K = (\Sigma K)^2(\Sigma K)^{D}$ which implies $\mathrm{Ind}(\Sigma K) \leq 1$, moreover $\mathrm{Ind}(A) \leq 2$.

Then let A be the form of (2.7). For $\mathrm{Ind}(A) \leq 2$, we obtain $N^2 = 0$. Representations (4.1) and (4.2) directly lead to

$$\begin{aligned} AA^{\diamond}=AA^{D} &\iff U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{*}\triangle & 0 \\ P_{N}S^{*}\triangle & 0 \end{bmatrix}U^{*}=U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{bmatrix}U^{*} \\ &\iff \begin{bmatrix} I_t & 0 \\ NP_{N}S^{*}\triangle & 0 \end{bmatrix}U^{*}=U\begin{bmatrix} I_t & (T^{k})^{-1}\widetilde{T} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence we get $\widetilde{T}=0$ which implies $S=0$.

$(a)\Longrightarrow(d)$. It can be directly checked.

$(a)\iff(e)$. From the definition of $A^{\diamond}$ and $A^{Ⓦ}$ together with Lemma 2.3, it follows that

$$\begin{aligned} A^{\diamond}=A^{Ⓦ} &\iff A^{2}A^{\dagger}=(A^{Ⓦ})^{\dagger} \\ &\iff U\begin{bmatrix} T & SP_{N} \\ 0 & NP_{N} \end{bmatrix}U^{*}=\left(U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^{*}\right)^{\dagger} \\ &\iff (T^{-2}S)^{*}=0,\ SP_{N}=0,\ NP_{N}=0 \\ &\iff S=0,\ N^{2}=0. \end{aligned}$$

□

From [7], it is shown that $A^{\diamond}=A^{\textcircled{\dagger}}$ is equivalent to $A^{\diamond}=A^{D,\dagger}$ by using the Hartwig-Spindelböck decomposition. Now we can verify the equivalence of $A^{\diamond}=A^{\textcircled{\dagger}}$ and $A^{\diamond}=A^{D,\dagger}$ by Core-EP decomposition.

Theorem 4.4. *Let $A\in\mathbb{C}^{n\times n}$ be decomposed by* (2.7). *Then the following statements are equivalent:*

(a) $A^{\diamond}=A^{\textcircled{\dagger}}$;

(b) $SN=0$ *and* $N^{2}=0$;

(c) $A^{\diamond}=A^{D,\dagger}$.

Proof. $(a)\iff(b)$. According to Corollary 3.3 in [9], we have that

$$A^{k}(A^{k})^{\dagger}=U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^{*}.$$

From the definition of $A^{\diamond}$, $A^{\textcircled{\dagger}}$ and (2.9) together with the equation above, it follows that

$$\begin{aligned} A^{\diamond}=A^{\textcircled{\dagger}} &\iff A^{2}A^{\dagger}=A^{k+1}(A^{k})^{\dagger} \\ &\iff U\begin{bmatrix} T & SP_{N} \\ 0 & NP_{N} \end{bmatrix}U^{*}=U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^{*} \\ &\iff SN=0,\ N^{2}=0. \end{aligned}$$

$(b)\iff(c)$. From the definition of $A^{\diamond}$ and $A^{D,\dagger}$ together with (4.2), by using Lemma 2.3, it follows that

$$\begin{aligned}
A^{\diamond} = A^{D,\dagger} &\iff A^2A^{\dagger} = (A^{D,\dagger})^{\dagger} \\
&\iff U\begin{bmatrix} T & SP_N \\ 0 & NP_N \end{bmatrix}U^* = \left(U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\tilde{T}P_N \\ 0 & 0 \end{bmatrix}U^*\right)^{\dagger} \\
&\iff SP_N = 0,\ NP_N = 0,\ \tilde{T}P_N = 0 \\
&\iff SN = 0,\ N^2 = 0,
\end{aligned}$$

where $\tilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$. □

Remark 4.5. *If A of the form* (2.7) *is nilpotent, it follows that $A = UNU^*$. Then the* (a) *of Theorem* 4.3 *and the* (b) *of the Theorem* 4.4 *are equivalent to $N^2 = 0$. In other words, if A is nilpotent, then it follows that the conditions $A^{\diamond} = A^D, A^{\diamond} = A^{\textcircled{\dagger}}, A^{\diamond} = A^{D,\dagger}, A^{\diamond} = A^{\dagger,D}$ and $A^{\diamond} = A^{\textcircled{w}}$ are equivalent.*

