AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2315–2329.
DOI: 10.3934/math.2021140
Received: 11 November 2020
Accepted: 11 December 2020
Published: 16 December 2020

Research article

Existence result for a Kirchhoff elliptic system involving p-Laplacian operator with variable parameters and additive right hand side via sub and super solution methods

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Abstract: The paper deals with the study of the existence result for a Kirchhoff elliptic system with additive right hand side and variable parameters involving $p-$Laplacian operator by using the sub-super solutions method. Our study is an natural extension result of our previous once in (Math. Methods Appl. Sci. 41 (2018), 5203–5210), where in the latter we discussed only the simple case when the parameters are constant.

Keywords: Kirchhoff elliptic systems; existence; positive solutions; sub-supersolution; multiple parameters
Mathematics Subject Classification: 35J60, 35B30, 35B40

1. Introduction

Consider the following system

$$\begin{cases} -M_1\left(\int\limits_{\Omega}|\nabla u|^p\,dx\right)\triangle_p u = \alpha(x)\,f(v)+\beta(x)\,g(u) \text{ in } \Omega, \\ -M_2\left(\int\limits_{\Omega}|\nabla v|^p\,dx\right)\triangle_p v = \gamma(x)\,h(u)+\eta(x)\,l(v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_p z = \mathrm{div}\left(|\nabla z|^{p-2}\,\nabla z\right)$, $1 < p < N$, the p-Laplacian operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $M_i : \mathbb{R}^+ \to \mathbb{R}^+, i = 1, 2$, are continuous functions with further conditions to be given later, $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$.

This nonlocal problem originates from the stationary version of Kirchhoff's work [15] in 1883.

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where Kirchhoff extended the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

Recently, Kirchhoff elliptic equations have been heavily studied, we refer to [1–21, 23, 24].

In [1], Alves and Correa proved the validity of Sub-super solutions method for problems of Kirchhoff class involving a single equation and a boundary condition

$$\begin{cases} -M\left(\|u\|^2\right)\Delta u = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $f \in C\left(\overline{\Omega} \times \mathbb{R}\right)$.
By using a comparison principle that requires M to be non-negative and non-increasing in $[0, +\infty)$, with $H(t) := M\left(t^2\right)t$ increasing and $H(\mathbb{R}) = \mathbb{R}$, they managed to prove the existence of positive solutions assuming f increasing in the variable u for each $x \in \Omega$ fixed.

For systems involving similar class of equations, this result can not be used directly, i.e. the existence of a subsolution and a supersolution does not guarantee the existence of the solution.

Therefore, a further construction is needed. As in [22], where we studied the system

$$\begin{cases} -A\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = \lambda_1 f(v) + \mu_1 g(u) \text{ in } \Omega, \\ \\ -B\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v = \lambda_2 h(u) + \mu_2(x)\, l(v) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Using a weak positive supersolution as first term of a constructed iterative sequence (u_n, v_n) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, and a comparison principle introduced in [1], the authors established the convergence of this sequence to a positive weak solution of the considered problem.

To complement our above works in [22], where we discussed only the simple case when the parameters are constant, we are working in this paper for proving the existence result for problem (1.1) by considering the complicated case when the parameters α, β, γ and η in the right hand side are variable. We also give a better subsolution providing easier computations compared with the last work in [22].

2. Existence result

Definition 1. *$(u, v) \in \left(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right)$, is called a weak solution of (1.1) if it satisfies*

$$M_1\left(\int\limits_{\Omega} |\nabla u|^p \, dx\right) \int\limits_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi dx = \int\limits_{\Omega} \alpha(x) f(v) \phi dx + \int\limits_{\Omega} \beta(x) g(u) \phi \, dx \text{ in } \Omega,$$

$$M_2\left(\int\limits_{\Omega} |\nabla v|^p \, dx\right) \int\limits_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \psi dx = \int\limits_{\Omega} \gamma(x) h(u) \psi dx + \int\limits_{\Omega} \eta(x) l(v) \psi \, dx \text{ in } \Omega$$

for all $(\phi, \psi) \in \left(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right)$.

