AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1991–2002.
DOI:10.3934/math.2021121
Received: 20 October 2020
Accepted: 01 December 2020
Published: 07 December 2020

Research article

On the nonexistence of some open immersions

Dandan Shi*

School of Mathematical Sciences, Capital Normal University, 105 Xisanhuanbeilu, Beijing 100048, People's Republic of China

* **Correspondence:** Email: shidan555@gmail.com.

Abstract: In this paper, we will prove a sufficient condition for that there does not exist an open immersion between two affine schemes of finite type over a field k with the same dimension. The proof is based on the following fact: the complement of an open affine subset in a noetherian integral separated scheme has pure codimension 1. We will first prove it when k is a finite field, the key to the proof of this part is Lang-Weil estimation. Then we'll prove a general result over an arbitrary field by reducing to the case when k is finite. And the proof of the general result is much more complicated. In order to use the result over a finite field, at some point we must produce a scheme that is defined over $\mathbf{F}_q$ and an open immersion, also defined over $\mathbf{F}_q$. One important lemma is that a morphism $f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ between two integral domains with the same quotient field K is an open immersion if and only if B is a birational extension of A in K and B is flat over A. According to the general result, the following open immersions do not exist: $SL_{n/k} \hookrightarrow \mathbf{A}_k^{n^2-1}$,$Sp_{n/k} \hookrightarrow \mathbf{A}_k^{2n^2+n}$,$SO_{n/k} \hookrightarrow \mathbf{A}_k^{\frac{n^2-n}{2}}$,$PGL_{n/k} \hookrightarrow \mathbf{A}_k^{n^2-1}$, where k is an arbitrary field.

Keywords: open immersion; flat; birational; finite type; the Zariski topology
Mathematics Subject Classification: 14A20

1. Introduction

The topology involved in this paper is the Zariski topology. The closed sets of affine space $\mathbf{A}_k^n$ are those of the form $V(S) = \{x \in \mathbf{A}_k^n \mid f(x) = 0, \forall f \in S\}$ where S is any set of polynomials in n variables over k. A variety is an integral separated scheme of finite type over an algebraically closed field.

It is well-known that $GL_{m/k}$ is an open subset of $\mathbf{A}_k^{m^2}$, where k is a field. A natural question is whether there exists an open immersion $\phi : SL_{m/k} \hookrightarrow \mathbf{A}_k^{m^2-1}$. If k is algebraically closed, it is easy to show that such an open immersion does not exist, by combining two classical results on algebraic groups. Indeed, if such an immersion exists, then the complement of its image is the zero locus of some polynomial function f. The restriction of this function to $SL_{m/k}$ is a regular invertible function on this connected algebraic group, and hence a scalar multiple of a character by a result of Rosenlicht(see [8]). Since every character of SL_m is trivial, f is constant, and hence $SL_{m/k}$ is an affine space. But this contradicts the fact that every algebraic group with underlying variety an affine space is unipotent(see [6]). In this paper we study the nonexistence of some open immersions between affine varieties over a field k, and we will prove a theorem which can explain the nonexistence of ϕ in another way. In addition, there is a result which is similar in spirit: any injective endomorphism of an affine variety is also surjective(see [7]). The main theorem is Theorem 2.15.

Since an open subset of an irreducible affine scheme is dense, we can reduce to the case where the open immersion $\phi : X \hookrightarrow Y$ is dominant.We first prove the theorem when k is a finite field and then we prove a more general result by reducing to this case. As a simple application, we have the following conclusions: open immersions $SL_{n/k} \hookrightarrow \mathbf{A}_k^{n^2-1}$,$Sp_{n/k} \hookrightarrow \mathbf{A}_k^{2n^2+n}$,$SO_{n/k} \hookrightarrow \mathbf{A}_k^{\frac{n^2-n}{2}}$,$PGL_{n/k} \hookrightarrow \mathbf{A}_k^{n^2-1}$ do not exist, and open immersions $SL_{n/k} \times_k \mathbf{G}_{a/k} \hookrightarrow \mathbf{A}_k^{n^2}$,$Sp_{n/k} \hookrightarrow SO_{2n+1/k}$,$Sp_{n/k} \hookrightarrow SL_{n/k} \times_k \mathbf{A}_k^{n^2+n+1}$ do not exist either, where k is an arbitrary field.

2. Results

2.1. Result over a finite field

To prove the main result of this subsection, we require the Lang-Weil estimate and some other lemmas.

Let X be an irreducible scheme defined over a finite field $\mathbf{F}_q$. Assume that X is embedded into a projective space of fixed dimension n, $\dim(X) = r$, $\deg(X) = d$, then we have universal estimates for $\#X(\mathbf{F}_{q^k})$, in terms of r, d, and q^k. More precisely, we show the following:

Lemma 2.1 ([Lang-Weil]). *Given nonnegative integers n, d and r, with $d > 0$, there is a positive constant $A(n, d, r)$, such that for every finite field $k = \mathbf{F}_q$, and every absolutely irreducible subscheme $X \subseteq \mathbf{P}_k^n$ of dimension r and degree d, we have*

$$|\#X(\mathbf{F}_{q^k}) - q^{kr}| \le (d-1)(d-2)q^{k(r-\frac{1}{2})} + A(n, d, r)q^{kr-k}. \tag{2.1}$$

Proof. See [11]. □

Lemma 2.2. *Given n, d and r as in Lemma 2.1, there is a positive constant $A_1(n, d, r)$, such that for every finite field $k = \mathbf{F}_q$, and every subscheme $X \subseteq \mathbf{P}_k^n$ of pure dimension r and degree $\le d$, we have*

$$\#X(\mathbf{F}_{q^l}) \le A_1(n, d, r)q^{lr}. \tag{2.2}$$

Proof. See Lemma 1 in [11]. □

If X is allowed to have components of smaller dimension, we still have

Corollary 2.3. *If X is an r-dimensional scheme over $k = \mathbf{F}_q$, then there is $c_X > 0$ such that $\#X(\mathbf{F}_{q^e}) \le c_X q^{er}$ for every $e \ge 1$.*

Proof. The proof is omitted. □

Corollary 2.4. *If X is an r-dimensional absolutely irreducible scheme over $\mathbf{F}_q$, then there is $c_X > 0$ such that $|\#X(\mathbf{F}_{q^e}) - q^{er}| \le c_X q^{e(r-\frac{1}{2})}$ for every $e \ge 1$.*

Proof. The proof is omitted. □

LEMMA 2.5. *Let A be an integrally closed noetherian domain. Then*

$$A = \bigcap_{ht\ \mathfrak{p}=1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1.

