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Research article

Power bounded and power bounded below composition operators on Dirichlet Type spaces

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Abstract: Motivated by [11, 12], under some conditions on weighted function K, we investigated power bounded and power bounded below composition operators on Dirichlet Type spaces $\mathcal{D}_K$.

Keywords: composition operator; power bounded; Dirichlet Type spaces $\mathcal{D}_K$
Mathematics Subject Classification: 30D45, 30D50

1. Introduction

As usual, let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$, $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$ and H^∞ be the set of bounded analytic functions in $\mathbb{D}$. Let $0 < p < \infty$. The Hardy space H^p (see [5]) is the sets of $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Suppose that $K : [0, \infty) \to [0, \infty)$ is a right-continuous and nondecreasing function with $K(0) = 0$. The Dirichlet Type spaces $\mathcal{D}_K$, consists of those functions $f \in H(\mathbb{D})$, such that

$$\|f\|_{D_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty.$$

The space $\mathcal{D}_K$ has been extensively studied. Note that $K(t) = t$, it is Hardy spaces H^2. When $K(t) = t^\alpha$, $0 \le \alpha < 1$, it give the classical weighted Dirichlet spaces $\mathcal{D}_\alpha$. For more information on $\mathcal{D}_K$, we refer to [3, 7–10, 14–16, 19, 23].

Let ϕ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_ϕ on $H(\mathbb{D})$ is defined by

$$C_\phi(f) = f \circ \phi, \quad f \in H(\mathbb{D}).$$

It is an interesting problem to studying the properties related to composition operator acting on analytic function spaces. For example: Shapiro [17] introduced Nevanlinna counting functions studied the compactness of composition operator acting on Hardy spaces. Zorboska [23] studied the boundedness and compactness of composition operator on weighted Dirichlet spaces $\mathcal{D}_\alpha$. El-Fallah, Kellay, Shabankhah and Youssfi [7] studied composition operator acting on Dirichlet type spaces D^p_α by level set and capacity. For general weighted function ω, Kellay and Lefèvre [9] using Nevanlinna type counting functions studied the boundedness and compactness of composition spaces on weighted Hilbert spaces $\mathcal{H}_\omega$. After Kellay and Lefèvre's work, Pau and Pérez investigate more properties of composition operators on weighted Dirichlet spaces $\mathcal{D}_\alpha$ in [14]. For more information on composition operator, we refer to [4, 18].

We assume that $\mathcal{H}$ is a separable Hilbert space of analytic functions in the unit disc. Composition operator C_ϕ is called power bounded on $\mathcal{H}$ if C_{ϕ^n} is bounded on $\mathcal{H}$ for all $n \in \mathbb{N}$.

Since power bounded composition operators is closely related to mean ergodic and some special properties (such as: stable orbits) of ϕ, it has attracted the attention of many scholars. Wolf [20, 21] studied power bounded composition operators acting on weighted type spaces H^∞_υ. Bonet and Domański [1, 2] proved that C_ϕ is power bounded if and only if C_ϕ is (uniformly) mean ergodic in real analytic manifold (or a connected domain of holomorphy in C^d). Keshavarzi and Khani-Robati [11] studied power bounded of composition operator acting on weighted Dirichlet spaces $\mathcal{D}_\alpha$. Keshavarzi [12] investigated the power bounded below of composition operator acting on weighted Dirichlet spaces $\mathcal{D}_\alpha$ later. For more results related to power bounded composition operators acting on other function spaces, we refer to the paper cited and referin [1, 2, 11, 12, 20, 21].

We always assume that $K(0) = 0$, otherwise, $\mathcal{D}_K$ is the Dirichlet space $\mathcal{D}$. The following conditions play a crucial role in the study of weighted function K during the last few years (see [22]):

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

Note that the weighted function K satisfies (1.1) and (1.2), it included many special case, such as $K(t) = t^p$, $0 < p < 1$, $K(t) = \log \frac{e}{t}$ and so on. Some special skills are needed in dealing with certain problems. Motivated by [11, 12], using several estimates on the weight function K, we studying power bounded composition operators acting on $\mathcal{D}_K$. In this paper, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$. We say that $a \lesssim b$ if there exists a constant C such that $a \le Cb$, where $a, b > 0$.

