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Research article

Tripled fixed point techniques for solving system of tripled-fractional differential equations

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Abstract: The intended goal of this manuscript is to discuss the existence of the solution to the below system of tripled-fractional differential equations (TFDEs, for short):

$$\begin{cases} \Theta^{\mu}\left[k(\alpha)-\gimel(\alpha,k(\alpha))\right]=\eth\left(\alpha,r(\alpha),I^{\tau}(r(\alpha))\right)+\eth\left(\alpha,l(\alpha),I^{\tau}(l(\alpha))\right), \\ \Theta^{\mu}\left[l(\alpha)-\gimel(\alpha,l(\alpha))\right]=\eth\left(\alpha,k(\alpha),I^{\tau}(k(\alpha))\right)+\eth\left(\alpha,r(\alpha),I^{\tau}(r(\alpha))\right), \\ \Theta^{\mu}\left[r(\alpha)-\gimel(\alpha,r(\alpha))\right]=\eth\left(\alpha,l(\alpha),I^{\tau}(l(\alpha))\right)+\eth\left(\alpha,k(\alpha),I^{\tau}(k(\alpha))\right), \\ \qquad k(0)=0,\ l(0)=0,\ r(0)=0, \end{cases} \quad a.e.\ \alpha\in\Omega,\ \tau>0,\ \mu\in(0,1),$$

where Θ^{μ} is RL-fractional derivative of order τ, $\Omega=[0,\Lambda]$, $\Lambda>0$, and $\gimel:\Omega\times\mathbb{R}\to\mathbb{R}$, with $\gimel(0,0)=0$, $\eth:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are functions taken under appropriate hypotheses. The method of the proof depends on a manner of a tripled fixed point (TFP), which generalize a fixed point theorem of Burton [1]. At last, a non-trivial example to strong our results is illustrated.

Keywords: tripled fixed point theorem; RL-fractional derivative; system of tripled-fractional differential equations; Banach space
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1. Nonlinear analysis and fractional differential equations

Fractional calculus has been given proper attention in the last few decades by researchers. This subject gained new structures on an unlimited scale and are mainly applied in all branches of basic sciences, especially engineering sciences.

Because of fractional differential equations (FDEs) frequent appearance, it was particularly important, in many applications such as fluid mechanics, viscoelasticity, biology, physics and engineering. Recently, the related literature has been developed for application in FDEs in nonlinear dynamics [2–6]. Another reason why these equations are widespread is most FDEs do not have exact analytic solutions, approximation and numerical techniques. Consequently, it is used to give the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest.

There is no doubt that non-linear analysis, especially the fixed-point technique, contributes greatly to find the existence of solutions of nonlinear initial-value problems of FDEs [7–15].

Fixed point theory (FPT) speaks about two variants of arguments, FPT on metric spaces and topological problems under FPT. Topological problems under FPT is of particular interest to topologists and theoretical computer scientists, while FPT on metric spaces is of great importance in computing, computational biology, bio-informatics. This is another reason why the strong relationship between the FPT and the rest of the disciplines is very strong, Which leads to widespread. The main advantage of using FDEs is related to the fact that we can describe the dynamics of complex non-local systems with memory.

Another direction, nonlinear analysis used in the study of dynamical systems represented by nonlinear differential and integral equations. Since some of these equations that represent a dynamical system do not have an analytical solution, therefore studying the turmoil of these problems is very beneficial. There are different types of turmoil differential equations and the important type here is called a hybrid differential equation (HDE) [16]. From this moment, this branch has become very important for many researchers see [17–19]. As well as, hybrid FPT can be used to improve the existence theory for the hybrid equations.

The below first-order hybrid DE with linear turmoils of second type introduced by Dhage and Jadhav [20]:

$$\begin{cases} \frac{d}{d\alpha}\left[k(\alpha) - \gimel(\alpha, k(\alpha))\right] = \Game\left(\alpha, k(\alpha)\right), \quad a.e.\ \alpha \in \Omega, \\ k(\alpha_0) = k_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Omega \in [\alpha_0, \alpha_0 + \rho)$, $\rho > 0$, for some fixed $\alpha_0, \rho \in \mathbb{R}$, and $\gimel, \Game \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Via this notions they discussed the existence of the minimal and maximal solution for it and obtained exciting results about the strict and nonstrict differential inequalities. The problem (1.1) developed in a fractional version under the title FHDE involving the Riemann-Liouville (RL) differential operators of order $0 < \mu < 1$ by Lu *et al.* [21] as follows:

$$\begin{cases} \Theta^{\mu}\left[k(\alpha) - \gimel(\alpha, k(\alpha))\right] = \Game\left(\alpha, l(\alpha)\right), \quad a.e.\ \alpha \in \Omega, \\ k(\alpha_0) = k_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

$\gimel, \Game \in C(\Omega \times \mathbb{R}, \mathbb{R})$. They showed the existence theorem for FHDEs by applying mixed Lipschitz and Carathéodory conditions. From this standpoint, the concept has become widely used in the field of fractional analysis and has become a huge turning point, see [22–30]. The problem (1.2) generalized to two-point boundary value problem, so-called a coupled system of FDEs and some massive results to find a solutions of coupled nonlinear fractional reaction-diffusion equations are presented, see [31,32].