Proof. See Theorem 38 in [1]. □

LEMMA 2.6. *For any noetherian, integral, normal affine scheme X=Spec A,and any nonempty closed subscheme Z of X with codimension at least 2, $X-Z$ is not an affine scheme.*

Proof. If $X-Z$ is affine, then $i : X-Z \to X$ is a morphism of affine schemes, hence i is totally determined by $i^{\#} : \Gamma(X, \mathcal{O}_X) \to \Gamma(X-Z, \mathcal{O}_{X-Z}) = \Gamma(X-Z, \mathcal{O}_X)$. Since X is integral, $i^{\#}$ is injective.

Moreover, since Z does not contain any codimension 1 point of X, so for any $f \in \Gamma(X-Z, \mathcal{O}_X)$, f is regular at all codimension 1 points.Hence by Lemma 2.5, f is regular on X bexause A is integrally closed. As a result, $i^{\#}$ is surjective.

In sum, $i^{\#}$ is both injective and surjective, so it is actualy an isomorphism. Therefore i is an isomorphism, which means Z is empty and we get a contradiction. □

COROLLARY 2.7. *If X is a noetherian, integral, separated scheme X, U is an affine open subset, then the complement of U has codimension 1 in X.*

Proof. Replace X by its normalization, and U by its preimage in$X$. Then the codimension of the complement of U doesn't change, and so we reduce to the normal case. Intersecting U with the members of an open affine cover of X, we reduce to the case when X is affine.(X is separated, so the intersection of two open affines is open affine.)Then Lemma 2.6 applies. □

We can now prove the main result of this subsection.

THEOREM 2.8. *Assume X and Y are affine n-dimensional schemes over $\mathbf{F}_q$, $\#X(\mathbf{F}_{q^m}) \neq \#Y(\mathbf{F}_{q^m})$, if $\left|\#X(\mathbf{F}_{q^m}) - \#Y(\mathbf{F}_{q^m})\right| = o(q^{m(n-1)})$ $(m \to \infty \quad m \in \mathbb{Z})$, then there does not exist an open immersion $X \hookrightarrow Y$.*

$o(q^{m(n-1)})$ represents a polynomial of q, denoted by f, such that $\lim\limits_{m\to\infty} \frac{f(q^m)}{q^{m(n-1)}} = 0$.

Proof. Suppose that there exists an open immersion $X \hookrightarrow Y$. Let $Y = \text{Spec}R$, $V = Y \setminus X$ which is associated with an induced scheme structure R/I that is reduced, $I \subset R$ is the defining ideal of V.

We denote by $V_{\overline{\mathbf{F}_q}}$ the scheme $V \times_{\text{Spec}\mathbf{F}_q} \text{Spec}\overline{\mathbf{F}}_q$.

Let $X_1, X_2, \ldots, X_l$ be the irreducible components of $V_{\overline{\mathbf{F}_q}}$ of maximal dimension r, there is a finite extention $\mathbf{F}_{q^e}$ of $\mathbf{F}_q$ such that for some closed subscheme V_i of $V_{\mathbf{F}_{q^e}}$, we have $V_i \times_{\text{Spec}\mathbf{F}_{q^e}} \text{Spec}\overline{\mathbf{F}_q} = X_i$, $1 \leq i \leq l$. This implies that each V_i is absolutely irreducible. Note that the dimension of any other irreducible component of $V_{\mathbf{F}_{q^e}}$ is smaller than r, and $\dim(V_i \bigcap V_j) < r$ when $i \neq j$.

Combining Corollary 2.3 with Corollary 2.4, we have

$$|\#V(\mathbf{F}_{q^{ek}}) - lq^{ekr}| \leq \alpha_X q^{ek(r-\frac{1}{2})},$$

That is

$$lq^{ekr} - \alpha_X q^{ek(r-\frac{1}{2})} \leq \#V(\mathbf{F}_{q^{ek}}) \leq lq^{ekr} + \alpha_X q^{ek(r-\frac{1}{2})}, \tag{2.3}$$

Then we have

$$\#V(\mathbf{F}_{q^{ek}}) \geq lq^{ekr} - \alpha_X q^{ek(r-\frac{1}{2})}. \tag{2.4}$$

According to Corollary 2.7, $r = n - 1$, we obtain

$$\#V(\mathbf{F}_{q^{ek}}) \geq lq^{(n-1)ek} - \alpha_X q^{(n-\frac{3}{2})ek}, \tag{2.5}$$

On the other hand,

$$\left|\#V(\mathbf{F}_{q^{ek}})\right| = \left|\#X(\mathbf{F}_{q^{ek}}) - \#Y(\mathbf{F}_{q^{ek}})\right| = o(q^{ek(n-1)}) \quad (k \to \infty \quad k \in \mathbb{Z}) \tag{2.6}$$

this implies

$$lq^{(n-1)ek} - \alpha_X q^{(n-\frac{3}{2})ek} \leq o(q^{ek(n-1)}), \quad (k \to \infty \quad k \in \mathbb{Z})$$

This contradiction shows that there does not exist an open immersion $X \hookrightarrow Y$.

□

2.2. *Result over an arbitrary field*

We need the following four lemmas to prove the main theorem.