Definition 2. *Let $\left(\underline{u}, \underline{v}\right), (\overline{u}, \overline{v})$ be a pair of nonnegative functions in $\left(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right)$, they are called positive weak subsolution and positive weak supersolution (respectively) of (1.1) if they satisfy the following*

$$M_1\left(\int\limits_{\Omega} \left|\nabla \underline{u}\right|^p dx\right) \int\limits_{\Omega} \left|\nabla \underline{u}\right|^{p-2} \nabla \underline{u} \nabla \phi dx \leq \int\limits_{\Omega} \alpha(x) f\left(\underline{v}\right) \phi \, dx + \int\limits_{\Omega} \beta(x) g\left(\underline{u}\right) \phi \, dx,$$

$$M_2\left(\int\limits_{\Omega} \left|\nabla \underline{v}\right|^2 dx\right) \int\limits_{\Omega} \left|\nabla \underline{v}\right|^{p-2} \nabla \underline{v} \nabla \psi dx \leq \int\limits_{\Omega} \gamma(x) h\left(\underline{u}\right) \psi \, dx + \int\limits_{\Omega} \eta(x) l\left(\underline{v}\right) \psi \, dx$$

and

$$M_1\left(\int_\Omega |\nabla \overline{u}|^p\,dx\right)\int_\Omega |\nabla \overline{u}|^{p-2}\,\nabla \overline{u}\nabla\phi dx \geq \int_\Omega \alpha\,(x)\,f\,(\overline{v})\,\phi\,dx + \int_\Omega \beta\,(x)\,g\,(\overline{u})\,\phi\,dx,$$

$$M_2\left(\int_\Omega |\nabla \overline{v}|^p\,dx\right)\int_\Omega |\nabla \overline{v}|^{p-2}\,\nabla \overline{v}\nabla\psi dx \geq \int_\Omega \gamma\,(x)\,h\,(\overline{u})\,\psi\,dx \;+ \int_\Omega \eta\,(x)\,l\,(\overline{v})\,\psi\,dx$$

for all $(\phi,\psi) \in \left(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right)$, with $\phi \geq 0$ and $\psi \geq 0$, and $\left(\underline{u},\underline{v}\right), (\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$.

Lemma 1. *(Comparison principle [24]) Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function such that*

$$M\,(s) > m_0 > 0, \textit{ for all } s \in \mathbb{R}^+. \tag{2.1}$$

If u, v are two non-negative functions verifying

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\,dx\right)\triangle_p u \geq -M\left(\int_\Omega |\nabla v|^p\,dx\right)\triangle_p v \textit{ in } \Omega, \\ u = v = 0 \textit{ on } \partial\Omega, \end{cases} \tag{2.2}$$

then $u \geq v$ a.e. in Ω.

Proof. Thanks to [24]. Define the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by the formula

$$J\,(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p\,dx\right),\; u \in W_0^{1,p}(\Omega)\,,$$

where

$$\widehat{M}\,(s) = \int_0^s M\,(\xi)\,d\xi.$$

It is obvious that the functional J is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in W_0^{1,p}(\Omega)$ is the functional $J' \in W_0^{-1,p}(\Omega)$, given by

$$J'\,(u)\,(\varphi) = M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla\varphi dx,\; \varphi \in W_0^{1,p}(\Omega)\,.$$

It is obvious that J' is continuous and bounded since the function M is continuous.

We will show that J' is strictly monotone in $W_0^{1,p}(\Omega)$.

