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Research article

The recurrence formula for the number of solutions of a equation in finite field

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Abstract: The main purpose of this paper is using analytic methods to give a recurrence formula of the number of solutions of an equation over finite field. We use analytic methods to give a recurrence formula for the number of solutions of the above equation. And our method is based on the properties of the Gauss sum. It is worth noting that we used a novel method to simplify the steps and avoid complicated calculations.

Keywords: recurrence formula; finite field; analytic methods; characters; Guass sum
Mathematics Subject Classification: 11D45, 11D72

1. Introduction

Let p be a prime, f be a polynomial with k variable and $\mathbf{F}_p = \mathbf{Z}/(p)$ be the finite field, where $\mathbf{Z}$ is the integer ring, and let

$$N(f;p) = \#\{(x_1, x_2, \cdots, x_k) \in \mathbf{F}_p^k | f(x_1, x_2, \cdots, x_k) = 0\}.$$

Many scholars studied the exact formula (including upper bound and lower bound) for $N(f;p)$ for many years, it is one of the main topics in the finite field theory, the most elementary upper bounds was given as follows (see [14])

$$N(f;p) \leq p^{k-1} deg f.$$

Let ord_p denote the p-adic additive valuation normalized such that $ord_p p = 1$. The famous Chevalley-Warning theorem shows that $ord_p N(f;p) > 0$ if $n > deg f$. Let [x] denote the least integer $\geq x$ and e denote the extension degree of $\mathbf{F}_q/\mathbf{F}_p$. Ax (see [2]) showed that

$$ord_p N(f;q) \geq e\left[\frac{n - deg f}{deg f}\right].$$

In 1977, S. Chowla et al. (see [7]) investigated a problem about the number of solutions of a equation in finite field $\mathbf{F}_p$ as follow,

$$x_1^3 + x_2^3 + \cdots + x_k^3 \equiv 0,$$

where p is a prime with $p \equiv 1 \bmod 3$ and $x_i \in \mathbf{F}_p$, $1 \le i \le k$.

Let M_k denotes the number of solutions of the above equation. They proved that

$$M_3 = p^2 + d(p-1), M_4 = p^2 + 6(p^2 - p),$$
$$\sum_{s=1}^{\infty} M_s x^s = \frac{x}{1-px} + \frac{x^2(p-1)(2+dx)}{1-3px^2-pdx^3},$$

where d is uniquely determined by $4p = d^2 + 27y^2$ and $d \equiv 1 \; mod \; 3$.

Myerson [12] extended the result in [2] to the field $\mathbf{F}_q$ and first studied the following equation over $\mathbf{F}_q$,

$$x_1^3 + x_2^3 + \cdots + x_k^3 \equiv 0.$$

Recently J. Zhao et al. (see [17]) investigated the following equations over field $\mathbf{F}_p$,

$$f_1 = x_1^4 + x_2^4 + x_3^4,$$
$$f_2 = x_1^4 + x_2^4 + x_3^4 + x_4^4.$$

And they give exact value of $N(f_1; p)$ and $N(f_2; p)$. For more general problem about this issue interested reader can see [6, 9–11].

In this paper, let $A(k, p)$ denotes the number of solutions of the following equation in $\mathbf{F}_p$,

$$x_1^6 + x_2^6 + \cdots + x_k^6 \equiv 0,$$

where p is a prime with $p \equiv 1 \bmod 3$ and $x_i \in \mathbf{F}_p$, $1 \le i \le k$, and for simplicity, in the rest of this paper, we assume there exists an integer z such that $z^3 \equiv 2 \bmod p$, we use analytic methods to give a recurrence formula for the number of solutions of the above equation. And our method is based on the properties of Gauss sum. It is worth noting that we used a novel method to simplify the steps and avoid a lot of complicated calculations. We proved the following:

Theorem 1. For any positive integer $k \ge 1$, we have the recurrence formula

$$\begin{aligned} A(k+6,p) = \; & 5pA(k+4,p) + 10dpA(k+3,p) \\ & +(46p^2 + 5d^2p + dp)A(k+2,p) \\ & +(2p^2 + 120dp^2 + 3d^3p + d^2p + dp)A(k+1,p) \\ & +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} \\ & +(129d^2 + 11d + 6)p^2 + d^4p)A(k,p) \\ & +p^{k+5} - p^{k+4} - (10dp + 2d^2)p^{k+3} - 64p^{k+5/2} \\ & -(429 + 121d + 5d^2)p^{k+2} \\ & -2d^2p^{k+3/2} - (3d^3 + 130d^2 + 12d + 6)p^{k+1} - d^4p^k, \end{aligned}$$