Lemma 2.9. *Let $A \subseteq B$ be two integral domains. If B is faithfullly flat over A and* qf*(A)=*qf*(B). Then $A = B$.*

qf(A) means the quotient field of A, and qf(B) is similar.

Proof. Take $x \in B$ with $x = b/a$ $(a, b \in A)$. B is faithfully flat over A, so it follows that $b = ax \in aB \cap A = aA$ (cf.[[1], (4.C)]). Hence $x = b/a \in A$. Therefore $A = B$. □

Lemma 2.10. *Let A be an integral domain with quotient field K and let B be an extension of A. B is a finitely generated A-algebra. Then the canonical morphism $f : Spec(B) \longrightarrow Spec(A)$ is an open immersion $\Longleftrightarrow$ B is a birational extension of A in K and B is flat over A.*

Proof. $\Rightarrow$ B is flat over A if and only if B_P is flat over A_p for every $P \in B$ $(p = P \cap A)$. Since f is an open immersion, it follows that $B_P \cong A_p$, and B_P is flat A_p. Furthermore f is birational, this implies that qf$(B) = K$.

$\Leftarrow$ Any flat morphism that is locally of finite type is open. Next we will show that this map is injective. Let $P, P' \in$ Spec(B) with $P \cap A = P' \cap A := p$, then $A_p \to B_P$ is flat. As a flat extension of rings satisfies Going-Down Theorem(cf.[[1], (5.D)]), Spec$(B_P) \to$ Spec(A_p) is surjective, it follows that B_P is faithfully flat over A_P(cf.[[1], (4.D)]). Hence $A_p = B_P$ by Lemma 2.9. Similary, $A_p = B_{P'} = B_P$. Hence $P = P'$. So Spec(B) is homeomorphic to an open subset U of Spec(A). Since open sets of the form $D(g)$ form a base for the topology of Spec(A), take $B_g \subset U$, it is enough to prove $A_g \cong B_g$. We have a surjective homomorphism $B_g \otimes_{A_g} B_g \overset{\phi}{\twoheadrightarrow} B_g$, since $B_g \otimes_{A_g} B_g \subset B_g \otimes_{A_g} K = K$, ϕ is also injective, namely ϕ is an isomorphism. As B_g is faithfully flat over A_g, we conclude that $A_g \cong B_g$. □

LEMMA 2.11. *Let $A \subset B \subset C$ be rings. Suppose that A is Noetherian, that C is finitely generated as an A-algebra and that C is either (1) finitely generated as a B-module or (2) integral over B. Then B is finitely generated as an A-algebra.*

Proof. See Proposition 7.8 in [5]. □

LEMMA 2.12. *Let k be a finitely generated $\mathbb{Z}$-algebra. If k is a field, then k is finite.*

Proof. We have a homomorphism $\mathbb{Z} \xrightarrow{f} k$, if ker($f$)= $p\mathbb{Z}$(p is a prime), then k is a finitely generated $\mathbf{F}_p$-algebra, so k is a finite algebraic extension of $\mathbf{F}_p$(cf.[[5], (7.9)]), and k is a finite field.

If ker(f)=(0), we have $\mathbb{Z} \subset \mathbb{Q} \subset k$. Since k is a finitely generated $\mathbb{Z}$-algebra, it is a finitely generated $\mathbb{Q}$-algebra, similarly, k is a finite algebraic extension of $\mathbb{Q}$, hence k is a finitely generated $\mathbb{Q}$-module, by Lemma 2.11, $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-algebra, a contradiction. Let $\mathbb{Q} = \mathbb{Z}[c_1, \ldots, c_s]$, $c_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathbb{Z}, 1 \leq i \leq s$, take p such that p and b_i are coprime, then $\frac{1}{p} \notin \mathbb{Z}[c_1, \ldots, c_s]$, this contradiction shows that k is a finite field. □

Suppose S_0 is a scheme, and $\mathscr{A}_\lambda$ are commutative quasi-coherent O_{S_0}-algebras, then $\mathscr{A} = \varinjlim \mathscr{A}_\lambda$ is a quasi-coherent O_{S_0}-algebra. Denote by S_λ(resp.S) the spectrum of the O_{S_0}-algebra $\mathscr{A}_\lambda$(resp.$\mathscr{A}$), and let $u_{\lambda\mu} : S_\mu \to S_\lambda$(for $\lambda \leq \mu$) and $u_\lambda : S \to S_\lambda$ be respectively the S_0-morphisms corresponding to homomorphisms $\varphi_{\mu\lambda} : \Gamma(S_0, \mathscr{A}_\lambda) \to \Gamma(S_0, \mathscr{A}_\mu)$ and $\varphi_\lambda : \Gamma(S_0, \mathscr{A}_\lambda) \to \Gamma(S_0, \mathscr{A})$;it is clear that $(S_\lambda, u_{\lambda\mu})$ is a projective system in the category of S_0-schemes.

Given two S_α-schemes X_α, Y_α, we define two projective systems of $(X_\lambda, v_{\lambda\mu})$ and $(Y_\lambda, w_{\lambda\mu})$ by setting $X_\lambda = X_\alpha \times_{S_\alpha} S_\lambda, Y_\lambda = Y_\alpha \times_{S_\alpha} S_\lambda, v_{\lambda\mu} = \mathrm{id}_{X_\alpha} \times u_{\lambda\mu}, w_{\lambda\mu} = \mathrm{id}_{Y_\alpha} \times u_{\lambda\mu}$(for $\alpha \leq \lambda \leq \mu$), whose projective limits are respectively $X = X_\alpha \times_{S_\alpha} S, Y = Y_\alpha \times_{S_\alpha} S$, the canonical morphisms $v_\lambda : X \to X_\lambda$ and $w_\lambda : Y \to Y_\lambda$ are respectively equal to $\mathrm{id}_{X_\alpha} \times u_\lambda$ and $\mathrm{id}_{Y_\alpha} \times u_\lambda$. We denote by f_λ the morphism $X_\lambda \to Y_\lambda$, and for $\alpha \leq \lambda \leq \mu$, we have $f_\mu = f_\lambda \times \mathrm{id}_{S_\mu} : X_\lambda \times_{S_\lambda} S_\mu \to Y_\lambda \times S_\mu, f = f_\lambda \times \mathrm{id}_S : X_\lambda \times_{S_\lambda} S \to Y_\lambda \times_{S_\lambda} S$.