Indeed, for any $u, v \in W_0^{1,p}(\Omega)$, $u \neq v$, without loss of generality, we may assume that

$$\int_\Omega |\nabla u|^p\,dx \geq \int_\Omega |\nabla v|^p\,dx.$$

Otherwise, changing the role of u and v in the following proof.
Therefore, we have

$$M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \geq M\left(\int_{\Omega} |\nabla v|^p \, dx\right). \tag{2.3}$$

Since $M(s)$ is a monotone function.
Using Cauchy's inequality, we have

$$\nabla u.\nabla v \leq |\nabla u| \, |\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right). \tag{2.4}$$

Using (2.4) we get

$$\int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} |\nabla u|^{p-2} \nabla u.\nabla v dx \geq \frac{1}{2}\int_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx, \tag{2.5}$$

and

$$\int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} |\nabla v|^{p-2} \nabla v.\nabla u dx \geq \frac{1}{2}\int_{\Omega} |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right) dx. \tag{2.6}$$

If $|\nabla u(x)| \geq |\nabla v(x)|$ for all $x \in \Omega$, using (2.3)–(2.6) we have

$$\begin{aligned}
I_1 \ : \ &= J'(u)(u) - J'(u)(v) - J'(v)(u) + J'(v)(v) \qquad (2.7)\\
&= M\left(\int_{\Omega} |\nabla u|^p \, dx\right)\left(\int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} |\nabla u|^{p-2} \nabla u.\nabla v dx\right)\\
&\quad - M\left(\int_{\Omega} |\nabla v|^p \, dx\right)\left(\int_{\Omega} |\nabla v|^{p-2} \nabla v.\nabla u dx - \int_{\Omega} |\nabla v|^p \, dx\right)\\
&\geq \frac{1}{2} M\left(\int_{\Omega} |\nabla u|^p \, dx\right)\int_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx\\
&\quad - \frac{1}{2} M\left(\int_{\Omega} |\nabla v|^p \, dx\right)\int_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx\\
&= \frac{1}{2} M\left(\int_{\Omega} |\nabla v|^p \, dx\right)\int_{\Omega} \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right) dx\\
&\geq \frac{m_0}{2}\int_{\Omega} \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right) dx.
\end{aligned}$$

If $|\nabla v(x)| \geq |\nabla u(x)|$ for all $x \in \Omega$, changing the role of u and v in (2.3)–(2.7), we have

$$\begin{aligned}
I_2 \quad : \quad &= J'(v)(v) - J'(v)(u) - J'(u)(v) + J'(u)(u) &(2.8)\\
&= M\left(\int_\Omega |\nabla v|^p\, dx\right)\left(\int_\Omega |\nabla v|^p\, dx - \int_\Omega |\nabla v|^{p-2}\,\nabla v.\nabla u dx\right)\\
&\quad -M\left(\int_\Omega |\nabla u|^p\, dx\right)\left(\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v dx - \int_\Omega |\nabla u|^p\, dx\right)\\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right)dx\\
&\quad -\frac{1}{2}M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right)dx\\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx\\
&\geq \frac{m_0}{2}\int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx.
\end{aligned}$$

From (2.6) and (2.7) we have

$$\left(J'(u) - J'(v)\right)(u-v) = I_1 = I_2 \geq 0, \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{2.9}$$

Moreover, if $u \neq v$ and $\left(J'(u) - J'(v)\right)(u-v) = 0$, then we have

$$\int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx = 0,$$

so $|\nabla u| = |\nabla v|$ in Ω. Thus, we deduce that

$$\begin{aligned}
\left(J'(u) - J'(v)\right)(u-v) &= J'(u)(u-v) - J'(v)(u-v) &(2.10)\\
&= M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\,|\nabla u - \nabla v|^2\, dx\\
&= 0,
\end{aligned}$$

i.e., $u - v$ is a constant.

In view of $u = v = 0$ on $\partial\Omega$ we have $u \equiv v$ which is contrary with $u \neq v$.

Therefore $\left(J'(u) - J'(v)\right)(u-v) > 0$ and J is strictly monotone in $W_0^{1,p}(\Omega)$.