2. Power bounded of C_ϕ

We assume that $\mathcal{H}$ is a separable Hilbert space of analytic functions in the unit disc. Let $R \in H(\mathbb{D})$ and $\{R_\zeta : \zeta \in \mathbb{D}\}$ be an independent collection of reproducing kernels for $\mathcal{H}$. Here $R_\zeta(z) = R(\bar{\zeta}z)$. The reproducing kernels mean that $f(\zeta) = \langle f, R_\zeta\rangle$ for any $f \in \mathcal{H}$. Let $R_{K,z}$ be the reproducing kernels for

$\mathcal{D}_K$. By [3], we see that if K satisfy (1.1) and (1.2), we have $\|R_{K,z}\|_{\mathcal{D}_K} \approx \frac{1}{\sqrt{K(1-|z|^2)}}$. Before we go into further, we need the following lemma.

Lemma 1. *Let K satisfies (1.1) and (1.2). Then*

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} \approx \frac{1}{(1-t)K(1-t)}$$

for all $0 \le t < 1$.

Proof. Without loss of generality, we can assume $4/5 < t < 1$. Since K is nondecreasing, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} &\approx \frac{1}{(\ln \frac{1}{t})K(\ln \frac{1}{t})} \int_{-\ln t}^{\infty} \frac{\gamma e^{-\gamma} K(\ln \frac{1}{t})}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\
&\gtrsim \frac{1}{(1-t)K(1-t)} \int_{\ln 2}^{\infty} \frac{\gamma e^{-\gamma} K(\ln \frac{1}{t})}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\
&\gtrsim \frac{1}{(1-t)K(1-t)} \int_{\ln 2}^{\infty} \gamma e^{-\gamma} d\gamma \\
&\approx \frac{1}{(1-t)K(1-t)}.
\end{aligned}$$

Conversely, make change of variables $y = \frac{1}{x}$, an easy computation gives

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} &\approx \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{t^{\frac{1}{x}}}{x^2 K(x)} dx \\
&\approx \int_0^1 \frac{t^{\frac{1}{x}}}{x^2 K(x)} dx \approx \int_1^{\infty} \frac{t^y}{K(\frac{1}{y})} dy.
\end{aligned}$$

Let $y = \frac{\gamma}{-\ln t}$. We can deduce that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} &\approx \frac{1}{(\ln \frac{1}{t})} \int_{-\ln t}^{\infty} \frac{\gamma e^{-\gamma}}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\
&= \frac{1}{(\ln \frac{1}{t})K(\ln \frac{1}{t})} \int_{-\ln t}^{\infty} \frac{\gamma e^{-\gamma} K(\ln \frac{1}{t})}{K(\frac{1}{\gamma} \ln \frac{1}{t})} d\gamma \\
&\lesssim \frac{1}{(1-t)K(1-t)} \int_{-\ln t}^{\infty} \gamma e^{-\gamma} \varphi_K(\gamma) d\gamma.
\end{aligned}$$

By [6], under conditions (1.1) and (1.2), there exists an enough small $c > 0$ only depending on K such that

$$\varphi_K(s) \lesssim s^c, \ 0 < s \le 1$$

and

$$\varphi_K(s) \lesssim s^{1-c},\ s \geq 1.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} &\lesssim \frac{1}{(1-t)K(1-t)} \int_{-\ln t}^{\infty} \gamma e^{-\gamma} \varphi_K(\gamma) d\gamma \\ &\lesssim \frac{1}{(1-t)K(1-t)} \left(\int_0^{\infty} e^{-\gamma} \gamma^{2-c} d\gamma + \int_0^{\infty} e^{-\gamma} \gamma^{1+c} d\gamma \right) \\ &\approx \frac{1}{(1-t)K(1-t)} \left(\Gamma(3-c) + \Gamma(2+c) \right),\end{aligned}$$

where $\Gamma(.)$ is the Gamma function. It follows that

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{K(\frac{1}{n+1})} \lesssim \frac{1}{(1-t)K(1-t)}.$$