LEMMA 2.13. *Suppose S_0 is quasi-compact, X_α and Y_α are of finite presentation over S_α, let $f_\alpha : X_\alpha \to Y_\alpha$ be an $S_\alpha -$ morphism.Then f is an open immersion if and only if there exists $\lambda \geq \alpha$ such that f_λ is an open immersion(in which case f_μ is also an open immersion for $\mu \geq \lambda$).*

Proof. See Theorem 8.10.5 in [13]. □

LEMMA 2.14. *Let X and Y be affine n-dimensional integral schemes of finite type over a field k. If we have an open immersion $X \overset{f}{\hookrightarrow} Y/k$, then there exists a finitely generated $\mathbb{Z}$-subalgebra R of k, affine scheme $\mathscr{X}, \mathscr{Y}$ of finite type over $S = SpecR$, and an open immersion $\mathscr{X} \overset{f_S}{\hookrightarrow} \mathscr{Y}/S$, such that $\mathscr{X} \times_S Speck \cong X$, $\mathscr{Y} \times_S Speck \cong Y$, and $f_S \times_S \mathrm{id}_k = f$.*

Proof. In order to use Lemma 2.13, we should find a field k',two affine schemes X', Y' that are defined over k',an open immersion $f' : X' \to Y'$,also defined over k', such that k' is the quotient field of a finitely generated $\mathbb{Z}$-subalgebra $R' \subset k$, and $X = X' \times_{\mathrm{Spec}k'} \mathrm{Spec}k, Y = Y' \times_{\mathrm{Spec}k'} \mathrm{Spec}k, f = f' \times \mathrm{id}_{\mathrm{Spec}k} : X' \times_{\mathrm{Spec}k'} \mathrm{Spec}k \to Y' \times_{\mathrm{Spec}k'} \mathrm{Spec}k$, because $k' = \varinjlim_{s \in R', s \neq 0} R'_s$.

Write $[\underline{x}] = [x_1, \ldots, x_l]$, $[\underline{y}] = [y_1, \ldots, y_m]$, $X = \mathrm{Spec}k[\underline{x}]/I$, $Y = \mathrm{Spec}k[\underline{y}]/J$, where $I = (\mathtt{f}_1, \ldots, \mathtt{f}_s)k[\underline{x}]$ and $J = (\mathtt{g}_1, \ldots, \mathtt{g}_t)k[\underline{y}]$ are respectively the defining ideals of X and Y.

If we have an open immersion $X \overset{f}{\hookrightarrow} Y/k$, by Lemma 2.10, we have a flat and birational k-morphism $f^{\#} : k[\underline{y}]/J \to k[\underline{x}]/I$. So we have an isomorphism $\mathrm{qf}\{k[\underline{x}]/I\} \overset{g}{\to} \mathrm{qf}\{k[\underline{y}]/J\}$.

Let g_0 be the restriction to $k[\underline{x}]/I$ of g.

Set $\varphi = g_0 \circ \pi$, $\pi : k[\underline{y}] \twoheadrightarrow k[\underline{x}]/I$.

We have a homomorphism

$$\begin{aligned} \varphi : k[\underline{x}] &\longrightarrow \mathrm{qf}\{k[\underline{y}]/J\} \\ x_i &\longmapsto \frac{\bar{f}_i}{\bar{g}_i} \end{aligned} \tag{2.7}$$

$\bar{f}_i, \bar{g}_i \in k[\underline{y}]/J$, $\bar{f}_i, \bar{g}_i$ have no non-unit common factor, $1 \le i \le l$.

Set R_0 = {the subring of k generated by identity 1, all the coefficients of $\mathtt{f}_1, \dots, \mathtt{f}_s$, $\mathtt{g}_1, \dots, \mathtt{g}_t$}.

Set R_1 = {the subring of k generated by S_0, all the coefficients of $\bar{f}_i$,$\bar{g}_i$ and their inverses}.

It is noted that R_1 is a finitely generated $\mathbb{Z}$-algebra.

Consider the restriction to $R_1[\underline{x}]$ of φ

$$\begin{array}{ccccccc} & & (\mathtt{f}_1, \dots, \mathtt{f}_s)R_1[\underline{x}] & & & & \\ & & \bigcap & & & & \\ 0 & \longrightarrow & \ker\varphi_1 & \longrightarrow & R_1[\underline{x}] & \overset{\varphi_1}{\longrightarrow} & \mathrm{qf}\{k[\underline{y}]/J\} \\ & & \bigcap & & \bigcap & & \\ 0 & \longrightarrow & (\mathtt{f}_1, \dots, \mathtt{f}_s)k[\underline{x}] & \longrightarrow & k[\underline{x}] & \overset{\varphi}{\longrightarrow} & \mathrm{qf}\{k[\underline{y}]/J\} \end{array}$$

$R_1[\underline{x}]$ is noetherian, so $\ker\varphi_1$ is finitely generated.

Let $b_1, b_2, \dots, b_\ell$ be the generators of $\ker\varphi_1$

$$b_1 = P_{11}\mathtt{f}_1 + \cdots + P_{1s}\mathtt{f}_s, \dots, b_\ell = P_{\ell 1}\mathtt{f}_1 + \cdots + P_{\ell s}\mathtt{f}_s \tag{2.8}$$

Set R_2 = {the subring of k generated by R_1, all the coefficients of P_{ij} and their inverses. $1 \le i \le \ell$, $1 \le j \le s$}

R_2 is still a finitely generated $\mathbb{Z}$-algebra.