with the initial condition

$$A(1,p)=1, A(2,p)=4(p-1)+p, A(3,p)=10d(p-1)+p^2,$$
$$A(4,p)=56p(p-1)+10d^2(p-1)+p^3,$$
$$A(5,p)=188dp(p-1)+5d^3(p-1)+16dC(p)(p-1)+p^4,$$
$$A(6,p)=p^5+1400p^2(p-1)+(388d^2+8d-576)p(p-1)+d^2p-d^2,$$

where d is uniquely determined by $4p=d^2+27y^2$ and $d\equiv 1 \; mod \; 3$, and $C(p)=\sum_{a=1}^{p} e_p(a^3)$.

Remark. Our method is suitable to calculus the number of solutions of the following equation in $\mathbf{F}_p$,

$$x_1^t+x_2^t+\cdots+x_k^t\equiv 0,$$

where p satisfied a certain congruence conditions, and t is any nature number.

Our Theorem 2 can be deduced from Theorem 1 and the theory of the Difference equations.

Theorem 2. Let t_i $(1\le i\le k)$ be the real root of the below equation with multiplicity s_i $(1\le i\le k)$ respectively, and $\rho_j e^{\pm i w_j}$ $(1\le j\le h)$ be the complex root of the below equation with multiplicity r_j $(1\le j\le h)$ respectively,

$$\begin{aligned} x^6 &= 5px^4+10dpx^3+(46p^2+5d^2p+dp)x^2\\ &+(2p^2+120dp^2+3d^3p+d^2p+dp)x\\ &+(-4p^5+2d^2p^4+64p^{7/2}+381p^3+2d^2p^{5/2}\\ &+(129d^2+11d+6)p^2+d^4p). \end{aligned}$$

We have

$$A(n,p)=\sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}n^{s_i-a}t_i^n+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}n^{r_j-b}\rho_j^n\cos nw_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}n^{r_j-b}\rho_j^n\sin nw_j,$$

where C_{ia}, D_{jb}, E_{jb}, are determined by

$$\begin{aligned}
A(6,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}6^{s_i-a}t_i^6+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}6^{r_j-b}\rho_j^6\cos 6w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}6^{r_j-b}\rho_j^6\sin 6w_j,\\
A(5,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}5^{s_i-a}t_i^5+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}5^{r_j-b}\rho_j^5\cos 5w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}5^{r_j-b}\rho_j^5\sin 5w_j,\\
A(4,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}4^{s_i-a}t_i^4+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}4^{r_j-b}\rho_j^4\cos 4w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}4^{r_j-b}\rho_j^4\sin 4w_j,\\
A(3,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}3^{s_i-a}t_i^3+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}3^{r_j-b}\rho_j^3\cos 3w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}3^{r_j-b}\rho_j^3\sin 3w_j,\\
A(2,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}2^{s_i-a}t_i^2+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}2^{r_j-b}\rho_j^2\cos 2w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}2^{r_j-b}\rho_j^2\sin 2w_j,\\
A(1,p) &= \sum_{i=1}^{k}\sum_{a=1}^{s_i}C_{ia}t_i+\sum_{j=1}^{h}\sum_{b=1}^{r_j}D_{jb}\rho_j\cos w_j+\sum_{j=1}^{h}\sum_{b=1}^{r_j}E_{jb}\rho_j\sin w_j.
\end{aligned} \tag{1.1}$$

2. Some Lemmas

Before we prove these lemmas, we give some notations, χ_2 denotes the second-order character of $\mathbf{F}_p$, χ denotes the third-order character of $\mathbf{F}_p$, ψ denotes the sixth order character of $\mathbf{F}_p$.

$$e_p(x) = e^{\frac{2\pi ix}{p}}, \tau(\chi) = \sum_{m=1}^{p}\chi(m)e_p(m), G(\chi, m) = \sum_{a=1}^{p}\chi(a)e_p(am).$$

We call $G(\chi, m)$ the Gauss sum, and we have the following:

$$G(\chi, m) = \tau(\chi)\overline{\chi}(m), (m, p) = 1. \tag{2.1}$$

And also we have

$$|\tau(\chi)| = \sqrt{p}, \tag{2.2}$$

where χ is a primitive character of $\mathbf{F}_p$. And let $G(m, 6; p) = \sum\limits_{a=0}^{p-1} e_p(ma^6)$. For the property of the exponential sum and the general Gauss sum, interested readers can see [1, 4, 5, 8, 13, 15].