In [13, Theorem 4], the author gave some equivalent conditions for $A^{\diamond} \in \mathbb{C}_n^{\mathrm{EP}}$. Then we will give some necessary and sufficient conditions for $A^{\diamond}$ which belongs to some special matrix classes by using Core-EP decomposition.

Theorem 4.6. *Let $A \in \mathbb{C}^{n\times n}$ be the form of* (2.7)*. Then,*
(a) $A^{\diamond} \in \mathbb{C}_n^{\mathrm{CM}} \iff N^2 = 0$;
(b) $A^{\diamond} \in \mathbb{C}_n^{\mathrm{P}} \iff N^2 = 0$ *and* $T = TT^* + SP_NS^*$;
(c) $A^{\diamond} \in \mathbb{C}_n^{\mathrm{OP}} \iff T = I_t, SN = 0$ *and* $N^2 = 0$ (*or* $A^2 = A_1$. *where* A_1 *is presented in Lemma 2.2.*)

Proof. (a). From the definition of BT-inverse, it follows that

$$A^{\diamond} \in \mathbb{C}_n^{\mathrm{CM}} \iff (A^2A^{\dagger})^{\dagger} \in \mathbb{C}_n^{\mathrm{CM}} \iff A^2A^{\dagger} \in \mathbb{C}_n^{\mathrm{CM}}.$$

By (2.7) and (2.9), we obtain that

$$A^2A^{\dagger} = U\begin{bmatrix} T & SP_N \\ 0 & NP_N \end{bmatrix}U^*.$$

Thus $A^{\diamond} \in \mathbb{C}_n^{\mathrm{CM}} \iff N^2N^{\dagger} = 0 \iff N^2 = 0$ which establishes point (a) of the theorem.

(b). For $A^{\diamond} \in \mathbb{C}_n^{\mathrm{P}} \subseteq \mathbb{C}_n^{\mathrm{CM}}$, we have $N^2 = 0$. From (4.1), now we have that

$$A^{\diamond} = U\begin{bmatrix} T^*\triangle & 0 \\ P_NS^*\triangle & 0 \end{bmatrix}U^*,$$

where $\triangle = (TT^* + SP_NS^*)^{-1}$.

Since $A^{\diamond} \in \mathbb{C}_n^{\mathrm{P}}$, we get that $T^*\triangle = I_t$, hence $T = (\triangle^*)^{-1} = \triangle^{-1}$. The sufficient condition of (b) can be directly checked, therefore point (b) of the theorem holds.

(c). It can be directly checked that $A^2 = A_1$ is equivalent to $T = I_t, SN = 0$ and $N^2 = 0$ by Core-EP decomposition. For $A^{\diamond} \in \mathbb{C}_n^{\mathrm{OP}} \subseteq \mathbb{C}_n^{\mathrm{P}}$, we have $N^2 = 0$ and $T = \triangle^{-1}$. From (4.1), we have

$$A^{\diamond} = U\begin{bmatrix} I_r & 0 \\ P_NS^*\triangle & 0 \end{bmatrix}U^*,$$

where $\triangle = (TT^* + SP_NS^*)^{-1}$.

Since $A^\diamond \in \mathbb{C}_n^{\mathrm{OP}}$, we get that $SP_N = 0$ which implies $T = I_t, SN = 0$. The sufficient condition of (c) can be directly checked, therefore point (c) of the theorem holds.