Based on the above work, our main goal in this manuscript is to find the existence solution to the system of TFDEs as the form:

$$\begin{cases} \Theta^{\mu}\left[k(\alpha)-\daleth(\alpha,k(\alpha))\right]=\eth\left(\alpha,r(\alpha),I^{\tau}(r(\alpha))\right)+\eth\left(\alpha,l(\alpha),I^{\tau}(l(\alpha))\right), \\ \Theta^{\mu}\left[l(\alpha)-\daleth(\alpha,l(\alpha))\right]=\eth\left(\alpha,k(\alpha),I^{\tau}(k(\alpha))\right)+\eth\left(\alpha,r(\alpha),I^{\tau}(r(\alpha))\right), \\ \Theta^{\mu}\left[r(\alpha)-\daleth(\alpha,r(\alpha))\right]=\eth\left(\alpha,l(\alpha),I^{\tau}(l(\alpha))\right)+\eth\left(\alpha,k(\alpha),I^{\tau}(k(\alpha))\right), \\ \qquad k(0)=0,\ l(0)=0,\ r(0)=0, \end{cases} \quad a.e.\ \alpha\in\Omega,\ \tau>0,\ \mu\in(0,1), \tag{1.3}$$

Mechanism of proof depends mainly on the manner of TFP theorem, which is an extension of the results Burton [1] in a Banach space.

2. Basic tools

We shall agree in this part on $\mathfrak{C}(\Omega\times\mathbb{R},\mathbb{R})$ refers to the class of continuous functions $\daleth:\Omega\times\mathbb{R}\to\mathbb{R}$, and $C(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ the class of functions $\eth:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that, the mapping

$\heartsuit_1$ $\alpha\to\eth(\alpha,k,l)$ is measurable, for all $k,l\in\mathbb{R}$,

$\heartsuit_2$ $k\to\eth(\alpha,k,l)$ is continuous, for all $k\in\mathbb{R}$,

$\heartsuit_3$ $l\to\eth(\alpha,k,l)$ is continuous, for all $l\in\mathbb{R}$.

Hence, the class $C(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is called Carathéodory class of functions on $\Omega\times\mathbb{R}\times\mathbb{R}$, and if it bounded by a Lebesgue integrable function on Ω, then it called Lebesgue integrable.

Now we shall present some previous results that are used in the next section.

Definition 2.1. [33] The usual form of the RL-fractional integral operator of order τ is

$$I^{\tau}\eth(\alpha)=\frac{1}{\Gamma(\tau)}\int_0^{\alpha}(\alpha-\hbar)^{\tau-1}\eth(\hbar)d\hbar,$$

where $\tau>0$, and the function $\eth$ defined on $L^1(\mathbb{R}^+)$.

Definition 2.2. [33] The usual form of the Caputo fractional derivative of the function $\eth$ is

$$^{C}\Theta^{\tau}\eth(k)=\frac{1}{\Gamma(\xi-\tau)}\int_0^{\alpha}(\alpha-\hbar)^{\xi-\tau-1}\eth^{(\xi)}(\hbar)d\hbar,$$

where $\tau\in\mathbb{R}^+$(a positive real number) such that $\xi-1<\tau\leq\xi$, $\xi\in\mathbb{N}$ and $\eth^{(\xi)}(\hbar)$ is exists, and function of class C.

Definition 2.3. [33] Let $\eth:(0,\infty)\to\mathbb{R}$ be a continuous function, the RL-fractional derivative of order τ is defined as

$$D^{\tau}\eth(\alpha)=\frac{1}{\Gamma(\eta-\tau)}\left(\frac{d}{d\alpha}\right)^{n}\int_0^{\alpha}(\alpha-\hbar)^{\eta-\tau-1}\eth(\hbar)d\hbar,$$

where $n=[\tau]+1$.

Lemma 2.4. *[33, 34] For $\tau\in(0,1)$ and $\eth\in L^1(0,1)$, we have*

(1) *the equation $I^{\tau}D^{\tau}\eth(\alpha)=\eth(\alpha)$ is fulfilled,*

(2) *the equation $I^{\tau}D^{\tau}\eth(\alpha)=\eth(\alpha)-\frac{\left[D^{\tau-1}\eth(\alpha)\right]_{\alpha=0}}{\Gamma(\tau)}\alpha^{\tau-1}$ is satisfied almost everywhere (a.e.) on Ω.*

The below result will be generalized in this paper as previously presented by Burton [1].

Lemma 2.5. *[1] Suppose that ∇ is a Banach space, $\wp \neq \emptyset$ is a closed convex bounded subset of it. Let $\mathfrak{I} : \nabla \to \nabla$ and $\mathfrak{R} : \wp \to \nabla$ be two operators such that*
(i) for all $k, l \in \nabla$, $\ell < 1$, we get $\left\|\mathfrak{I}k - \mathfrak{I}l\right\| \leq \ell \|k - l\|$,
(ii) the completely continuous property hold for the operators $\mathfrak{R}$,
(iii) $k = \mathfrak{I}k + \mathfrak{R}l$ implies $k \in \wp$, for all $l \in \wp$.
Then the the operator equation $k = \mathfrak{I}k + \mathfrak{R}l$ has a solution in $\wp$.

In 2011, Coupled fixed point notion is generalized to TFP concept by Berinde and Borcut [35] in the setting of partially ordered metric spaces. Via the mentioned spaces they presented pivotal results about TFP theorems. For the authors contributions in this direction, see [36–42].

Definition 2.6. [35] It is said that a trio $(k, l, r) \in \nabla^\star$ is a TFP of a self-mapping $\mathfrak{I} : \nabla^\star \to \nabla$ if

$$k = \mathfrak{I}(k, l, r),\ l = \mathfrak{I}(l, k, l) \text{ and } r = \mathfrak{I}(r, l, k).$$

Definition 2.7. [36] A trio $(k, l, r) \in \nabla^\star$ on a non-empty set ∇, is called a tripled coincidence point of the two self-mappings $\mathfrak{I} : \nabla^\star \to \nabla$ and $\mathfrak{R} : \nabla \to \nabla$ if $\mathfrak{R}k = \mathfrak{I}(k, l, r)$, $\mathfrak{R}l = \mathfrak{I}(l, r, k)$ and $\mathfrak{R}r = \mathfrak{I}(r, k, l)$.