The proof is completed. □

Theorem 1. *Let K satisfy (1.1) and (1.2). Suppose that ϕ is an analytic selt-map of unit disk which is not the identity or an elliptic automorphism. Then C_ϕ is power bounded on $\mathcal{D}_K$ if and only if ϕ has its Denjoy-Wolff point in $\mathbb{D}$ and for every $0 < r < 1$, we have*

$$\sup_{n\in\mathbb{N}, a\in\mathbb{D}} \frac{\int_{D(a,r)} N_{\phi_n,K}(z) dA(z)}{(1-|a|^2)^2 K(1-|a|^2)} < \infty, \tag{A}$$

where

$$D(a,r) = \left\{ z : \left| \frac{a-z}{1-\overline{a}z} \right| < r \right\},\ \ 0 < r < 1$$

and

$$N_{\phi_n,K}(z) = \sum_{\phi_n(z_j)=w} K(1-|z_j(w)|^2).$$

Proof. Suppose that $w \in \mathbb{D}$ is the Denjoy-Wolff point of ϕ and (A) holds. Then $\lim_{n\to\infty} \phi_n(0) = w$. Hence, there is some $0 < r < 1$ such that $\{\phi_n(0)\}_{n\in\mathbb{N}} \subseteq r\mathbb{D}$. Thus,

$$|f(\phi_n(0))|^2 \lesssim \|R_{K,\phi_n(0)}\|^2_{\mathcal{D}_K} \lesssim \|R_{K,r}\|^2_{\mathcal{D}_K},\ \ f \in \mathcal{D}_K.$$

From [24], we see that

$$1-|a| \approx 1-|z| \approx |1-\overline{a}z|,\ \ z \in D(a,r). \tag{B}$$

Let $\{a_i\}$ be a r-lattice. By sub-mean properties of $|f'|$, combine with (B), we deduce

$$
\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K} dA(z) \lesssim \sum_{i=1}^{\infty} \int_{D(a_i,r)} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
&\lesssim \sum_{i=1}^{\infty} \int_{D(a_i,r)} \frac{1}{(1-|a_i|)^2} \int_{D(a_i,l)} |f'(w)|^2 dA(w) N_{\phi_n,K}(z) dA(z) \\
&\lesssim \sum_{i=1}^{\infty} \int_{D(a_i,l)} |f'(w)|^2 \left(\int_{D(a_i,r)} \frac{N_{\phi_n,K}(z) dA(z)}{(1-|a_i|)^2 K(1-|a_i|^2)} \right) K(1-|w|^2) dA(w) \\
&\lesssim \sum_{i=1}^{\infty} \int_{D(a_i,l)} |f'(w)|^2 K(1-|w|^2) dA(w) < \infty.
\end{aligned}
$$

Thus,

$$
\|f \circ \phi_n\|_{\mathcal{D}_K}^2 = |f(\phi_n(0))|^2 + \int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K} dA(z) < \infty.
$$

On the other hand. Suppose that C_ϕ is power bounded on $\mathcal{D}_K$. Hence, for any $f \in \mathcal{D}_K$ and any $n \in \mathbb{N}$, we have $|f(\phi_n(0))| \lesssim 1$. Hence, by [3], it is easily to see that $\|R_{K,\phi_n(0)}\|_{\mathcal{D}_K} \approx \frac{1}{\sqrt{K(1-|\phi_n(0)|^2)}} \lesssim 1$. Note that

$$
\lim_{|z|\to 1} \|R_{K,z}\|_{\mathcal{D}_K} \approx \lim_{|z|\to 1} \frac{1}{\sqrt{K(1-|z|)}} = \infty.
$$