□

Remark 4.7. *If A of the form* (2.7) *is nilpotent which implies $A = UNU^*$, then $A^\diamond \in \mathbb{C}_n^{\mathrm{CM}}$ or $\mathbb{C}_n^{\mathrm{P}}$ or $\mathbb{C}_n^{\mathrm{OP}}$ is equivalent to $A^2 = 0$ (or $N^2 = 0$).*

From [13], it is known that $A^\diamond A = AA^\diamond$ and $(A^\diamond)^\dagger = (A^\dagger)^\diamond$ are both satisfied when $A \in \mathbb{C}_n^{\mathrm{EP}}$, but we can't conclude $A \in \mathbb{C}_n^{\mathrm{EP}}$ when $A^\diamond A = AA^\diamond$ or $(A^\diamond)^\dagger = (A^\dagger)^\diamond$ holds. How to establish an equivalence relation between them, the following theorem will give.

Theorem 4.8. *Let $A \in \mathbb{C}^{n\times n}$ written as in* (2.1). *Then the following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{\mathrm{EP}}$;

(b) $AA^\diamond = A^\diamond A$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$;

(c) $(A^\diamond)^\dagger = (A^\dagger)^\diamond$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$;

(d) $(A^\diamond)^m = (A^\dagger)^m$ *for some* $m \geq 2$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Proof. That (a) implies items $(b), (c)$ and (d) can be checked directly by the definition of $A^\diamond$.

$(b) \Rightarrow (a)$. For $A \in \mathbb{C}_n^{\mathrm{CM}}$, we get that K is nonsingular. By (2.5) and (2.6), we get that $A^\diamond = A^{\textcircled{\#}}$ and $AA^{\textcircled{\#}} = A^{\textcircled{\#}}A$. Hence it follows that $A \in \mathbb{C}_n^{\mathrm{EP}}$ by [3, Theorem 3].

$(c) \Rightarrow (a)$. This follows similarly as in the part $(b) \Rightarrow (a)$.

$(d) \Rightarrow (a)$. It is known that $A \in \mathbb{C}_n^{\mathrm{EP}}$ is equivalent to $L = 0$. Combining (2.3), (2.5) with $(A^\diamond)^m = (A^\dagger)^m$ leads to $L = 0$ which means $A \in \mathbb{C}_n^{\mathrm{EP}}$. □

5. Representations of BT-inverse by the maximal classes

Finally, we study the representations for the BT-inverse. In [4], let $A \in \mathbb{C}_n^{\mathrm{CM}}$. While $A^{\textcircled{\#}} = A^\# AA^\dagger$ or $(A^2A^\dagger)^\dagger$, the author gave new representations by the maximal matrix classes such as $A^{\textcircled{\#}} = XAY$ or $(A^2Z)^\dagger$ where $\mathcal{R}(XA) \subseteq \mathcal{R}(A)$ and $Y \in A\{1,3\}$ or $Z \in A\{1,3\}$. Similarly, the author in [21] gave the representations of $A^{\textcircled{\dagger}}$, $A^{D,\dagger}$ by the maximal classes. Now, we will derive the representations of BT-inverse by the maximal classes. We first give the important lemma as follows.

Lemma 5.1. [22] *Let $A, B, C \in \mathbb{C}^{n\times n}$. Then the matrix equation $AXB = C$ is consistent if and only if for some $A^{(1)} \in A\{1\}$, $B^{(1)} \in B\{1\}$,*

$$AA^{(1)}CB^{(1)}B = C,$$

in which case the general solution is

$$X = A^{(1)}CB^{(1)} + Z - A^{(1)}AZBB^{(1)},$$

for arbitrary $Z \in \mathbb{C}^{n\times n}$.