Let u, v be two functions such that (2.2) is verified. Taking $\varphi = (u-v)^+$, the positive part of $u - v$ as a test function of (2.2), we have

$$\left(J'(u) - J'(v)\right)(\varphi) = M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla\varphi dx \tag{2.11}$$

$$-M\left(\int_{\Omega} |\nabla v|^p \, dx\right) \int_{\Omega} |\nabla v|^{p-2} \nabla v . \nabla \varphi dx$$
$$\leq \quad 0.$$

Relations (2.10) and (2.11) mean that $u \leq v$. □

Before stating and proving our main result, here are the conditions we need.

$(H1)$ $M_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, are two continuous and increasing functions that satisfy the monotonicity conditions of lemma 2.2 so that we can use the Comparison principle, and assume further that there exists $m_1, m_2 > 0$ such that

$$M_1(s) \geq m_1, \; M_2(s) \geq m_2, \quad \text{for all } s \in \mathbb{R}^+.$$

$(H2)$ $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$ and

$$\alpha(x) \geq \alpha_0 > 0, \; \beta(x) \geq \beta_0 > 0, \; \gamma(x) \geq \gamma_0 > 0, \; \eta(x) \geq \eta_0 > 0$$

for all $x \in \Omega$.

$(H3)$ f, g, h, and l are continuous on $[0, +\infty[$, C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t \to +\infty} f(t) = +\infty, \lim_{t \to +\infty} l(t) = +\infty, \lim_{t \to +\infty} g(t) = +\infty, \quad \lim_{t \to +\infty} h(t) = +\infty.$$

$(H4)$ For all $K > 0$

$$\lim_{t \to +\infty} \frac{f\left(K\left((h(t))^{\frac{1}{p-1}}\right)\right)}{t^{p-1}} = 0.$$

$(H5)$

$$\lim_{t \to +\infty} \frac{g(t)}{t^{p-1}} = \lim_{t \to +\infty} \frac{l(t)}{t^{p-1}} = 0.$$

Theorem 1. *For large values of $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$, system (1.1) admits a large positive weak solution if conditions $(H1) - (H5)$ are satisfied.*

Proof of Theorem 1. Consider σ_p the first eigenvalue of $-\triangle_p$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\| = 1$ and $\phi_1 \in C^\infty\left(\overline{\Omega}\right)$ (see [10]).

Let $S = \sup_{x \in \Omega}\{\sigma_p \phi_1^p - |\nabla \phi_1|^p\}$, then from growth conditions (H3)

$$f(t) \geq S, \; g(t) \geq S, \; h(t) \geq S, \; l(t) \geq S, \; \text{for } t \text{ large enough}.$$

For each $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$ large, let us define

$$\underline{u} = \left(\frac{\alpha_0 + \beta_0}{m_1}\right)^{\frac{1}{p-1}} \frac{p-1}{p} \phi_1^{\frac{p}{p-1}},$$

and

$$\underline{v} = \left(\frac{\gamma_0 + \eta_0}{m_2}\right)^{\frac{1}{p-1}} \frac{p-1}{p} \phi_1^{\frac{p}{p-1}},$$

where m_1, m_2 are given by condition $(H1)$. Let us show that $\left(\underline{u}, \underline{v}\right)$ is a subsolution of problem (1.1) for $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$ large enough. Indeed, let $\phi \in W_0^{1,p}(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1) - (H3)$, we get