Consider the restriction to $R_2[\underline{x}]$ of φ

$$\begin{array}{ccccccc} 0 & \longrightarrow & (b_1, \dots, b_\ell)R_1[\underline{x}] & \longrightarrow & R_1[\underline{x}] & \overset{\varphi_1}{\longrightarrow} & \mathrm{qf}\{k[\underline{y}]/J\} \\ & & \bigcap & & \bigcap & & \bigcap \\ 0 & \longrightarrow & \ker\varphi_2 & \longrightarrow & R_2[\underline{x}] & \overset{\varphi_2}{\longrightarrow} & \mathrm{qf}\{k[\underline{y}]/J\} \\ & & \bigcap & & \bigcap & & \bigcap \\ 0 & \longrightarrow & (\mathtt{f}_1, \dots, \mathtt{f}_s)k[\underline{x}] & \longrightarrow & k[\underline{x}] & \overset{\varphi}{\longrightarrow} & \mathrm{qf}\{k[\underline{y}]/J\} \end{array}$$

Clearly we have $\ker\varphi_2 \supset (\mathtt{f}_1, \dots, \mathtt{f}_s)R_2[\underline{x}]$.

We denote by $\mathrm{IM}(\varphi_1)$ the image of φ_1 in $\mathrm{qf}\{k[\underline{y}]/J\}$, it is a subring of $\mathrm{qf}\{k[\underline{y}]/J\}$ generated by R_1 and $\varphi_1(x_i)$, so we have an exact sequence

$$
\begin{array}{l}
0 \longrightarrow (b_1,\ldots,b_\ell)R_1[\underline{x}] \longrightarrow \quad R_1[\underline{x}] \quad \longrightarrow \quad \mathrm{IM}(\varphi_1) \quad \longrightarrow 0 \\
(b_1,\ldots,b_\ell)R_1[\underline{x}]\otimes_{R_1} R_2 \longrightarrow R_1[\underline{x}]\otimes_{R_1} R_2 \longrightarrow \mathrm{IM}(\varphi_1)\otimes_{R_1} R_2 \longrightarrow 0
\end{array}
$$

$\mathrm{IM}(\varphi_1)\otimes_{R_1} R_2$ is the subring of $\mathrm{qf}\{k[\underline{y}]/J\}$ generated by R_2 and $\varphi_1(x_i)$, so $\ker\varphi_2$ is the image of $(b_1,\ldots,b_\ell)R_1[\underline{x}]\otimes_{R_1} R_2$ in $R_1[\underline{x}]\otimes_{R_1} R_2 \cong R_2[\underline{x}]$, namely $(b_1,\ldots,b_\ell)R_2[\underline{x}]$.

By (2.8) we have

$(\mathtt{f}_1,\ldots,\mathtt{f}_s)R_2[\underline{x}] \supset (b_1,\ldots,b_\ell)R_2[\underline{x}] = \ker\varphi_2$.

Hence $\ker\varphi_2 = (\mathtt{f}_1,\ldots,\mathtt{f}_s)R_2[\underline{x}]$.

On the other hand, we have a homomorphism of rings:

$f^\# : k[\underline{y}]/J \to k[\underline{x}]/I$

Set $\psi = f^\# \circ \rho$, $\rho : k[\underline{y}] \twoheadrightarrow k[\underline{y}]/J$.

We have a homomorphism

$$
\begin{array}{rl}
\psi : k[\underline{y}] & \longrightarrow \mathrm{qf}\{k[\underline{x}]/I\} \\
y_i & \longmapsto \dfrac{\bar{f}_i'}{\bar{g}_i'}
\end{array}
\tag{2.9}
$$

$\bar{f}_i', \bar{g}_i' \in k[\underline{x}]/I$, $\bar{f}_i', \bar{g}_i'$ have no non-unit common factor, $1 \le i \le m$.

Set $R_3 = \{$the subring of k generated by R_2 and all the coefficients of $\bar{f}_i', \bar{g}_i'$ and their inverses$\}$.

Consider the restriction to $R_3[\underline{y}]$ of ψ

$$
\begin{array}{ccccccc}
 & & (\mathtt{g}_1,\ldots,\mathtt{g}_t)R_3[\underline{y}] & & & & \\
 & & \bigcap & & & & \\
0 & \longrightarrow & \ker\psi_1 & \longrightarrow & R_3[\underline{y}] & \xrightarrow{\psi_1} & \mathrm{qf}\{k[\underline{x}]/I\} \\
 & & \bigcap & & & & \\
0 & \longrightarrow & (\mathtt{g}_1,\ldots,\mathtt{g}_t)k[\underline{y}] & \longrightarrow & k[\underline{y}] & \xrightarrow{\psi} & \mathrm{qf}\{k[\underline{x}]/I\}
\end{array}
$$

Let $c_1, c_2, \ldots, c_{\ell'}$ be the generators of $\ker\psi_1$

$$
c_1 = Q_{11}\mathtt{g}_1 + \cdots + Q_{1t}\mathtt{g}_t, \ldots, c_{\ell'} = Q_{\ell' 1}\mathtt{g}_1 + \cdots + Q_{\ell' t}\mathtt{g}_t \tag{2.10}
$$

$Q_{ij} \in k[\underline{y}]$,$1 \le i \le \ell'$,$1 \le j \le t$.

Set $R_4 = \{$the subring of k generated by R_3 and all the coefficients of Q_{ij} and their inverses. $1 \le i \le \ell'$, $1 \le j \le t\}$

Let ψ_2 be the restriction to $R_4[\underline{y}]$ of ψ. Similarly, R_4 is a finitely generated $\mathbb{Z}$-algebra and $\ker\psi_2 = (\mathtt{g}_1,\ldots,\mathtt{g}_t)R_4[\underline{y}]$.