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ of rank r has the form* (2.1). *Then the following conditions are equivalent:*

(*a*) $A^{\diamond} = (A^2X)^{\dagger}$;
(*b*) $A^2X = AP_A$;
(*c*) $X = P_{(A^2)^{\dagger}}A^{\dagger} + (I_n - P_{(A^2)^{\dagger}})Z$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$;
(*d*) *X can be expressed as*

$$X = U\begin{bmatrix} P^*R^{\dagger}\Sigma K + (I_r - P^*R^{\dagger}P)Z_1 - P^*R^{\dagger}QZ_3 & (I_r - P^*R^{\dagger}P)Z_2 - P^*R^{\dagger}QZ_4 \\ Q^*R^{\dagger}\Sigma K - Q^*R^{\dagger}PZ_1 + (I_{n-r} - Q^*R^{\dagger}Q)Z_3 & -Q^*R^{\dagger}PZ_2 + (I_{n-r} - Q^*R^{\dagger}Q)Z_4 \end{bmatrix}U^*,$$

where $R = PP^* + QQ^*$, $P = (\Sigma K)^2$ *and* $Q = \Sigma K\Sigma L$, *for arbitrary* Z_1, Z_2, Z_3, Z_4.

Proof. (*a*) ⇒ (*b*). Since $A^{\diamond} = (AP_A)^{\dagger} = (A^2X)^{\dagger}$, we have $A^2X = AP_A$.

(*b*) ⇒ (*c*). It is evident that $P_{(A^2)^{\dagger}}A^{\dagger}$ satisfies the equation

$$A^2X = AP_A. \tag{5.1}$$

Applying Lemma 5.1 to this equation, the general solution of (4.3) is given by

$$X = P_{(A^2)^{\dagger}}A^{\dagger} + (I_n - P_{(A^2)^{\dagger}})Z,$$

for arbitrary $Z \in \mathbb{C}^{n\times n}$.

(*c*) ⟺ (*d*). From (2.1), it follows that

$$A^2 = U\begin{bmatrix} (\Sigma K)^2 & \Sigma K\Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{5.2}$$

and applying [23, Lemma 1] to (5.2), we obtain that

$$(A^2)^{\dagger} = U\begin{bmatrix} P^*R^{\dagger} & 0 \\ Q^*R^{\dagger} & 0 \end{bmatrix}U^*,$$

where $R = PP^* + QQ^*$, $P = (\Sigma K)^2$ and $Q = \Sigma K\Sigma L$. Next, partitioning accordingly

$$Z = U\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}U^*,$$

a straightforward computation shows that $X = P_{(A^2)^{\dagger}}A^{\dagger} + (I_n - P_{(A^2)^{\dagger}})Z$ is equivalent to

$$X = U\begin{bmatrix} P^*R^{\dagger}\Sigma K + (I_r - P^*R^{\dagger}P)Z_1 - P^*R^{\dagger}QZ_3 & (I_r - P^*R^{\dagger}P)Z_2 - P^*R^{\dagger}QZ_4 \\ Q^*R^{\dagger}\Sigma K - Q^*R^{\dagger}PZ_1 + (I_{n-r} - Q^*R^{\dagger}Q)Z_3 & -Q^*R^{\dagger}PZ_2 + (I_{n-r} - Q^*R^{\dagger}Q)Z_4 \end{bmatrix}U^*, \tag{5.3}$$

where $R = PP^* + QQ^*$, $P = (\Sigma K)^2$ and $Q = \Sigma K\Sigma L$, for arbitrary Z_1, Z_2, Z_3, Z_4.

(*c*) ⇒ (*a*). By a direct calculation, we have that $A^2X = A^2A^{\dagger}$. Therefore

$$(A^2X)^{\dagger} = (A^2A^{\dagger})^{\dagger} = A^{\diamond}.$$

□

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (2.1)*, $X, Y \in AP_A\{1\}$. Then the following conditions are equivalent:*

(a) $A^{\diamond} = XAP_AY$;

(b) $XAP_A = P_{(A^2)^{\dagger}}$ *and* $AP_AY = A(AP_A)^{\dagger}$;

(c) $X = (AP_A)^{\dagger} + Z(I_n - P_{AP_A})$ *and* $Y = (AP_A)^{\dagger} + (I_n - P_{(AP_A)^{\dagger}})W$, *for arbitrary* $Z, W \in \mathbb{C}^{n\times n}$;

(d) *X, Y can be expressed as*

$$X = U\begin{bmatrix} (\Sigma K)^{\dagger} + Z_1(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_2 \\ Z_3(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_4 \end{bmatrix} U^*,$$

for arbitrary Z_1, Z_2, Z_3, Z_4;

$$Y = U\begin{bmatrix} (\Sigma K)^{\dagger} + (I_r - (\Sigma K)^{\dagger}\Sigma K)W_1 & (I_r - (\Sigma K)^{\dagger}\Sigma K)W_2 \\ W_3 & W_4 \end{bmatrix} U^*,$$

for arbitrary W_1, W_2, W_3, W_4.