$$\begin{aligned}
M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \int_\Omega \left|\nabla \underline{u}\right|^{p-2} \nabla \underline{u} \nabla \phi dx &= M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \frac{\alpha_0 + \beta_0}{m_1} \int_\Omega \phi_1 \left|\nabla \phi_1\right|^{p-2} \nabla \phi_1 . \nabla \phi dx \\
&= \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \times \\
&\quad \left\{\int_\Omega \left|\nabla \phi_1\right|^{p-2} \nabla \phi_1 \left[\nabla (\phi_1 \phi) - \phi \nabla \phi_1\right] dx\right\} \\
&= \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \times \\
&\quad \left\{\int_\Omega \left|\nabla \phi_1\right|^{p-2} \nabla \phi_1 \nabla (\phi_1 \phi) dx\right\} \\
&\quad - \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \left\{\int_\Omega \left|\nabla \phi_1\right|^p \phi dx\right\} \\
&= \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \left\{\int_\Omega \sigma_p \left|\phi_1\right|^{p-2} \phi_1 . (\phi_1 \phi) dx\right\} \\
&\quad - \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \left\{\int_\Omega \left|\nabla \phi_1\right|^p \phi dx\right\} \\
&= \frac{\alpha_0 + \beta_0}{m_1} M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) \int_\Omega \left(\sigma_p \left|\phi_1\right|^p - \left|\nabla \phi_1\right|^p\right) \phi dx \\
&\leq (\alpha_0 + \beta_0) \int_\Omega S \phi dx \\
&\leq \int_\Omega \alpha(x) f\left(\underline{v}\right) \phi dx + \int_\Omega \beta(x) g\left(\underline{u}\right) \phi dx
\end{aligned}$$

for $\alpha_0 + \beta_0 > 0$ large enough, and all $\phi \in W_0^{1,p}(\Omega)$ with $\phi \geq 0$ in Ω.

Similarly,

$$M_2\left(\int_\Omega \left|\nabla \underline{v}\right|^2 dx\right)\int_\Omega \left|\nabla \underline{v}\right|^{p-2}\nabla \underline{v}.\nabla \psi dx \leq \int_\Omega \gamma(x)\, h\left(\underline{u}\right)\psi dx \;+\int_\Omega \eta(x)\, \text{ł}\left(\underline{v}\right)\psi dx \text{ in } \Omega$$

for $\gamma_0 + \eta_0 > 0$ large enough and all $\psi \in W_0^{1,p}(\Omega)$ with $\psi \geq 0$ in Ω.

Also notice that $\underline{u} > 0$ and $\underline{v} > 0$ in Ω,
$\underline{u} \to +\infty$ and $\underline{v} \to +\infty$ as $\alpha_0 + \beta_0 \to +\infty$ and $\gamma_0 + \eta_0 \to +\infty$.

For the supersolution part, consider e_p the solution of the following problem

$$\begin{cases} -\triangle_p e_p = 1 \text{ in } \Omega, \\ e_p = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.12}$$

We give the supersolution of problem (2.12) by

$$\overline{u} = Ce_p, \quad \overline{v} = \left(\frac{\|\gamma\|_\infty + \|\eta\|_\infty}{m_2}\right)^{\frac{1}{p-1}}\left(h\left(C\left\|e_p\right\|_\infty\right)\right)^{\frac{1}{p-1}} e_p,$$

where $C > 0$ is a large positive real number to be given later.
Indeed, for all $\phi \in W_0^{1,p}(\Omega)$ with $\phi \geq 0$ in Ω, we get from (2.12) and the condition $(H1)$

$$\begin{aligned} M_1\left(\int_\Omega |\nabla\overline{u}|^p dx\right)\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}.\nabla\phi dx &= M_1\left(\int_\Omega |\nabla\overline{u}|^p dx\right)\int_\Omega \left|\nabla e_p\right|^{p-2}\nabla e_p.\nabla\phi dx \\ &= C^{p-1}M_1\left(\int_\Omega |\nabla\overline{u}|^p dx\right)\int_\Omega \phi dx \\ &\geq m_1 C^{p-1}\int_\Omega \phi dx. \end{aligned}$$