The restriction to $R_4[\underline{y}]/J$ of $f^\#$

$$
f^\#_{R_4} : R_4[\underline{y}]/J \longrightarrow R_4[\underline{x}]/I.
$$

is birational according to the construction of R_4.

Write $k_4 = \mathrm{qf}(R_4)$. Obviously $k_4 \subset k$, the restriction to $k_4[\underline{y}]/J$ of $f^\#$

$$f^\#_{k_4} : k_4[\underline{y}]/J \longrightarrow k_4[\underline{x}]/I.$$

is still birational.

We have a commutative diagram

$$\begin{array}{ccc} k_4[\underline{y}]/J & \xrightarrow{f^\#_{k_4}} & k_4[\underline{x}]/I \\ \downarrow{\scriptstyle i_1} & \searrow & \downarrow{\scriptstyle i_2} \\ k[\underline{y}]/J & \xrightarrow{f^\#} & k[\underline{x}]/I \end{array}$$

i_1 is flat and $f^\#$ is flat, so $f \circ i_1 = i_2 \circ f^\#_{k_4}$ is flat. As i_2 is faithfully flat, $f^\#_{k_4}$ is flat.

According Lemma 2.10, f_{k_4} is an open immersion.

Let $k' = k_4$, $R' = R_4$, $X' = \mathrm{Spec}R'[\underline{x}]/I$, $Y' = \mathrm{Spec}R'[\underline{y}]/J$, $f' = f_{k_4}$, then Lemma 2.13 applies. □

Theorem 2.15. *Assume R is an integral domain which is a finitely generated $\mathbb{Z}$-algebra, $S = \mathrm{Spec}R$, $\xi \in S$ is the generic point. X and Y are affine integral S-schemes of finite type, such that X_ξ and Y_ξ are n-dimensional affine schemes. If for a sufficiently general closed point $t \in S$, $\#X_t(\kappa(t)^m) \neq \#Y_t(\kappa(t)^m)$, and $|\#X_t(\kappa(t)^m) - \#Y_t(\kappa(t)^m)| = o(|\kappa(t)|^{m(n-1)})$ $(m \to \infty \quad m \in \mathbb{Z})$, then there does not exist an open immersion $X_\xi \hookrightarrow Y_\xi$("sufficiently general" is made precise in the proof).*

Proof. $\kappa(t))$ is the residue field of the point t, because t is a closed point of S, $\kappa(t))$ is a finitely generated $\mathbb{Z}$-algebra. According to Lemma 2.12, $\kappa(t))$ is a finite field. Set $\kappa(t) = \mathbf{F}_q$, $\kappa(t)^m = \mathbf{F}_{q^m}$. So $\#X_t(\kappa(t)^m) = \#(X \times_S \kappa(t)^m)$ makes sense, so does $\#Y_t(\kappa(t)^m)$.

Write $[\underline{x}] = [x_1, \ldots, x_l]$, $[\underline{y}] = [y_1, \ldots, y_m]$, $X = \mathrm{Spec}R[\underline{x}]/I$, $Y = \mathrm{Spec}R[\underline{y}]/J$, where $I = (\mathtt{f}_1, \ldots, \mathtt{f}_s)R[\underline{x}]$ and $J = (\mathtt{g}_1, \ldots, \mathtt{g}_t)R[\underline{y}]$ are respectively the defining ideals of X and Y.

Suppose there is an open immersion $X_\xi \overset{f_\xi}{\hookrightarrow} Y_\xi/\kappa(\xi)$, by Lemma 2.14, there exists a finitely generated $\mathbb{Z}$-subalgebra R' of $\kappa(\xi)$, an open immersion

$$\mathrm{Spec}R'[\underline{x}]/I \overset{f_{R'}}{\hookrightarrow} \mathrm{Spec}R'[\underline{y}]/J,$$

such that $\mathrm{Spec}R'[\underline{x}]/I \times_{\mathrm{Spec}R'} \mathrm{Spec}\kappa(\xi) \cong X_\xi$, $\mathrm{Spec}R'[\underline{y}]/J \times_{\mathrm{Spec}R'} \mathrm{Spec}\kappa(\xi) \cong Y_\xi$, and $f_{R'} \times_{\mathrm{Spec}R'} \mathrm{id}_{\kappa(\xi)} = f_\xi$.

Set $R'' = \{$the subring of $\kappa(\xi)$ generated by R and $R'\}$, then $\mathrm{Spec}R''[\underline{x}]/I \times_{\mathrm{Spec}R'} \mathrm{Spec}\kappa(\xi) \cong X_\xi$, $\mathrm{Spec}R''[\underline{y}]/J \times_{\mathrm{Spec}R'} \mathrm{Spec}\kappa(\xi) \cong Y_\xi$, $\mathrm{Spec}R''[\underline{x}]/I \overset{f_{R''}}{\hookrightarrow} \mathrm{Spec}R'[\underline{y}]/J$ is an open immersion, and $f_{R''} \times_{\mathrm{Spec}R''} \mathrm{id}_{\kappa(\xi)} = f_\xi$.

$\mathrm{Spec}R'' \subset \mathrm{Spec}R$ is a dense open subset, for a sufficiently general closed point $t \in \mathrm{Spec}R$, we have $t \in \mathrm{Spec}R''$.

Let $P \subset R''$ be the maximal ideal corresponding to $t \in S$, we have

$$\bar{f}^\#_{R''} : R''/P \otimes_{R''} R''[\underline{y}]/J \longrightarrow R''/P \otimes_{R''} R''[\underline{x}]/I.$$

The property of being an open immersion is stable under base change, so $\bar{f}_{R''}$ is an open immersion.

$R^{''}/P$ is a finitely generated $\mathbb{Z}$-algebra which is a field, according to 2.12, it is a finite field, set $R^{''}/P \cong \mathbf{F}_q$.