Therefore, we deduce that $\phi_n(0) \in r\mathbb{D}$, where $0 < r < 1$ and $n \in \mathbb{N}$. Also note that if $w \in \overline{\mathbb{D}}$ is the Denjoy-Wolff point of ϕ, we have $\lim_{n\to\infty} \phi_n(0) = w$. Thus, $w \in \mathbb{D}$. Let

$$
f_a(z) = \frac{1-|a|^2}{\overline{a}\sqrt{K(1-|a|^2)}(1-\overline{a}z)}.
$$

It is easily to verify that $f_a \in \mathcal{D}_K$ and $f_a'(z) = \frac{1-|a|^2}{\sqrt{K(1-|a|^2)}(1-\overline{a}z)^2}$. Thus, combine with (B), we have

$$
\begin{aligned}
\frac{\int_{D(a,r)} N_{\phi_n,K}(z) dA(z)}{(1-|a|^2)^2 K(1-|a|^2)} &\lesssim \int_{D(a,r)} \frac{(1-|a|^2)^2}{K(1-|a|^2)|1-\overline{a}z|^4} N_{\phi_n,K}(z) dA(z) \\
&\leq \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)|1-\overline{a}z|^4} N_{\phi_n,K}(z) dA(z) \\
&\lesssim \|f_a \circ \phi_n\|_{\mathcal{D}_K}^2 < \infty.
\end{aligned}
$$

Thus, (A) hold. The proof is completed. □

Theorem 2. *Let K satisfy (1.1) and (1.2). Suppose that ϕ is an analytic selt-map of unit disk which is not the identity or an elliptic automorphism with w as its Denjoy-wolff point. Then C_ϕ is power bounded on $\mathcal{D}_K$ if and only if*
(1). $w \in \mathbb{D}$.
(2). $\{\phi_n\}$ is a bounded sequence in $\mathcal{D}_K$.
(3). If $n \in \mathbb{N}$ and $|a| \geq \frac{1+|\phi_n(0)|}{2}$, then $\frac{N_{\phi_n,K}(a)}{K(1-|a|^2)} \lesssim 1$.

Proof. Suppose that C_ϕ is power bounded on $\mathcal{D}_K$. By Theorem 1, we see that $w \in \mathbb{D}$. Note that $z \in \mathcal{D}_K$ and $\phi_n = C_{\phi_n} z$, we have (2) hold. Now, we are going to show (3) hold. Let $|a| \geq \frac{1+|\phi_n(0)|}{2}$ and $\Delta(a) = \{z : |z-a| < \frac{1}{2}(1-|a|)\}$. Thus,

$$|\phi_n(0)| < |z|, \quad z \in \Delta(a).$$

If K satisfy (1.1) and (1.2). By [9], $N_{\phi_n,K}$ has sub-mean properties. Thus,

$$\begin{aligned}
&\frac{N_{\phi_n,K}(a)}{K(1-|a|^2)} \lesssim \frac{\int_{\Delta(a)} N_{\phi_n,K}(z)dA(z)}{(1-|a|^2)^2 K(1-|a|^2)} \\
&\lesssim \int_{\Delta(a)} \frac{(1-|a|^2)^2}{K(1-|a|^2)|1-\overline{a}z|^4} N_{\phi_n,K}(z)dA(z) \\
&\lesssim \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)|1-\overline{a}z|^4} N_{\phi_n,K}(z)dA(z) \\
&\lesssim \|f \circ \phi_n\|^2_{\mathcal{D}_K} < \infty.
\end{aligned}$$

Conversely. Suppose that (1)–(3) holds. Let $f \in \mathcal{D}_K$. Note that $z \in \mathcal{D}_K$, $z' = 1$ and $\frac{1+|\phi_n(0)|}{2} < 1$. By Lemma 1, we see that

$$\|R'_{K,\frac{1+|\phi_n(0)|}{2}}\|^2_{\mathcal{D}_K} \approx \frac{1}{(1-\frac{1+|\phi_n(0)|}{2})K(1-\frac{1+|\phi_n(0)|}{2})} < \infty.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z)dA(z) \\
&= \int_{|z|\geq \frac{1+|\phi_n(0)|}{2}} |f'(z)|^2 N_{\phi_n,K}(z)dA(z) + \int_{|z|< \frac{1+|\phi_n(0)|}{2}} |f'(z)|^2 N_{\phi_n,K}(z)dA(z) \\
&\lesssim \int_{|z|\geq \frac{1+|\phi_n(0)|}{2}} |f'(z)|^2 K(1-|z|^2)dA(z) + \|R'_{K,\frac{1+|\phi_n(0)|}{2}}\|^2_{\mathcal{D}_K} \int_{|z|< \frac{1+|\phi_n(0)|}{2}} N_{\phi_n,K}(z)dA(z) \\
&\lesssim \|f\|^2_{\mathcal{D}_K} + \|R'_{K,\frac{1+|\phi_n(0)|}{2}}\|^2_{\mathcal{D}_K} \|\phi_n\|^2_{\mathcal{D}_K} < \infty.
\end{aligned}$$