Proof. (a) $\Rightarrow$ (b). Postmultiplying $A^{\diamond} = XAP_AY$ by AP_A. For $Y \in AP_A\{1\}$, it follows that $XAP_A = P_{(A^2)^{\dagger}}$. Premultiplying $A^{\diamond} = XAP_AY$ by AP_A. Since $X \in AP_A\{1\}$, it follows that $AP_AY = AP_AA^{\diamond} = AA^{\diamond}$.

(b) $\Rightarrow$ (c). Applying Lemma 5.1 to two equations $XAP_A = (AP_A)^{\dagger}AP_A$ and $AP_AY = A(AP_A)^{\dagger}$ respectively, the general solutions are given by $X = (AP_A)^{\dagger} + Z(I_n - P_{AP_A})$ for arbitrary $Z \in \mathbb{C}^{n\times n}$ and $Y = (AP_A)^{\dagger} + (I_n - P_{(AP_A)^{\dagger}})W$ for arbitrary $W \in \mathbb{C}^{n\times n}$.

(c) $\Rightarrow$ (d). Assume that A has the form given in (2.1), we have

$$I_n - P_{AP_A} = U\begin{bmatrix} I_r - \Sigma K(\Sigma K)^{\dagger} & 0 \\ 0 & I_{n-r} \end{bmatrix} U^*,$$

$$I_n - P_{(AP_A)^{\dagger}} = U\begin{bmatrix} I_r - (\Sigma K)^{\dagger}\Sigma K & 0 \\ 0 & I_{n-r} \end{bmatrix} U^*.$$

Next, partitioning accordingly

$$Z = U\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^*, W = U\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} U^*,$$

a straightforward shows that $X = (AP_A)^{\dagger} + Z(I_n - P_{AP_A})$ is equivalent to

$$X = U\begin{bmatrix} (\Sigma K)^{\dagger} + Z_1(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_2 \\ Z_3(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_4 \end{bmatrix} U^*, \tag{5.4}$$

for arbitrary Z_1, Z_2, Z_3, Z_4. $Y = (AP_A)^{\dagger} + (I_n - P_{(AP_A)^{\dagger}})W$ is equivalent to

$$Y = U\begin{bmatrix} (\Sigma K)^{\dagger} + (I_r - (\Sigma K)^{\dagger}\Sigma K)W_1 & (I_r - (\Sigma K)^{\dagger}\Sigma K)W_2 \\ W_3 & W_4 \end{bmatrix} U^*, \tag{5.5}$$

for arbitrary W_1, W_2, W_3, W_4.

$(d) \Rightarrow (a)$. According to (5.4) and (5.5), a straightforward computation shows that

$$\begin{aligned} XAP_AY &= U\begin{bmatrix} (\Sigma K)^{\dagger} + Z_1(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_2 \\ Z_3(I_r - \Sigma K(\Sigma K)^{\dagger}) & Z_4 \end{bmatrix}\begin{bmatrix} \Sigma K(\Sigma K)^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (\Sigma K)^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= A^{\diamond}. \end{aligned}$$

□

6. Conclusions

In this work, different characteristics of the BT-inverse of a square matrix have been developed. Some necessary and sufficient conditions for a matrix to be the BT-inverse have been derived. The Core-EP decomposition is efficient for investigating the relationships between the BT-inverse and other generalized inverses. The expression of BT-inverse has been extended to more general ones by the maximal classes of matrices.

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Conflict of interest

The authors declare no conflict of interest.

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