By $(H4)$ and $(H5)$, we can choose C large enough so that

$$m_1 C^{p-1} \geq \|\alpha\|_\infty f\left[\left(\frac{\|\gamma\|_\infty + \|\eta\|_\infty}{m_2}\right)^{\frac{1}{p-1}}\left(h\left(C\left\|e_p\right\|_\infty\right)\right)^{\frac{1}{p-1}} e_p\right] + \|\beta\|_\infty g\left(C\left\|e_p\right\|_\infty\right).$$

Therefore,

$$
\begin{aligned}
&M_1\left(\int\limits_\Omega |\nabla \overline{u}|^p\,dx\right)\int\limits_\Omega |\nabla \overline{u}|^{p-2}\,\nabla \overline{u}.\nabla \phi dx \\
&\geq \left\{\|\alpha\|_\infty f\left[\left(\tfrac{\|\gamma\|_\infty+\|\eta\|_\infty}{m_2}\right)^{\frac{1}{p-1}}\left(h\left(C\left\|e_p\right\|_\infty\right)\right)^{\frac{1}{p-1}} e_p\right]+\|\beta\|_\infty g\left(C\left\|e_p\right\|_\infty\right)\right\}\int\limits_\Omega \phi dx \\
&\geq \|\alpha\|_\infty \int\limits_\Omega f\left[\left(\tfrac{\|\gamma\|_\infty+\|\eta\|_\infty}{m_2}\right)^{\frac{1}{p-1}}\left(h\left(C\left\|e_p\right\|_\infty\right)\right)^{\frac{1}{p-1}} e_p\right]\phi dx+\|\beta\|_\infty \int\limits_\Omega g\left(C\left\|e_p\right\|_\infty\right)\phi dx \\
&\geq \int\limits_\Omega \alpha(x)\, f(\overline{v})\,\phi dx+\int\limits_\Omega \beta(x)\, g(\overline{u})\,\phi dx.
\end{aligned}
\tag{2.13}
$$

Also

$$
\begin{aligned}
M_2\left(\int\limits_\Omega |\nabla \overline{v}|^p\,dx\right)\int\limits_\Omega |\nabla \overline{v}|^{p-2}\,\nabla \overline{v}.\nabla \psi dx &= \left(\|\gamma\|_\infty+\|\eta\|_\infty\right)\int\limits_\Omega h\left(C\left\|e_p\right\|_\infty\right)\psi dx \\
&\geq \int\limits_\Omega \gamma(x)\, h(\overline{u})\,\psi dx+\int\limits_\Omega \eta(x)\, h\left(C\left\|e_p\right\|_\infty\right)\psi dx.
\end{aligned}
\tag{2.14}
$$

Using $(H4)$ and $(H5)$ again for C large enough we get

$$
h\left(C\left\|e_p\right\|_\infty\right)\geq l\left[\left(\frac{\|\gamma\|_\infty+\|\eta\|_\infty}{m_2}\right)^{\frac{1}{p-1}}\left(h\left(C\left\|e_p\right\|_\infty\right)\right)^{\frac{1}{p-1}}\left\|e_p\right\|_\infty\right]\geq l(\overline{v}). \tag{2.15}
$$

Combining (2.13) and (2.14), we obtain

$$
M_2\left(\int\limits_\Omega |\nabla \overline{v}|^p\,dx\right)\int\limits_\Omega |\nabla \overline{v}|^{p-2}\,\nabla \overline{v}.\nabla \psi dx \geq \int\limits_\Omega \gamma(x)\, h(\overline{u})\,\psi dx+\int\limits_\Omega \eta(x)\, l(\overline{v})\,\psi dx. \tag{2.16}
$$

By (2.12) and (2.15), we conclude that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1). Furthermore, $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C chosen large enough.