Namely there exists an open immersion $X_t/\kappa(t) \hookrightarrow Y_t/\kappa(t)$.

$|\#X_t(\kappa(t)^m) - \#Y_t(\kappa(t)^m)| = o(|\kappa(t)|^{m(n-1)})(m \to \infty \quad m \in \mathbb{Z})$ implies

$$\left|\#X_t/\kappa(t)(\mathbf{F}_{q^m}) - \#Y_t/\kappa(t)(\mathbf{F}_{q^m})\right| = o(q^{m(n-1)})(m \to \infty \quad m \in \mathbb{Z}),$$

by Theorem 2.8, this can not happen. This contradiction shows that there does not exist an open immersion $X_\xi \hookrightarrow Y_\xi$. □

Remark 2.16. If it happens that $\#X_t/\kappa(t) > \#Y_t/\kappa(t)$, there does not exist an open immersion $X \hookrightarrow Y$.

Remark 2.17. Denote $|\#X_t(\kappa(t)^m) - \#Y_t(\kappa(t)^m)|$ by f_m, if $\deg f_m \neq m(\dim X - 1)(\forall m \in \mathbb{Z})$, there does not exist an open immersion $X \hookrightarrow Y$.

Remark 2.18. Given X and Y as in Theorem 2.15, if Y is quasi-projective, there does not exist an open immersion $X \hookrightarrow Y$. Regard X as an affine open subscheme of Y, for any affine open subscheme $U \subset Y$, $U \cap X$ is still affine because Y is separated over S. Then we come to the conclusion.

Corollary 2.19. *Let X and Y be affine n-dimensional integral schemes of finite type over a field k. If there exists a finitely generated $\mathbb{Z}$-subalgebra R of k and affine schemes $\mathscr{X}, \mathscr{Y}$ of finite type over $S = SpecR$, such that $\mathscr{X} \times_S Speck \cong X$,$\mathscr{Y} \times_S Speck \cong Y$,and for a sufficiently general closed point $t \in S$, $\#\mathscr{X}_t(\kappa(t)^m) \neq \#\mathscr{Y}_t(\kappa(t)^m)$ and $|\#\mathscr{X}_t(\kappa(t)^m) - \#\mathscr{Y}_t(\kappa(t)^m)| = o(|\kappa(t)|^{m(n-1)}) \quad (m \to \infty \quad m \in \mathbb{Z})$, then there does not exist an open immersion $X \overset{f}{\hookrightarrow} Y/k$.*

Proof. Write $[\underline{x}] = [x_1, \ldots, x_l]$, $[\underline{y}] = [y_1, \ldots, y_m]$, $X = \mathrm{Spec}k[\underline{x}]/I$, $Y = \mathrm{Spec}k[\underline{y}]/J$, where $I = (\mathtt{f}_1, \ldots, \mathtt{f}_s)k[\underline{x}]$ and $J = (\mathtt{g}_1, \ldots, \mathtt{g}_t)k[\underline{y}]$ are respectively the defining ideals of X and Y.

Assume there is an open immersion $X \overset{f}{\hookrightarrow} Y/k$, by Lemma 2.14, there exists a finitely generated $\mathbb{Z}$-subalgebra $R^{'}$ of k, an open immersion

$$\mathrm{Spec}R^{'}[\underline{x}]/I \overset{f_{R'}}{\hookrightarrow} \mathrm{Spec}R^{'}[\underline{y}]/J,$$

such that $\mathrm{Spec}R^{'}[\underline{x}]/I \times_{\mathrm{Spec}R'} \mathrm{Spec}k \cong X$, $\mathrm{Spec}R^{'}[\underline{y}]/J \times_{\mathrm{Spec}R'} \mathrm{Spec}k \cong Y$, and $f_{R'} \times_{\mathrm{Spec}R'} \mathrm{id}_k = f$.

Set $R^{''} = \{$the subring of k generated by R and $R^{'}\}$, we have $\mathrm{Spec}R^{''}[\underline{x}]/I \times_{\mathrm{Spec}R'} \mathrm{Spec}k \cong X$, $\mathrm{Spec}R^{''}[\underline{y}]/J \times_{\mathrm{Spec}R'} \mathrm{Spec}k \cong Y$, $\mathrm{Spec}R^{''}[\underline{x}]/I \overset{f_{R''}}{\hookrightarrow} \mathrm{Spec}R^{'}[\underline{y}]/J$ is an open immersion, and $f_{R''} \times_{\mathrm{Spec}R''} \mathrm{id}_{\kappa(\xi)} = f_\xi$.

It is noted that $\mathrm{Spec}R^{''} \subset \mathrm{Spec}R$, then we come to the conclusion according to Theorem 2.15. □

3. Examples

Example 3.1. Since

$$\begin{aligned} \#SL_{n/\mathbf{F}_q}(\mathbf{F}_{q^m}) &= (q^{mn}-1)(q^{mn}-q^m)\cdots(q^{mn}-q^{mn-2m})q^{mn-m} \\ \#\mathbf{A}^{n^2-1}_{\mathbf{F}_q}(\mathbf{F}_{q^m}) &= q^{mn^2-m} \end{aligned}$$

$\left|\#SL_{n/\mathbf{F}_q}(\mathbf{F}_{q^m}) - \#\mathbf{A}^{n^2-1}_{\mathbf{F}_q}(\mathbf{F}_{q^m})\right|$
$= q^{mn^2-m} - (q^{mn}-1)\cdots(q^{mn}-q^{mn-2m})q^{mn-m}$
$= q^{m(n^2-3)}$ + lower-degree terms + ...

We have $\dim SL_{n/\mathbf{F}_q} = \dim \mathbf{A}^{n^2-1}_{\mathbf{F}_q} = n^2-1$, so for an arbitrary field k, there does not exist an open immersion $SL_{n/k} \hookrightarrow \mathbf{A}^{n^2-1}_k$.