Lemma 1. Let p be a prime with $p \equiv 1 \mod 3$. Then for any third-order character χ of $\mathbf{F}_p$, we have the identity

$$\tau^3(\chi) + \tau^3(\overline{\chi}) = dp,$$

where d is uniquely determined by $4p = d^2 + 27y^2$ and $d \equiv 1 \mod 3$.

Proof. For the proof of this lemma see [3].

Lemma 2. Let χ be a third-order character of $\mathbf{F}_p$ with $p \equiv 1 \mod 3$, and $C(p) = \tau(\chi) + \tau(\overline{\chi})$, then $C(p) = \sum\limits_{a=1}^{p} e_p(a^3)$.

Proof.

$$A = \tau(\chi) + \tau(\overline{\chi}) = \sum_{a=1}^{p}(1 + \chi(a) + \overline{\chi}(a))e(\frac{a}{p}) = \sum_{a=1}^{p} e(\frac{a^3}{p}).$$

Lemma 3. Let $p \equiv 1 \mod 6$, $2 \equiv z^3 \mod p$ for some z, and let χ be a third-order character of $\mathbf{F}_p$, ψ be a sixth-order character of $\mathbf{F}_p$, then we have the identity

$$\tau(\psi) = \frac{\tau^2(\chi)}{\sqrt{p}}.$$

Proof. This is Lemma 3 in [16].

Lemma 4. As the definition above, we have the identity

$$G(m, 6; p) = \sqrt{p}\chi_2(m) + \frac{\overline{x^2}}{\sqrt{p}}\psi(m) + \frac{x^2}{\sqrt{p}}\overline{\psi}(m) + \overline{x}\chi(m) + x\overline{\chi}(m),$$

where $(m, p) = 1$ and $x = \tau(\chi)$.

Proof. Firstly we have the identity

$$1 + \chi_2(m) + \chi(m) + \overline{\chi}(m) + \psi(m) + \overline{\psi}(m) = \begin{cases} 6, & \text{if } m \equiv a^6 \mod p; \\ 0, & \text{otherwise.} \end{cases}$$

So we have

$$G(m,6;p) = \sum_{a=0}^{p-1}(1+\chi_2(a)+\chi(a)+\overline{\chi}(a)+\psi(a)+\overline{\psi}(a))e_p(ma)$$
$$= G(\chi_2,m)+G(\psi,m)+G(\overline{\psi},m)+G(\chi,m)+G(\overline{\chi},m)$$

By (2.1) and Lemma 3, we have

$$G(m,6;p) = \tau(\chi_2)\chi_2(m)+\tau(\overline{\psi})\psi(m)+\tau(\psi)\overline{\psi}(m)+\tau(\overline{\chi})\chi(m)+\tau(\chi)\overline{\chi}(m)$$
$$= \sqrt{p}\chi_2(m)+\frac{\overline{x^2}}{\sqrt{p}}\psi(m)+\frac{x^2}{\sqrt{p}}\overline{\psi}(m)+\overline{x}\chi(m)+x\overline{\chi}(m). \tag{2.3}$$

By (2.3), we complete the proof of our lemma.

Next we let,

$$G^n(m,6;p) = a_n + b_n\chi_2(m) + c_n\psi(m) + d_n\overline{\psi}(m) + e_n\chi(m) + f_n\overline{\chi}(m). \tag{2.4}$$

Then we have following Lemma 5.

Lemma 5. Let $a_n, b_n, c_n, d_n, e_n, f_n$ are defined as above, then we have that $a_n, b_n, c_n, d_n, e_n, f_n$ are uniquely determined by n, where $n \geq 1$.

Proof. By the orthogonality of characters of $\mathbf{F}_p$, we have

$$\sum_{a=1}^{p-1}\chi(a) = \begin{cases} p-1, & \text{if } \chi=\chi_0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

By (2.4) and (2.5) we have

$$\sum_{m=1}^{p-1}G^n(m,6;p) = (p-1)a_n + b_n\sum_{m=1}^{p-1}\chi_2(m) + c_n\sum_{m=1}^{p-1}\psi(m) + d_n\sum_{m=1}^{p-1}\overline{\psi}(m)$$
$$+e_n\sum_{m=1}^{p-1}\chi(m) + f_n\sum_{m=1}^{p-1}\overline{\chi}(m)$$
$$= (p-1)a_n.$$

So we have

$$a_n = \frac{1}{p-1}\sum_{m=1}^{p-1}G^n(m,6;p). \tag{2.6}$$

By the same method, we have

$$b_n = \frac{1}{p-1}\sum_{m=1}^{p-1}\chi_2(m)G^n(m,6;p),$$

$$c_n = \frac{1}{p-1}\sum_{m=1}^{p-1}\overline{\psi}(m)G^n(m,6;p),$$

$$d_n = \frac{1}{p-1}\sum_{m=1}^{p-1}\psi(m)G^n(m,6;p),$$

$$e_n = \frac{1}{p-1}\sum_{m=1}^{p-1}\overline{\chi}(m)G^n(m,6;p),$$

$$f_n = \frac{1}{p-1}\sum_{m=1}^{p-1}\chi(m)G^n(m,6;p).$$

So now it is easy to see the conclusion of the lemma.