The proof is completed. □

Theorem 3. *Let K satisfy (1.1) and (1.2). Suppose that ϕ is an analytic selt-map of $\mathbb{D}$ with Denjoy-Wolff point w and C_ϕ is power bounded on $\mathcal{D}_K$. Then $f \in \Gamma_{c,K}(\phi)$ if and only if for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \int_{\Omega_\epsilon(f)} \frac{N_{\phi_n,K}(z)dA(z)}{(1-|z|^2)^2 K(1-|z|^2)} = 0, \tag{C}$$

where $\Gamma_{c,K}(\phi) = \{f \in \mathcal{D}_K : \ C_{\phi_n} f$ is convergent$\}$ and $\Omega_\epsilon(f) = \{z : (1-|z|^2)^2 K(1-|z|^2)|f'(z)|^2 \geq \epsilon\}$.

Proof. Let $f \in \mathcal{D}_K$ and (C) hold. For any $\delta > 0$, we choose $0 < \epsilon < \delta$ and ϵ is small enough such that

$$\int_{\Omega_\epsilon(f)^c} |f'(z)|^2 K(1-|z|^2)dA(z) < \delta.$$

By our assumption, we also know that for this ϵ, there is some $N \in \mathbb{N}$ such that for each $n \geq N$, we have

$$\int_{\Omega_\epsilon(f)} \frac{N_{\phi_n,K}(z)}{(1-|z|^2)^2 K(1-|z|^2)} dA(z) < \delta.$$

Since

$$|f'(z)| \lesssim \frac{\|f\|_{\mathcal{D}_K}}{(1-|z|^2)\sqrt{K(1-|z|^2)}}, \ f \in \mathcal{D}_K.$$

We obtain

$$\begin{aligned}
&\int_{\Omega_\epsilon(f)} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
\lesssim &\|f\|^2_{\mathcal{D}_K} \int_{\Omega_\epsilon(f)} \frac{N_{\phi_n,K}(z)}{(1-|z|^2)^2 K(1-|z|^2)} dA(z) < \delta \|f\|^2_{\mathcal{D}_K}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_\epsilon(f)^c} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
= &\int_{\Omega_\epsilon(f)^c \cap r\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) + \int_{\Omega_\epsilon(f)^c \setminus r\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
\lesssim &\epsilon \int_{\Omega_\epsilon(f)^c \cap r\mathbb{D}} \frac{N_{\phi_n,K}(z)}{(1-|z|^2)^2 K(1-|z|^2)} dA(z) + \int_{\Omega_\epsilon(f)^c \setminus r\mathbb{D}} |f'(z)|^2 K(1-|z|^2) dA(z) \\
\lesssim &\epsilon \int_{\Omega_\epsilon(f)^c \cap r\mathbb{D}} \frac{N_{\phi_n,K}(z)}{(1-r^2)^2 K(1-r^2)} dA(z) + \int_{\Omega_\epsilon(f)^c} |f'(z)|^2 K(1-|z|^2) dA(z) \\
< &\delta \frac{\|\phi_n\|^2_{\mathcal{D}_K}}{(1-r^2)^2 K(1-r^2)} + \delta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
= &\int_{\Omega_\epsilon(f)} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) + \int_{\Omega_\epsilon(f)^c} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) \\
\lesssim &\left(\|f\|^2_{\mathcal{D}_K} + \frac{\|\phi_n\|^2_{\mathcal{D}_K}}{(1-r^2)^2 K(1-r^2)} + 1 \right) \delta.
\end{aligned}$$