Definition 2.8. [36] Assume that $\nabla \neq \emptyset$ is a set, a trio $(k, l, r) \in \nabla^\star$ is called a tripled common fixed point of $\mathfrak{I} : \nabla^\star \to \nabla$ and $\mathfrak{R} : \nabla \to \nabla$, if $k = \mathfrak{R}k = \mathfrak{I}(k, l, r)$, $l = \mathfrak{R}l = \mathfrak{I}(l, r, k)$ and $r = \mathfrak{R}r = \mathfrak{I}(r, k, l)$.

Here, consider Ψ refers to the family of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ fulfilling $\psi(v) < v$ for $v > 0$ and $\psi(0) = 0$.

3. Main theorem

In the beginning of this part, we know that $\nabla = C(\Omega, \mathbb{R})$ is a Banach space with respect to the supremum norm and the pointwise operations, if it defined on the supremum norm.

The two operations defined here are scalar multiplication and a sum on $\nabla \times \nabla \times \nabla = \nabla^3$ as follows:

$$(k_1, l_1, r_1) + (k_2, l_2, r_2) + (k_3, l_3, r_3) = (k_1 + k_2 + k_3, l_1 + l_2 + l_3, r_1 + r_1 + r_3),$$

and

$$\varpi(k, l, r) = (\varpi k, \varpi l, \varpi r),$$

for all $k, l, r \in \nabla$, $\varpi \in \mathbb{R}$. Then ∇^3 is a vector space.

The below Lemma are very important in the sequel and his proof is clear:

Lemma 3.1. *Let $\nabla^\star = \nabla^3$. Define*

$$\|(k, l, r)\| = \|k\| + \|l\| + \|r\|.$$

Then with respect to this norm, $\nabla^\star$ is a Banach space.

Now our main theorem in this section is valid for viewing.

Theorem 3.2. *Assume that ∇ is a Banach space, $\wp \neq \emptyset$ is a closed, convex, and bounded subset of it and $\wp^\star = \wp^3$. Let $\mathfrak{I} : \nabla \to \nabla$ and $\mathfrak{R}, \Upsilon : \wp \to \nabla$ be three operators such that*

($\dagger_i$) *there is $\psi_R \in \Psi$ such that for all $k, l \in \nabla$, and for some $\ell > 0$, we get*

$$\left\|\mathfrak{I}k - \mathfrak{I}l\right\| \leq \ell\psi_R\left(\frac{\|k-l\|}{1+\|k-l\|}\right),$$

($\dagger_{ii}$) *the completely continuous property hold for the oberators $\mathfrak{R}$ and Υ;*

($\dagger_{iii}$) *$k = \mathfrak{I}k + \mathfrak{R}l + \Upsilon r$ implies $k \in \wp$, for all $l, r \in \wp$.*

Then there exists at least a tripled fixed point (tfp) of the operator $Z(k,l,r) = \mathfrak{I}k + \mathfrak{R}l + \Upsilon r$ in $\wp^*$, whenever $\ell \in (0,1)$.

Proof. Check that $\wp^\star \neq \emptyset$ is a closed, convex, and bounded subset of a Banach space $\nabla^\star$ are easy. Define $\mathfrak{I}^\star : \nabla^\star \to \nabla^\star$, and $\mathfrak{R}^\star, \Upsilon^\star : \wp^\star \to \nabla^\star$ by

$$\mathfrak{I}^\star(k,l,r) = (\mathfrak{I}k, \mathfrak{I}l, \mathfrak{I}r),\ \mathfrak{R}^\star(k,l,r) = (\mathfrak{R}l, \mathfrak{R}k, \mathfrak{R}l) \text{ and } \Upsilon^\star(k,l,r) = (\Upsilon r, \Upsilon l, \Upsilon k).$$

The proof follows if $\mathfrak{I}^\star(k,l,r) + \mathfrak{I}^\star(k,l,r) + \Upsilon^\star(k,l,r) = (k,l,r)$ has at least one solution. Because

$$\begin{aligned}
(Z(k,l,r), Z(l,k,l), Z(r,l,k)) &= (\mathfrak{I}k + \mathfrak{R}l + \Upsilon r, \mathfrak{I}l + \mathfrak{R}k + \Upsilon l, \mathfrak{I}r + \mathfrak{I}l + \Upsilon k) \\
&= (\mathfrak{I}k, \mathfrak{I}l, \mathfrak{I}r) + (\mathfrak{R}l, \mathfrak{R}k, \mathfrak{R}l) + (\Upsilon r, \Upsilon l, \Upsilon k) \\
&= \mathfrak{I}^\star(k,l,r) + \mathfrak{R}^\star(k,l,r) + \Upsilon^\star(k,l,r) = (k,l,r).
\end{aligned}$$

this leads to the operator $Z(k,l,r)$ has at least one TFP. Now we prove that the operators $\mathfrak{I}^\star, \mathfrak{R}^\star$ and $\Upsilon^\star$ satisfy the conditions of Theorem 3.2 as follows:

• Prove that $\mathfrak{I}^\star$ is a contraction. Apply assumption ($\dagger_i$) for all $k = (k_1, k_2, k_3)$, $l = (l_{1,} l_2, l_3), r = (r_1, r_2, r_3) \in \wp^\star$, one can werite

$$\begin{aligned}
\left\|\mathfrak{I}^\star k - \mathfrak{I}^\star l\right\| &= \left\|(\mathfrak{I}k_1, \mathfrak{I}k_2, \mathfrak{I}k_3) - (\mathfrak{I}l_1, \mathfrak{I}l_2, \mathfrak{I}l_3)\right\| \\
&= \left\|(\mathfrak{I}k_1 - \mathfrak{I}l_1, \mathfrak{I}k_2 - \mathfrak{I}l_2, \mathfrak{I}k_3 - \mathfrak{I}l_3)\right\| \\
&= \left\|\mathfrak{I}k_1 - \mathfrak{I}l_1\right\| + \left\|\mathfrak{I}k_2 - \mathfrak{I}l_2\right\| + \left\|\mathfrak{I}k_3 - \mathfrak{I}l_3\right\| \\
&\leq \ell\left[\psi_R\left(\frac{\|k_1 - l_1\|}{1+\|k_1 - l_1\|}\right) + \psi_R\left(\frac{\|k_2 - l_2\|}{1+\|k_2 - l_2\|}\right) + \psi_R\left(\frac{\|k_3 - l_3\|}{1+\|k_3 - l_3\|}\right)\right] \\
&< \ell\left(\frac{\|k_1 - l_1\|}{1+\|k_1 - l_1\|} + \frac{\|k_2 - l_2\|}{1+\|k_2 - l_2\|} + \frac{\|k_3 - l_3\|}{1+\|k_3 - l_3\|}\right) \\
&\leq \ell\left(\|k_1 - l_1\| + \|k_2 - l_2\| + \|k_3 - l_3\|\right) \\
&= \ell\left(\|k_1 - l_1, k_2 - l_2, k_3 - l_3\|\right) \\
&= \ell\,\|k - l\|,
\end{aligned}$$

which leads to $\mathfrak{I}^\star$ is Lipschitzian, hence it is a contraction with a constant ℓ.

• Show that $\mathfrak{R}^\star$ and $\Upsilon^\star$ are compact and continuous operators on $\wp^\star$. Assume the sequence $(k_n) = (k_{1n}, k_{2n}, k_{3n}) \in \wp^\star$ converging to a point $k = (k_1, k_2, k_3) \in \wp^\star$, it follows by the continuity of $\mathfrak{R}$ and Υ that

$$\lim_{n\to\infty} \mathfrak{R}^\star k_n = \left(\lim_{n\to\infty} \mathfrak{R}k_{2n}, \lim_{n\to\infty} \mathfrak{R}k_{1n}, \lim_{n\to\infty} \mathfrak{R}k_{2n}\right) = (\mathfrak{R}k_2, \mathfrak{R}k_1, \mathfrak{R}k_2) = \mathfrak{R}^\star(k_2, k_1, k_2) = \mathfrak{R}^\star k,$$

$$\lim_{n\to\infty} \Upsilon^{\star} k_n \;=\; \left(\lim_{n\to\infty} \Upsilon k_{3n}, \lim_{n\to\infty} \Upsilon k_{2n}, \lim_{n\to\infty} \Upsilon k_{1n}\right) = (\Upsilon k_3, \Upsilon k_2, \Upsilon k_1) = \Upsilon^{\star}(k_3, k_2, k_1) = \Upsilon^{\star} k.$$

Hence, the operators $\mathfrak{R}^{\star}$ and $\Upsilon^{\star}$ are continuous. Also, we have

$$\begin{aligned}
\left\|\mathfrak{R}^{\star}(k_1, k_2, k_3)\right\| &= \left\|(\mathfrak{R}k_2, \mathfrak{R}k_1, \mathfrak{R}k_2)\right\| \\
&= \left(2\left\|\mathfrak{R}k_2\right\| + \left\|\mathfrak{R}k_1\right\|\right) \\
&\leq 3\left\|\mathfrak{R}\wp\right\|,
\end{aligned}$$

similarly,

$$\begin{aligned}
\left\|\Upsilon^{\star}(k_1, k_2, k_3)\right\| &= \|(\Upsilon k_3, \Upsilon k_2, \Upsilon k_1)\| \\
&= (\|\Upsilon k_3\| + \|\Upsilon k_2\| + \|\Upsilon k_1\|) \\
&\leq 3\|\Upsilon\wp\|.
\end{aligned}$$

or all $k \in \wp^{\star}$, where $\left\|\mathfrak{R}\wp\right\| = \sup\{\left\|\mathfrak{R}k\right\| : k \in \wp\}$ and $\|\Upsilon\wp\| = \sup\{\|\Upsilon k\| : k \in \wp\}$. This shows that $\mathfrak{R}^{\star}$ and $\Upsilon^{\star}$ are uniformly bounded on $\wp^{\star}$.

Since $\mathfrak{R}(\wp)$ and $\Upsilon(\wp)$ are equi-continuous sets in ∇, then for every $\epsilon > 0$, there is $\delta > 0$ such that for $\alpha_1, \alpha_2 \in \Omega$, $|\alpha_1 - \alpha_2| < \delta$ implies $\begin{cases} \left|\mathfrak{R}k(\alpha_1) - \mathfrak{R}k(\alpha_2)\right| \leq \epsilon, \\ |\Upsilon k(\alpha_1) - \Upsilon k(\alpha_2)| \leq \epsilon \end{cases}$ for all $k \in \wp$. Thus for any $k = (k_1, k_2, k_3) \in \wp^{\star}$, we can get

$$\begin{aligned}
\left|\mathfrak{R}^{\star}k(\alpha_1) - \mathfrak{R}^{\star}k(\alpha_2)\right| &= \left|(\mathfrak{R}k_2(\alpha_1), \mathfrak{R}k_1(\alpha_1), \mathfrak{R}k_2(\alpha_1)) - (\mathfrak{R}k_2(\alpha_2), \mathfrak{R}k_1(\alpha_2), \mathfrak{R}k_2(\alpha_2))\right| \\
&= \left|(\mathfrak{R}k_2(\alpha_1) - \mathfrak{R}k_2(\alpha_2), \mathfrak{R}k_1(\alpha_1) - \mathfrak{R}k_1(\alpha_2), \mathfrak{R}k_2(\alpha_1) - \mathfrak{R}k_2(\alpha_2))\right| \\
&= \sqrt{2\left(\mathfrak{R}k_2(\alpha_1) - \mathfrak{R}k_2(\alpha_2)\right)^2 + \left(\mathfrak{R}k_1(\alpha_1) - \mathfrak{R}k_1(\alpha_2)\right)^2} \\
&\leq \sqrt{3}\epsilon. \qquad (3.1)
\end{aligned}$$