Lemma 6. The sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ are defined above, then they satisfied the following recurrence formulae ($n \geq 0$):

$$a_{n+1} = \sqrt{p}b_n + \frac{\overline{x^2}}{\sqrt{p}}d_n + \frac{x^2}{\sqrt{p}}c_n + xe_n + \overline{x}f_n, \tag{2.7}$$

$$b_{n+1} = \sqrt{p}a_n + \frac{\overline{x^2}}{\sqrt{p}}e_n + \frac{x^2}{\sqrt{p}}f_n + xd_n + \overline{x}c_n, \tag{2.8}$$

$$c_{n+1} = \sqrt{p}f_n + \frac{\overline{x^2}}{\sqrt{p}}a_n + \frac{x^2}{\sqrt{p}}e_n + xb_n + \overline{x}d_n, \tag{2.9}$$

$$d_{n+1} = \sqrt{p}e_n + \frac{\overline{x^2}}{\sqrt{p}}f_n + \frac{x^2}{\sqrt{p}}a_n + xc_n + \overline{x}b_n, \tag{2.10}$$

$$e_{n+1} = \sqrt{p}d_n + \frac{\overline{x^2}}{\sqrt{p}}c_n + \frac{x^2}{\sqrt{p}}b_n + xf_n + \overline{x}a_n, \tag{2.11}$$

$$f_{n+1} = \sqrt{p}c_n + \frac{\overline{x^2}}{\sqrt{p}}b_n + \frac{x^2}{\sqrt{p}}d_n + xa_n + \overline{x}e_n, \tag{2.12}$$

with the initial condition

$$a_0 = 1, b_0 = c_0 = d_0 = e_0 = f_0 = 0.$$

Proof. We only prove (2.7), the rest can be proved in the same way. By Lemma 5, we know a_n is unique determined by n. We can compare the coefficient of the equation

$$G^{n+1}(m,6;p) = G^n(m,6;p)G(m,6;p).$$

We have

$$a_{n+1} = \sqrt{p}b_n + \frac{\overline{x^2}}{\sqrt{p}}d_n + \frac{x^2}{\sqrt{p}}c_n + xe_n + \overline{x}f_n.$$

So we complete the proof of the lemma.