Conversely. Suppose that $f \in \mathcal{D}_K$ and w is the Denjoy-Wolff point of ϕ. Thus, $f \circ \phi_n \to f(w)$ uniform convergent and $f \in \Gamma_{c,K}(\phi)$ if and only if

$$\lim_{n\to\infty} \int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z) dA(z) = 0.$$

Suppose there exist $\epsilon > 0$ such that (C) dose not hold. There is a sequence $\{n_k\} \subseteq \mathbb{N}$ and some $\eta > 0$ such that for any $k \in \mathbb{N}$, we have

$$\int_{\Omega_\epsilon(f)} \frac{N_{\phi_n,K}(z) dA(z)}{(1-|z|^2)^2 K(1-|z|^2)} > \eta.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{D}}|f'(z)|^2N_{\phi_n,K}(z)dA(z)\\ \geq&\int_{\Omega_\epsilon(f)}|f'(z)|^2N_{\phi_n,K}(z)dA(z)\\ \geq&\epsilon\int_{\Omega_\epsilon(f)}\frac{N_{\phi_n,K}(z)}{(1-|z|^2)^2K(1-|z|^2)}dA(z)>\eta\epsilon.\end{aligned}$$

That is a contradiction. The proof is completed. □

3. Power bounded below of C_ϕ

The composition operator C_ϕ is called power bounded below if there exists some $C > 0$ such that $\|C_{\phi_n}f\|_{\mathcal{H}} \geq C\|f\|_{\mathcal{H}}$, for all $f \in \mathcal{H}$ and $n \in \mathbb{N}$.

In this section, we are going to show the equivalent characterizations of composition operator C_ϕ power bounded below on $\mathcal{D}_K$. Before we get into prove, let us recall some notions.

(1) We say that $\{G_n\}$, a sequence of Borel subsets of $\mathbb{D}$ satisfies the reverse Carleson condition on $\mathcal{D}_K$ if there exists some positive constant δ such that for each $f \in \mathcal{D}_K$,

$$\delta\int_{G_n}|f'(z)|^2K(1-|z|^2)dA(z)\geq\int_{\mathbb{D}}|f'(z)|^2K(1-|z|^2)dA(z).$$

(2) We say that $\{\mu_n\}$, a sequence of Carleson measure on $\mathbb{D}$ satisfies the reverse Carleson condition, if there exists some positive constant δ and $0 < r < 1$ such that

$$\mu_n(D(a,r)) > \delta|D(a,r)|$$

for each $a \in \mathbb{D}$ and $n \in \mathbb{N}$.

Theorem 4. *Let K satisfy (1.1) and (1.2). Suppose that ϕ is an analytic selt-map of $\mathbb{D}$ and C_ϕ is power bounded on $\mathcal{D}_K$. Then the following are equivalent.*

(1). C_ϕ is power bounded below.

(2). There exists some $\delta > 0$ such that $\|C_{\phi_n}f_a\| \geq \delta$ for all $a \in \mathbb{D}$ and $n \in \mathbb{N}$.

(3). There exists some $\delta > 0$ and $\epsilon > 0$ such that for all $a \in \mathbb{D}$ and $n \in \mathbb{N}$,

$$\int_{G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z)>\delta,$$

where $G_\epsilon(n) = \{z \in \mathbb{D} : \frac{N_{\phi_n,K}(z)}{K(1-|z|^2)} \geq \epsilon\}$.

(4). There is some $\epsilon > 0$ such that the sequence of measures $\{\chi_{G_\epsilon(n)}dA\}$ satisfies the reverse Carleson condition.

(5). The sequence of measures $\{\frac{N_{\phi_n,K}(z)}{K(1-|z|^2)}dA\}$ satisfies the reverse Carleson condition.

(6). There is some $\epsilon > 0$ such that the sequence of Borel sets $\{G_\epsilon(n)\}$ satisfies the reverse Carleson condition.

Proof. Suppose that w is the Denjoy-Wolff point of ϕ. By Theorem 2, $w \in \mathbb{D}$. Without loss of generality, we use $\varphi_w \circ \phi \circ \varphi_w$ instead of ϕ.