Again

$$\begin{aligned}
\left|\Upsilon^{\star}k(\alpha_1) - \Upsilon^{\star}k(\alpha_2)\right| &= |(\Upsilon k_3(\alpha_1), \Upsilon k_2(\alpha_1), \Upsilon k_1(\alpha_1)) - (\Upsilon k_3(\alpha_2), \Upsilon k_2(\alpha_2), \Upsilon k_1(\alpha_2))| \\
&= |(\Upsilon k_3(\alpha_1) - \Upsilon k_3(\alpha_2), \Upsilon k_2(\alpha_1) - \Upsilon k_2(\alpha_2), \Upsilon k_1(\alpha_1) - \Upsilon k_1(\alpha_2))| \\
&= \sqrt{(\Upsilon k_3(\alpha_1) - \Upsilon k_3(\alpha_2))^2 + (\Upsilon k_2(\alpha_1) - \Upsilon k_2(\alpha_2))^2 + (\Upsilon k_1(\alpha_1) - \Upsilon k_1(\alpha_2))^2} \\
&\leq \sqrt{3}\epsilon. \qquad (3.2)
\end{aligned}$$

It follows from (3.1) and (3.2) that $\mathfrak{R}^{\star}(\wp^{\star})$ and $\Upsilon^{\star}(\wp^{\star})$ are equi-continuous sets in $\nabla^{\star}$. Hence by the Arzelà-Ascoli theorem, $\mathfrak{R}^{\star}(\wp^{\star})$ and $\Upsilon^{\star}(\wp^{\star})$ are compact. This implies that $\mathfrak{R}^{\star}$ and $\Upsilon^{\star}$ continuous and compact o on $\wp^{\star}$, i.e., $\mathfrak{R}^{\star}$ and $\Upsilon^{\star}$ is completely continuous on $\wp^{\star}$.

• Finally, we fulfill the assumption $(\dagger_{iii})$ of Theorem 3.2. Suppose that $k = (k_1, k_2, k_3) \in \wp^{\star}$, $l = (l_{1,} l_2, l_3) \in \wp^{\star}$ and $r = (r_1, r_2, r_3) \in \wp^{\star}$ such that $k = \mathfrak{I}^{\star}k + \mathfrak{R}^{\star}l + \Upsilon^{\star}r$, then by hypothesis $(\dagger_{iii})$, we can write

$$\begin{aligned}
(k_1, k_2, k_3) &= \mathfrak{I}^{\star}(k_1, k_2, k_3) + \mathfrak{R}^{\star}(l_{1,} l_2, l_3) + \Upsilon^{\star}(r_1, r_2, r_3) \\
&= (\mathfrak{I}k_1, \mathfrak{I}k_2, \mathfrak{I}k_3) + (\mathfrak{R}l_2, \mathfrak{R}l_1, \mathfrak{R}l_2) + (\Upsilon r_3, \Upsilon r_2, \Upsilon r_1)
\end{aligned}$$

$$= \left(\Im k_1 + \Re l_2 + \Upsilon r_3, \Im k_2 + \Re l_1 + \Upsilon r_2, \Im k_3 + \Re l_2 + \Upsilon r_1\right),$$

which leads to $k_1 = \Im k_1 + \Re l_2 + \Upsilon r_3$, $k_2 = \Im k_2 + \Re l_1 + \Upsilon r_2$ and $k_3 = \Im k_3 + \Re l_2 + \Upsilon r_1$. So, by hypothesis ($\dagger_{iii}$), we get $k_1, k_2, k_3 \in \wp$, thus $k \in \wp^\star$. Therefore all hypotheses of Theorem 3.2 are fulfilled, hence the equation $k = \Im^\star k + \Re^\star l + \Upsilon^\star r$ has at least one solution on $\wp^\star$. Thus the operator $Z(k, l, r)$ has at least one TFP and this ends the proof. □

4. Solve a system of tripled-fractional differential equations

In this section, we discuss the existence solution of the system (1.3) under the below hypotheses:
(t_i) For all $\alpha \in \Omega$, $k \to k - \eth(\alpha, k)$ is increasing function in $\mathbb{R}$.
(t_{ii}) For all $\alpha \in \Omega$, and $k, l \in \mathbb{R}$, there is a constant $\varpi \geq \gamma > 0$ such that

$$|\daleth(\alpha, k(\alpha)) - \daleth(\alpha, l(\alpha))| \leq \frac{\gamma\,|k(\alpha) - l(\alpha)|}{4\left(\varpi + |k(\alpha) - l(\alpha)|\right)}.$$

(t_{iii}) Set $\mho_0 = \max_{\alpha\in\Omega} |\daleth(\alpha, 0)|$.
(t_{iv}) For a continuous function $\omega \in C(\Omega, \mathbb{R})$, we have

$$\eth\left(\alpha, k(\alpha), l(\alpha)\right) \leq \omega(\alpha),\ k, l \in \mathbb{R},\ \alpha \in \Omega.$$

The below lemma is very important in the existence results.