EXAMPLE 3.2. Since

$$\begin{aligned}\#Sp_{n/\mathbf{F}_q}(\mathbf{F}_{q^m}) &= (q^{2mn}-1)(q^{2mn-2m}-1)\cdots(q^{2m}-1)q^{mn^2}\\ &= q^{2mn^2+mn} - q^{2mn^2+mn-2m} + \text{lower-degree terms} + \ldots\\ \#\mathbf{A}^{2n^2+n}_{\mathbf{F}_q}(\mathbf{F}_{q^m}) &= q^{2mn^2+mn}\end{aligned}$$

$\left|\#Sp_{n/\mathbf{F}_q}(\mathbf{F}_{q^m}) - \#\mathbf{A}^{2n^2+n}_{\mathbf{F}_q}(\mathbf{F}_{q^m})\right|$
$= q^{m(2n^2+n-2)}$ + lower-degree terms + ...

We have $\dim Sp_{n/\mathbf{F}_q} = \mathbf{A}^{2n^2+n}_{\mathbf{F}_q} = 2n^2+n$, so for an arbitrary field k, there does not exist an open immersion $Sp_{n/k} \hookrightarrow \mathbf{A}^{2n^2+n}_k$.

EXAMPLE 3.3. Since

$$\#SO_{2t+1/\mathbf{F}_q}(\mathbf{F}_{q^m}) = q^{mt}\prod_{i=0}^{t-1}(q^{2mt}-q^{2mi}) \quad p=2 \tag{3.1}$$

$$\#SO_{2t/\mathbf{F}_q}(\mathbf{F}_{q^m}) = q^{mt}\prod_{i=1}^{t-1}(q^{2mt}-q^{2mi}) \quad p=2 \tag{3.2}$$

$$\#SO_{2t+1/\mathbf{F}_q}(\mathbf{F}_{q^m}) = q^{mt}\prod_{i=0}^{t-1}(q^{2mt}-q^{2mi}) \quad p>2 \tag{3.3}$$

$$\#SO_{2t/\mathbf{F}_q}(\mathbf{F}_{q^m}) = (q^{mt}-1)\prod_{i=1}^{t-1}(q^{2mt}-q^{2mi}) \tag{3.4}$$

($p > 2$ and -1 is a square in $\mathbf{F}_q$)

$$\#SO_{2t/\mathbf{F}_q}(\mathbf{F}_{q^m}) = (q^{mt}+(-1)^{t+1})\prod_{i=1}^{t-1}(q^{2mt}-q^{2mi}) \tag{3.5}$$

($p > 2$ and -1 is a nonsquare in $\mathbf{F}_q$)

We have $\dim SO_{n/\mathbf{F}_q} = \frac{n^2-n}{2}$, for $n > 2$, so for an arbitrary field k, there does not exist an open immersion $SO_{n/k} \hookrightarrow \mathbf{A}^{\frac{n^2-n}{2}}_k$.

EXAMPLE 3.4. Since

$$\begin{aligned}\#PGL_{n/\mathbf{F}_q}(\mathbf{F}_{q^m}) &= (q^{mn}-1)(q^{mn}-q^m)\cdots(q^{mn}-q^{mn-2m})q^{mn-m}\\ &= q^{m(n^2-1)} - q^{m(n^2-3)} + \text{lower-degree terms} + \ldots\end{aligned}$$

We have $\dim PGL_{n/\mathbf{F}_q} = n^2 - 1$, so there does not exist an open immersion $PGL_{n/k} \hookrightarrow \mathbf{A}_k^{n^2-1}$ for an arbitrary field k.

Example 3.5. Since

$$\begin{aligned} \#SL_{n/\mathbf{F}_q} \times \mathbf{G}_{a/\mathbf{F}_q}(\mathbf{F}_{q^m}) &= (q^{mn}-1)(q^{mn}-q^m)\cdots(q^{mn}-q^{mn-2m})q^{mn} \\ \#\mathbf{A}_{\mathbf{F}_q}^{n^2}(\mathbf{F}_{q^m}) &= q^{mn^2} \end{aligned}$$

$$\begin{aligned} &\left|\#SL_{n/\mathbf{F}_q} \times \mathbf{G}_{a/\mathbf{F}_q}(\mathbf{F}_{q^m}) - \#\mathbf{A}_{\mathbf{F}_q}^{n^2-1}(\mathbf{F}_{q^m})\right| \\ &= q^{mn^2-m} - (q^{mn}-1)\cdots(q^{mn}-q^{mn-2m})q^{mn} \\ &= q^{m(n^2-2)} + \text{lower-degree terms} + \ldots \end{aligned}$$

We have $\dim SL_{n/\mathbf{F}_q} \times \mathbf{G}_{a/\mathbf{F}_q} = \dim \mathbf{A}_{\mathbf{F}_q}^{n^2} = n^2$, so for an arbitrary field k, there does not exist an open immersion $SL_{n/k} \times_k \mathbf{G}_{a/k} \hookrightarrow \mathbf{A}_k^{n^2}$, and Example 1 is a corollary of this.

Example 3.6. We have $\dim Sp_{n/\mathbf{F}_q} = \dim SO_{2n+1/\mathbf{F}_q} = 2n^2 + n$, $\#SO_{2n+1/\mathbf{F}_q} > \#Sp_{n/\mathbf{F}_q}$ for $n > 2$, so when $n > 2$, there does not exist an open immersion $Sp_{n/k} \hookrightarrow SO_{2n+1/k}$ for an arbitrary field k.

Example 3.7. There does not exist an open immersion $Sp_{n/k} \hookrightarrow SL_{n/k} \times_k \mathbf{A}_k^{n^2+n+1}$ for any field k.

Acknowledgments

The author thanks Professor KeZheng Li for his guide and useful discussions with him. The author thanks Professor Fei Xu and Professor Lei Fu.

Conflict of interest

The author declares that there are no conflicts of interest.

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