(1) $\Rightarrow$ (2). It is obvious.

(3) $\Rightarrow$ (4). By [6], there exist a small $c > 0$ such that $\frac{K(t)}{t^c}$ is nondecreasing $(0 < t < 1)$. Thus, the proof is similar to [18, page 5]. Let $0 < r < 1$ and $C > 0$ such that

$$\int_{\mathbb{D}\setminus r\mathbb{D}} K(1-|z|^2)dA(z) \geq \frac{K(1-r^2)}{(1-r^2)^c}\int_{\mathbb{D}\setminus r\mathbb{D}}(1-|z|^2)^c dA(z) > 1 - \frac{C\delta}{2}.$$

Making change of variable $z = \varphi_a(w) = \frac{a-z}{1-\overline{a}z}$, we obtain

$$\begin{aligned}\frac{C\delta}{2} \geq &\int_{r\mathbb{D}} K(1-|z|^2)dA(z))\\ =&\int_{D(a,r)} \frac{(1-|a|^2)^2}{|1-\overline{a}w|^4}K(1-|\varphi_a(w)|^2)dA(w)\\ \geq&C\int_{D(a,r)} \frac{(1-|a|^2)^2}{K(1-|a|^2)|1-\overline{a}w|^4}K(1-|w|^2)dA(w)\\ =&C\int_{D(a,r)}|f_a'(w)|^2K(1-|w|^2)dA(w).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{D(a,r)\cap G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z)\\ =&\int_{G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z) - \int_{D(a,r)}|f_a'(z)|^2K(1-|z|^2)dA(z)\\ \geq&\ \delta - \frac{\delta}{2} = \frac{\delta}{2}.\end{aligned}$$

(2) $\Rightarrow$ (3). Let $r = \sup_{n\in\mathbb{N}} \frac{1+|\phi_n(0)|}{2}$. We claim that: there exists some $\epsilon > 0$ and some $\delta > 0$ such that for all $a \in \mathbb{D}$ and $n \in \mathbb{N}$,

$$\int_{r\mathbb{D}}|f_a'(z)|^2N_{\phi_n,K}(z)dA(z) > \delta$$

or

$$\int_{G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z) > \delta.$$

Suppose that there are no $\epsilon, \delta > 0$ such that the above inequalities hold. Thus, there exists sequences $\{a_k\} \subseteq \mathbb{D}$ and $\{n_k\} \subseteq \mathbb{N}$ such that

$$\int_{r\mathbb{D}}|f_{a_k}'(z)|^2N_{\phi_{n_k},K}(z)dA(z) < \frac{1}{k}$$

or

$$\int_{G_\epsilon(n)}|f_{a_k}'(z)|^2K(1-|z|^2)dA(z) < \frac{1}{k}.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{D}}|f_a'(z)|^2N_{\phi_{n_k},K}(z)dA(z)=\int_{r\mathbb{D}}|f_a'(z)|^2N_{\phi_{n_k},K}(z)dA(z)\\&+\int_{G_{\frac{1}{k}}(n_k)\backslash r\mathbb{D}}|f_a'(z)|^2N_{\phi_{n_k},K}(z)dA(z)+\int_{\mathbb{D}\backslash\left(G_{\frac{1}{k}}(n_k)\backslash r\mathbb{D}\right)}|f_a'(z)|^2N_{\phi_{n_k},K}(z)dA(z)\\&\leq\frac{1}{k}+\frac{L}{k}+\frac{\eta}{k}\to 0,\end{aligned}$$

as $k\to\infty$. Where

$$L=\sup_{|a|\geq\frac{1+|\phi_n(0)|}{2},n\in\mathbb{N}}\frac{N_{\phi_n,K}(a)}{K(1-|a|^2)},\quad \eta=\sup_{a\in\mathbb{D}}\|f_a\|^2_{\mathcal{D}_K}.$$