Now, we use a similar argument to [22] in order to obtain a weak solution of our problem. Consider the following sequence

$$
\{(u_n,v_n)\}\subset\left(W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)\right),
$$

where: $u_0:=\overline{u}, v_0=\overline{v}$ and (u_n,v_n) is the unique solution of the system

$$
\begin{cases}
-M_1\left(\int\limits_\Omega |\nabla u_n|^p\,dx\right)\triangle_p u_n=\alpha(x)\, f(v_{n-1})+\beta(x)\, g(u_{n-1}) \text{ in } \Omega, \\
-M_2\left(\int\limits_\Omega |\nabla v_n|^p\,dx\right)\triangle_p v_n=\gamma(x)\, h(u_{n-1})+\eta(x)\, l(v_{n-1}) \text{ in } \Omega, \\
u=v=0 \text{ on } \partial\Omega.
\end{cases}
\tag{2.17}
$$

Since M_1 and M_2 satisfy $(H1)$ and $\alpha(x) f(v_{n-1}), \beta(x) g(u_{n-1}), \gamma(x) h(u_{n-1})$, and $\eta(x) l(v_{n-1}) \in L^p(\Omega)$ (in x),

we deduce from a result in [1] that system (2.16) has a unique solution $(u_n, v_n) \in \left(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right)$.

Using (2.16) and the fact that (u_0, v_0) is a supersolution of (1.1), we get

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u_0|^p\,dx\right)\triangle_p u_0 \geq \alpha(x) f(v_0) + \beta(x) g(u_0) = -M_1\left(\int_\Omega |\nabla u_1|^p\,dx\right)\triangle_p u_1, \\ -M_2\left(\int_\Omega |\nabla v_0|^p\,dx\right)\triangle_p v_0 \geq \gamma(x) h(u_0) + \eta(x) l(v_0) = -M_2\left(\int_\Omega |\nabla v_1|^p\,dx\right)\triangle_p v_1. \end{cases}$$

Then by Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and the monotonicity of f, g, h, and l one has

$$\begin{aligned} -M_1\left(\int_\Omega |\nabla u_1|^p\,dx\right)\triangle_p u_1 &= \alpha(x) f(v_0) + \beta(x) g(u_0) \\ &\geq \alpha(x) f\left(\underline{v}\right) + \beta(x) g\left(\underline{u}\right) \geq -M_1\left(\int_\Omega \left|\nabla \underline{u}\right|^p\,dx\right)\triangle_p \underline{u}, \\ -M_2\left(\int_\Omega |\nabla v_1|^p\,dx\right)\triangle_p v_1 &= \gamma(x) h(u_0) + \eta(x) l(v_0) \\ &\geq \gamma(x) h\left(\underline{u}\right) + \eta(x) l\left(\underline{v}\right) \geq -M_2\left(\int_\Omega \left|\nabla \underline{v}\right|^p\,dx\right)\triangle_p \underline{v}. \end{aligned}$$

According to Lemma 1 again, we obtain $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$.

Repeating the same argument for u_2, v_2, observe that

$$\begin{aligned} -M_1\left(\int_\Omega |\nabla u_1|^p\,dx\right)\triangle_p u_1 &= \alpha(x) f(v_0) + \beta(x) g(u_0) \\ &\geq \alpha(x) f(v_1) + \beta(x) g(u_1) = -M_1\left(\int_\Omega |\nabla u_2|^p\,dx\right)\triangle_p u_2, \\ -M_2\left(\int_\Omega |\nabla v_1|^p\,dx\right)\triangle_p v_1 &= \gamma(x) h(u_0) + \eta(x) l(v_0) \\ &\geq \gamma(x) h(u_1) + \eta(x) l(v_1) = -M_2\left(\int_\Omega |\nabla v_2|^p\,dx\right)\triangle_p v_2, \end{aligned}$$

then $u_1 \geq u_2$, $v_1 \geq v_2$.
Similarly, we get $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ from