Lemma 4.1. *[21] For $l \in C(\Omega, \mathbb{R})$, The following problem*

$$\begin{cases} \Theta^\mu\left[k(\alpha) - \daleth(\alpha, k(\alpha))\right] = l(\alpha), & \alpha \in \Omega, \\ k(0) = 0, \end{cases}$$

has a unique solution as the form

$$k(\alpha) = \daleth(\alpha, k(\alpha)) + \frac{1}{\Gamma(\mu)} \int_0^\alpha \frac{l(\hbar)}{(\alpha - \hbar)^{1-\mu}} d\hbar,\ \alpha \in \Omega,\ \mu \in (0, 1),$$

provided that the hypothesis (t_i) is fulfilled, where $\daleth \in C(\Omega \times \mathbb{R}, \mathbb{R})$ with $\daleth(0, 0) = 0$.

Now we are ready to present our basic theory for this part.

Theorem 4.2. *Via assumptions (t_{ii}) − (t_{iv}), a system of TFDEs (1.3) has a solution on Ω.*

Proof. Put $\nabla = C(\Omega, \mathbb{R})$ and $\wp \subseteq \nabla$ defined by

$$\wp = \{k \in \nabla : \|k\| \leq \sqsupset\},$$

where $\sqsupset \geq \gamma + \mho_0 + \frac{2\Lambda^\mu}{\Gamma(1+\mu)} \|\omega\|_{L^1}$. It's obvious that $\wp$ is a closed, convex, and bounded subset of Banach space ∇.

Certainly the pair $(k(\alpha), l(\alpha), r(\alpha))$ is a unique solution to TFDEs system (1.3) iff a trio $(k(\alpha), l(\alpha), r(\alpha))$ justify the below system of of integral equations:

$$\begin{cases} k(\alpha) = \beth(\alpha, k(\alpha)) + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, r(\hbar), I^\tau(r(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, l(\hbar), I^\tau(l(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar, \\ l(\alpha) = \beth(\alpha, l(\alpha)) + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, k(\hbar), I^\tau(k(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, r(\hbar), I^\tau(r(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar, \\ r(\alpha) = \beth(\alpha, r(\alpha)) + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, l(\hbar), I^\tau(l(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar + \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha \frac{\eth(\hbar, k(\hbar), I^\tau(k(\hbar)))}{(\alpha-\hbar)^{1-\mu}} d\hbar, \quad \alpha \in \Omega. \end{cases} \tag{4.1}$$

Define three operators $\Im : \nabla \to \nabla$ and $\Re, \Upsilon : \wp \to \nabla$ by

$$\begin{cases} \Im k(\alpha) = \beth(\alpha, k(\alpha)), \\ \Re l(\alpha) = \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha (\alpha - \hbar)^{\mu-1} \eth(\hbar, l(\hbar), I^\tau(l(\hbar))) \, d\hbar, \\ \Upsilon r(\alpha) = \frac{1}{\Gamma(\mu)} \int\limits_0^\alpha (\alpha - \hbar)^{\mu-1} \eth(\hbar, r(\hbar), I^\tau(r(\hbar))) \, d\hbar, \quad \alpha \in \Omega. \end{cases}$$

So, system (4.1) convert onto the following system of operator equations:

$$\begin{cases} k(\alpha) = \Im k(\alpha) + \Re r(\alpha) + \Upsilon l(\alpha), \\ l(\alpha) = \Im l(\alpha) + \Re k(\alpha) + \Upsilon r(\alpha), \\ r(\alpha) = \Im r(\alpha) + \Re l(\alpha) + \Upsilon k(\alpha), \quad \alpha \in \Omega. \end{cases}$$

Now, we shall show that the operators $\Im, \Re$ and Υ justify all hypotheses of Theorem 3.2.

By assumption (t_{ii}), for $k, l \in \nabla$, $\alpha \in \Omega$, we have

$$\begin{aligned} \left|\Im k(\alpha) - \Im l(\alpha)\right| &= |\beth(\alpha, k(\alpha)) - \beth(\alpha, l(\alpha))| \\ &\leq \frac{\gamma\,|k(\alpha) - l(\alpha)|}{4\,(\varpi + |k(\alpha) - l(\alpha)|)} \\ &\leq \frac{\gamma\,\|k - l\|}{4\,(\varpi + \|k - l\|)}, \end{aligned}$$

Passing the the supremum over α, one can write

$$\left\|\Im k - \Im l\right\| \leq \frac{\gamma\,\|k - l\|}{4\,(\varpi + \|k - l\|)}. \tag{4.2}$$

It follows from (4.2) that $\Im$ is a nonlinear contraction on ∇ with a control function $\frac{1}{4}\psi$, where $\psi(v) = \frac{\gamma r}{\varpi + r}$.

Next, we prove that $\Re$ and Υ are compact and continuous on $\wp$. Assume the sequence $\{k_n\} \in \wp$ converging to a point $k \in \wp$, then for all $\alpha \in \Omega$ and by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \Re k_n(\alpha) = \frac{1}{\Gamma(\mu)} \lim_{n\to\infty} \left(\int\limits_0^\alpha (\alpha - \hbar)^{\mu-1} \eth(\hbar, k_n(\hbar), I^\tau(k_n(\hbar))) \, d\hbar \right)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\mu)} \int_0^{\alpha} (\alpha-\hbar)^{\mu-1} \lim_{n\to\infty} \eth\left(\hbar, k_n(\hbar), I^{\tau}(k_n(\hbar))\right) d\hbar \\
&= \frac{1}{\Gamma(\mu)} \int_0^{\alpha} (\alpha-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar = \Re k(\alpha).
\end{aligned}
$$

Likewise, we can clarify that $\lim_{n\to\infty} \Upsilon k_n(\alpha) = \Upsilon k(\alpha)$, for all $\alpha \in \Omega$. Hence $\Re$ and Υ are continuous.