This contradict (2), so our claim hold. Let $\epsilon,\delta>0$ be as in above. Since $f_a'\to 0$, uniformly on the compact subsets of $\mathbb{D}$, as $|a|\to 1$, there exists some $0<s<1$ such that for all $|a|>s$, we have

$$\int_{r\mathbb{D}}|f_a'(z)|^2N_{\phi_{n_k},K}(z)dA(z)\leq\|f_a'|_{r\mathbb{D}}\|^2_{H^\infty}\|\phi_n\|^2_{\mathcal{D}_K}\leq\delta.$$

That is, for $|a|>s$, we deduce that

$$\int_{G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z)>\delta.$$

Similar to the proof of (3) $\Rightarrow$ (4), there must be $\alpha,\beta>0$ such that

$$|G_\epsilon(n)\cap D(a,\alpha)|>\beta|D(a,\alpha)|,\ \forall|a|>s,\quad\forall n\in\mathbb{N}.$$

Therefore,

$$\int_{G_\epsilon(n)\cap D(a,\alpha)}K(1-|z|^2)dA(z)\gtrsim\beta\int_{D(a,\alpha)}K(1-|z|^2)dA(z),\ \forall|a|>s,\quad\forall n\in\mathbb{N}.$$

Now if $\{a_k\}$ is a α-lattice for $\mathbb{D}$, we have

$$\sum_{k=1}^{\infty}\int_{G_\epsilon(n)\cap D(a_k,\alpha)}K(1-|z|^2)dA(z)\gtrsim\beta\sum_{k=1}^{\infty}\int_{D(a_k,\alpha)}K(1-|z|^2)dA(z),\ \forall|a|>s,\quad\forall n\in\mathbb{N}.$$

Therefore,

$$\int_{G_\epsilon(n)}K(1-|z|^2)dA(z)\gtrsim 1\ \ \forall n\in\mathbb{N}.$$

For any $|a|\leq s$, we obtain $|f_a'(z)|\gtrsim(1-s^2)^2K^2(1-s^2)$. Hence,

$$\int_{G_\epsilon(n)}|f_a'(z)|^2K(1-|z|^2)dA(z)\gtrsim(1-s^2)^2K^2(1-s^2)\int_{G_\epsilon(n)}K(1-|z|^2)dA(z)\gtrsim 1.$$

Therefore, (3) hold.

(5) ⇒ (2). Let $a \in \mathbb{D}$. Then

$$\begin{aligned}
&\int_{\mathbb{D}} |f_a'(z)|^2 N_{\phi_{n_k},K}(z)dA(z) \\
&\geq \int_{D(a,r)} |f_a'(z)|^2 N_{\phi_{n_k},K}(z)dA(z) \\
&\gtrsim \int_{D(a,r)} \frac{N_{\phi_{n_k},K}(z)}{K(1-|z|^2)} dA(z) \gtrsim 1.
\end{aligned}$$

(4) ⇒ (6). Note that Luecking using a long proof to show that G satisfies the reverse Carleson condition if and only if the measure $\chi_G dA(z)$ is a reverse Carleson measure. Simlar to the proof of [13], we omited here.

(6) ⇒ (1). Let $f \in \mathcal{D}_K$. Then

$$\begin{aligned}
&\|C_{\phi_n} f\|^2_{\mathcal{D}_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 N_{\phi_n,K}(z)dA(z) \\
&\geq |f(0)|^2 + \int_{G_\epsilon(n)} |f_a'(z)|^2 N_{\phi_n,K}(z)dA(z) \\
&\geq |f(0)|^2 + \epsilon \int_{G_\epsilon(n)} |f_a'(z)|^2 K(1-|z|^2)dA(z) \\
&\geq |f(0)|^2 + \frac{\epsilon}{\delta} \int_{\mathbb{D}} |f_a'(z)|^2 K(1-|z|^2)dA(z) \gtrsim \|f\|^2_{\mathcal{D}_K}.
\end{aligned}$$

Thus, it is easily to get our result. The proof is completed.

□

4. Conclusions

In this paper, we give some equivalent characterizations of power bounded and power bounded below composition operator C_ϕ on Dirichlet Type spaces, which generalize the main results in [11, 12].

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Conflict of interest

We declare that we have no conflict of interest.

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