$$
\begin{aligned}
-M_1\left(\int_\Omega |\nabla u_2|^p\,dx\right)\triangle_p u_2 &= \alpha(x)\,f(v_1)+\beta(x)\,g(u_1)\\
&\geq \alpha(x)\,f\left(\underline{v}\right)+\beta(x)\,g\left(\underline{u}\right)\geq -M_1\left(\int_\Omega \left|\nabla\underline{u}\right|^p dx\right)\triangle_p\underline{u},\\
-M_2\left(\int_\Omega |\nabla v_2|^p\,dx\right)\triangle_p v_2 &= \gamma(x)\,h(u_1)+\eta(x)\,l(v_1)\\
&\geq \gamma(x)\,h\left(\underline{u}\right)+\eta(x)\,l\left(\underline{v}\right)\geq -M_2\left(\int_\Omega \left|\nabla\underline{v}\right|^p dx\right)\triangle_p\underline{v}.
\end{aligned}
$$

By repeating these implementations we construct a bounded decreasing sequence $\{(u_n, v_n)\} \subset \left(W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)\right)$ verifying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq ... \geq u_n \geq ... \geq \underline{u} > 0, \tag{2.18}$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq ... \geq v_n \geq ... \geq \underline{v} > 0. \tag{2.19}$$

By continuity of functions f, g, h, and l and the definition of the sequences (u_n) and (v_n), there exist positive constants $C_i > 0$, $i = 1, ..., 4$ such that

$$|f(v_{n-1})| \leq C_1, \quad |g(u_{n-1})| \ \leq C_2,\ |h(u_{n-1})| \leq C_3, \tag{2.20}$$

and

$$|l(u_{n-1})| \ \leq C_4 \text{ for all } n.$$

From (2.19), multiplying the first equation of (2.16) by u_n, integrating, using Holder inequality and Sobolev embedding we check that

$$
\begin{aligned}
m_1\int_\Omega |\nabla u_n|^p\,dx &\leq M_1\left(\int_\Omega |\nabla u_n|^p\,dx\right)\int_\Omega |\nabla u_n|^p\,dx \qquad (2.21)\\
&= \int_\Omega \alpha(x)\,f(v_{n-1})\,u_n dx+\int_\Omega \beta(x)\,g(u_{n-1})\,u_n dx\\
&\leq \|\alpha\|_\infty\int_\Omega |f(v_{n-1})|\ \ |u_n|\,dx+\|\beta\|_\infty\int_\Omega |g(u_{n-1})|\ \ |u_n|\,dx\\
&\leq C_1\int_\Omega\ |u_n|\,dx+C_2\int_\Omega |u_n|\,dx
\end{aligned}
$$

$$\leq \quad C_5 \|u_n\|_{W_0^{1,p}(\Omega)},$$

or

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq C_5, \ \forall n, \tag{2.22}$$

where $C_5 > 0$ is a constant independent of n.
Similarly, there exist $C_6 > 0$ independent of n such that

$$\|v_n\|_{W_0^{1,p}(\Omega)} \leq C_6, \ \ \forall n. \tag{2.23}$$

From (2.20) and (2.21), we deduce that $\{(u_n, v_n)\}$ admits a weakly converging subsequence in $W_0^{1,p}\left(\Omega, \mathbb{R}^2\right) \times W_0^{1,p}\left(\Omega, \mathbb{R}^2\right)$ to a limit (u, v) satisfying $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v). Now, letting $n \to +\infty$ in (2.16), we conclude that (u, v) is a positive weak solution of system (1.1). □

3. Conclusions

In [22], we discussed only the simple case when the parameters are constant, in this current work, we have proved the existence result for problem (1.1) by considering the complicated case when the parameters α, β, γ and η in the right hand side are variable. We also give a better subsolution providing easier computations compared with the last work in [22]. In the next work, we will try to apply the same techniques in the Hall-MHD equations which is nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see for example [2, 8, 9]).

Acknowledgments

The authors are grateful to the Editor in Chief and anonymous referee for the insightful and constructive comments that improved this paper

Conflict of interest

All authors declare no conflicts of interest in this paper.

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