Consider $k \in S$, by hypothesis (t_{iii}), we can get

$$
\begin{aligned}
\left|\Re k(\alpha)\right| &= \frac{1}{\Gamma(\mu)} \left| \int_0^{\alpha} (\alpha-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar \right| \\
&\le \frac{1}{\Gamma(\mu)} \int_0^{\alpha} (\alpha-\hbar)^{\mu-1} \left|\eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right)\right| d\hbar \\
&\le \frac{1}{\Gamma(\mu)} \int_0^{\alpha} (\alpha-\hbar)^{\mu-1} \omega(\hbar) d\hbar \\
&\le \frac{\Lambda^{\mu}}{\Gamma(1+\mu)} \|\omega\|_{L^1}
\end{aligned}
$$

Passing the supremum over α, we get

$$
\left\|\Re k(\alpha)\right\| \le \frac{\Lambda^{\mu}}{\Gamma(1+\mu)} \|\omega\|_{L^1}.
$$

Similarly, we can get the same result of the operator Υ, i.e.,

$$
\|\Upsilon k(\alpha)\| \le \frac{\Lambda^{\mu}}{\Gamma(1+\mu)} \|\omega\|_{L^1},
$$

for all $\alpha \in \Omega$. This prove the uniformly boundedness of $\Re$ and Υ on $\wp$. Let $\alpha_1, \alpha_2 \in \Omega$, for any $k \in \wp$, we can write

$$
\begin{aligned}
\left|\Re k(\alpha_1) - \Re k(\alpha_2)\right| &= \frac{1}{\Gamma(\mu)} \left| \int_0^{\alpha_1} (\alpha_1-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar \right. \\
&\quad \left. - \int_0^{\alpha_2} (\alpha_2-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar \right| \\
&\le \frac{1}{\Gamma(\mu)} \left| \int_0^{\alpha_1} (\alpha_1-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar \right. \\
&\quad \left. - \int_0^{\alpha_1} (\alpha_2-\hbar)^{\mu-1} \eth\left(\hbar, k(\hbar), I^{\tau}(k(\hbar))\right) d\hbar \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\mu)}\left|\int_0^{\alpha_1}(\alpha_2-\hbar)^{\mu-1}\,\eth\,(\hbar,k(\hbar),I^{\tau}(k(\hbar)))\,d\hbar\right.\\
&\left.-\int_0^{\alpha_2}(\alpha_2-\hbar)^{\mu-1}\,\eth\,(\hbar,k(\hbar),I^{\tau}(k(\hbar)))\,d\hbar\right|\\
\leq\;& \frac{\|\omega\|_{L^1}}{\Gamma(\mu)}\left(\left|\int_0^{\alpha_1}\left[(\alpha_1-\hbar)^{\mu-1}-(\alpha_2-\hbar)^{\mu-1}\right]d\hbar\right|+\left|\int_{\alpha_1}^{\alpha_2}(\alpha_2-\hbar)^{\mu-1}\,d\hbar\right|\right)\\
\leq\;& \frac{\|\omega\|_{L^1}}{\Gamma(\mu+1)}\left[\left|\alpha_1^{\mu}-\alpha_2^{\mu}\right|+(|\alpha_2-\alpha_1|)^{\mu}\right].
\end{aligned}
$$

The uniformly continuous of α^{μ} for $\mu\in(0,1)$ on Ω, implies that there is $\theta>0$, for given $\epsilon>0$ such that if $|\alpha_1-\alpha_2|<\theta$, we get

$$\left|\alpha_1^{\mu}-\alpha_2^{\mu}\right|\leq\frac{\Gamma(\mu+1)}{2\,\|\omega\|_{L^1}}\epsilon.$$

Set $\delta=\min\{\left(\frac{\Gamma(\mu+1)}{2\|\omega\|_{L^1}}\epsilon\right)^{\frac{1}{\mu}},\theta\}$, if $|\alpha_1-\alpha_2|<\delta$, we obtain that

$$\left|\mathfrak{R}k(\alpha_1)-\mathfrak{R}k(\alpha_2)\right|<\frac{\|\omega\|_{L^1}}{\Gamma(\mu+1)}\left[\frac{\Gamma(\mu+1)}{2\,\|\omega\|_{L^1}}\epsilon+\frac{\Gamma(\mu+1)}{2\,\|\omega\|_{L^1}}\epsilon\right]=\epsilon.$$

By a similar way, one can deduce that $|\Upsilon k(\alpha_1)-\Upsilon k(\alpha_2)|<\epsilon$. This show that $\mathfrak{R}(\wp)$ and $\Upsilon(\wp)$ are equi-continuous. Hence, $\mathfrak{R}$ and Υ are completely continuous on $\wp$.

Finally, for proving the stipulation $(\dagger_{iii})$ of Theorem 3.2, assume that $k\in\nabla$ and $l,r\in\wp$ such that $k=\mathfrak{I}k+\mathfrak{R}l+\Upsilon r$, then by hypotheses (t_{iii}) and (t_{iv}), we get

$$
\begin{aligned}
|k(\alpha)| \leq\;& \left|\mathfrak{I}k(\alpha)\right|+\left|\mathfrak{R}l(\alpha)\right|+|\Upsilon r(\alpha)|\\
\leq\;& (|\daleth(\alpha,k(\alpha))-\daleth(\alpha,0)|+|\daleth(\alpha,0)|)\\
&+\frac{1}{\Gamma(\mu)}\int_0^{\alpha}(\alpha-\hbar)^{\mu-1}\,|\eth\,(\hbar,l(\hbar),I^{\tau}(l(\hbar)))|\,d\hbar\\
&+\frac{1}{\Gamma(\mu)}\int_0^{\alpha}(\alpha-\hbar)^{\mu-1}\,|\eth\,(\hbar,r(\hbar),I^{\tau}(r(\hbar)))|\,d\hbar\\
\leq\;& \mho_0+\gamma+\frac{2}{\Gamma(\mu)}\int_0^{\alpha}(\alpha-\hbar)^{\mu-1}\,\omega(\alpha)d\hbar\\
\leq\;& \mho_0+\gamma+\frac{2\Lambda^{\mu}}{\Gamma(1+\mu)}\,\|\omega\|_{L^1}.
\end{aligned}
$$

Passing the supremum over α on Ω, we conclude that

$$\|k\|\leq\mho_0+\gamma+\frac{\Lambda^{\mu}}{\Gamma(1+\mu)}\,\|\omega\|_{L^1}\leq\beth.$$

It follows that $k \in \wp$. By the same manner we can get the same result if we choose $l(\alpha) = \Im l(\alpha) + \Re r(\alpha) + \Upsilon l(\alpha)$, $l \in \nabla$ and $k, r \in \wp$ or $r(\alpha) = \Im r(\alpha) + \Re l(\alpha) + \Upsilon k(\alpha)$, $r \in \nabla$ and $l, k \in \wp$. Based on what was discussed, we conclude that the hypothesis $(\dagger_{iii})$ has been proven. Thus all requirements hypotheses of Theorem 3.2 are fulfilled, so the operator $Z(k, l, r) = \Im k + \Re l + \Upsilon r$ has TFP on $\wp^{\star}$ which serves as a solution of to TFDEs system (1.3) on Ω. $\square$

The non-trivial below example support Theorem 4.2.

Example 4.3. Consider the below TFDEs system:

$$\begin{cases} \Theta^{\frac{1}{4}}\left[k(\alpha) - \frac{\sin(\alpha)|k(\alpha)|}{4(4+|k(\alpha)|)}\right] = \frac{\alpha|r(\alpha)|}{1+|r(\alpha)|} + \frac{\alpha|l(\alpha)|}{1+|l(\alpha)|}, \\ \Theta^{\frac{1}{4}}\left[l(\alpha) - \frac{\sin(\alpha)|l(\alpha)|}{4(4+|l(\alpha)|)}\right] = \frac{\alpha|k(\alpha)|}{1+|k(\alpha)|} + \frac{\alpha|r(\alpha)|}{1+|r(\alpha)|}, \\ \Theta^{\frac{1}{4}}\left[r(\alpha) - \frac{\sin(\alpha)|r(\alpha)|}{4(4+|r(\alpha)|)}\right] = \frac{\alpha|l(\alpha)|}{1+|l(\alpha)|} + \frac{\alpha|k(\alpha)|}{1+|k(\alpha)|}, \\ k(0) = 0,\ l(0) = 0,\ r(0) = 0,\ \alpha \in \Omega = [0, \pi]. \end{cases} \tag{4.3}$$

Clearly problem (4.3) is a special case of problem (1.3) if we set

$$\gimel(\alpha, k(\alpha)) = \frac{\sin(\alpha)\,|k(\alpha)|}{4(4 + |k(\alpha)|)}, \text{ and } Ↄ\,(\alpha, k(\alpha), I^{\tau}(k(\alpha))) = \frac{\alpha\,|k(\alpha)|}{1 + |k(\alpha)|},$$

for chosen $k, l, r \in \nabla$ and $\alpha \in \Omega$, we can write

$$\begin{aligned} |\gimel(\alpha, k(\alpha)) - \gimel(\alpha, l(\alpha))| &\leq \frac{1}{4}\left(\frac{|k(\alpha)|}{4 + |k(\alpha)|} - \frac{|l(\alpha)|}{4 + |l(\alpha)|}\right) \\ &\leq \frac{1}{4}\left(\frac{|k(\alpha) - l(\alpha)| + |l(\alpha)|}{4 + |k(\alpha) - l(\alpha)| + |l(\alpha)|} - \frac{|l(\alpha)|}{4 + |k(\alpha) - l(\alpha)| + |l(\alpha)|}\right) \\ &\leq \frac{1}{4}\left(\frac{|k(\alpha) - l(\alpha)|}{4 + |k(\alpha) - l(\alpha)| + |l(\alpha)|}\right) \\ &\leq \frac{|k(\alpha) - l(\alpha)|}{4(4 + |k(\alpha) - l(\alpha)|)}, \end{aligned}$$

also, $Ↄ\,(\alpha, l(\alpha), I^{\tau}(l(\alpha))) = \frac{\alpha|l(\alpha)|}{1+|l(\alpha)|} \leq \alpha$ and $Ↄ\,(\alpha, r(\alpha), I^{\tau}(r(\alpha))) \leq \alpha$. From the above setting we $\mho_0 = 0$, $\gamma = 1$, $\varpi = 4$, $\mu = \frac{1}{4}$, $\Lambda = \pi$, and $\omega(\alpha) = \alpha$. In addition to, $\gamma + \mho_0 + \frac{2\Lambda^{\mu}}{\Gamma(1+\mu)}\,\|\omega\|_{L^1} = 1 + 2\pi^{\frac{1}{4}}\Gamma(1.25) \leq 3$. Thus $\beth \geq 3$. Therefore the hypotheses $(t_{ii}) - (t_{iv})$ of Theorem 4.2 are fulfilled, so there is a solution of the system (4.3).

5. Conclusion

Undoubtedly, the theory of FDEs attracted many scientists and mathematicians to work on. The results have been obtained by using FPTs. FP technique play an important role in solutions of nonlinear initial-value problems of FDEs. From this point, in this manuscript, a tfp theorem and some lemmas to discuss the theoretical results are obtained. Also as an application, system of TFDEs has been created and a solution was obtained for it. Lastly, non-trivial example are presented here to support our application.

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Conflict of interest

The authors declare that they have no competing interests.

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