Lemma 7. Let a_n is defined as above, then we have

$$a_0 = 1, a_1 = 0, a_2 = 5p, a_3 = 10dp, a_4 = 56p^2 + 10d^2p,$$

$$a_5 = 188dp^2 + 5d^3p + 16dpC(p).$$

Proof. By Lemma 4 and after some elementary calculations we have

$$\begin{aligned}
G^2(m,6;p) &= 5p + 2dp^{1/2}\chi_2(m) + 4p^{1/2}x\psi(m) + 4p^{1/2}\overline{x}\overline{\psi}(m) \\
&\quad +(p^{-1}\overline{x^4} + 3x^2)\chi(m) + (p^{-1}x^4 + 3\overline{x^2})\overline{\chi}(m), \\
G^3(m,6;p) &= 10dp + (16p^{3/2} + dp^{1/2})\chi_2(m) + (15p\overline{x} + 2dx^2 + p^{-1}x^5)\chi(m) \\
&\quad +(15px + 2d\overline{x^2} + p^{-1}\overline{x^5})\overline{\chi}(m) \\
&\quad +(4p^{-1/2}x^4 + 12p^{1/2}\overline{x^2} + 2dp^{1/2}x)\psi(m) \\
&\quad +(4p^{-1/2}\overline{x^4} + 12p^{1/2}x^2 + 2dp^{1/2}\overline{x})\overline{\psi}(m), \\
G^4(m,6;p) &= 60p^2 + 9d^2p + dp + 48dp^{3/2}\chi_2(m) \\
&\quad +(p^{-2}x^8 + 17\overline{x^4} + 46px^2 + 16dp)\chi(m) \\
&\quad +(p^{-2}\overline{x^8} + 17x^4 + 46p\overline{x^2} + 16dp)\overline{\chi}(m) \\
&\quad +(56p^{3/2}x + 4dp^{-1/2}x^4 + 12dp^{1/2}\overline{x^2} + 8p^{-1/2}\overline{x^5})\psi(m) \\
&\quad +(56p^{3/2}\overline{x} + 4dp^{-1/2}\overline{x^4} + 12dp^{1/2}x^2 + 8p^{-1/2}x^5)\overline{\psi}(m), \\
G^5(m,6;p) &= 188dp^2 + 5d^3p + 16dpC(p) \\
&\quad +(52d^2p^{3/2} + 208p^{5/2} + 16dp^{1/2}(x^2 + \overline{x^2}))\chi_2(m) \\
&\quad +(p^{-2/5}x^{10} + p^{-3/2}\overline{x^8} + 4dp^{-1/2}\overline{x^5} + 71p^{1/2}x^4 \\
&\quad +(46p^{3/2} + 16p^{1/2})x^2 \\
&\quad +(129p^{3/2} + 10d^2p^{1/2})\overline{x^2} + 60dp^{3/2}x + 16dp^{3/2})\psi(m) \\
&\quad +(p^{-2/5}\overline{x^{10}} + p^{-3/2}x^8 + 4dp^{-1/2}x^5 + 71p^{1/2}\overline{x^4} \\
&\quad +(46p^{3/2} + 16p^{1/2})\overline{x^2} \\
&\quad +(129p^{3/2} + 10d^2p^{1/2})x^2 + 60dp^{3/2}\overline{x} + 16dp^{3/2})\overline{\psi}(m) \\
&\quad +(8p^{-1}\overline{x^7} + p^{-1}x^7 + 25x^5 + 52dpx^2 \\
&\quad +(28dp + 46p^2)x + 16d\overline{x^4} + 112p^2\overline{x})\chi(m) \\
&\quad +(8p^{-1}x^7 + p^{-1}\overline{x^7} + 25\overline{x^5} + 52dp\overline{x^2} \\
&\quad +(28dp + 46p^2)\overline{x} + 16dx^4 + 112p^2x)\overline{\chi}(m),
\end{aligned}$$

and comparing the above formulae with (2.6), we have

$$a_0 = 1, a_1 = 0, a_2 = 5p, a_3 = 10dp, a_4 = 60p^2 + 9d^2p + dp, a_5 = 188dp^2 + 5d^3p + 16dpC(p).$$

Lemma 8. Let a_n, b_n, c_n, d_n, e_n, f_n are defined as above, then we have

$$\begin{aligned}
a_6 &= 5pa_4 + 10dpa_3 + (46p^2 + 5d^2p + dp)a_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)a_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)a_0 \\
b_6 &= 5pb_4 + 10dpb_3 + (46p^2 + 5d^2p + dp)b_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)b_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)b_0 \\
c_6 &= 5pc_4 + 10dpc_3 + (46p^2 + 5d^2p + dp)c_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)c_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)c_0
\end{aligned}$$

$$\begin{aligned}
d_6 &= 5pd_4 + 10dpd_3 + (46p^2 + 5d^2p + dp)d_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)d_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)d_0 \\
e_6 &= 5pe_4 + 10dpe_3 + (46p^2 + 5d^2p + dp)e_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)e_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)e_0 \\
f_6 &= 5pf_4 + 10dpf_3 + (46p^2 + 5d^2p + dp)f_2 + (2p^2 + 120dp^2 + 3d^3p + d^2p + dp)f_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} + (129d^2 + 11d + 6)p^2 + d^4p)f_0
\end{aligned}$$

Proof. We only proof the first formula, the rest can be proof in the same way. By Lemma 6, we have

$$\begin{aligned}
a_6 &= \sqrt{p}b_5 + \frac{\overline{x^2}}{\sqrt{p}}d_5 + \frac{x^2}{\sqrt{p}}c_5 + xe_5 + \overline{x}f_5 \\
&= 5pa_4 + 2dp^{1/2}b_4 + 4p^{1/2}\overline{x}c_4 + 4p^{1/2}xd_4 \\
&\quad +(3\overline{x^2} + p^{-1}x^4)e_4 + (3x^2 + p^{-1}\overline{x^4})f_4 \\
&= 5pa_4 + 10dpa_3 + (d^2p^{1/2} + 12p^{3/2})b_3 + (2dp^{1/2}\overline{x} \\
&\quad +8p^{1/2}x^2 + p^{-1/2}\overline{x^4})c_3 \\
&\quad +(2dp^{1/2}x + 8p^{1/2}\overline{x^2} + p^{-1/2}x^4)d_3 + (11px + \overline{x^2} + p^{-1}\overline{x^5})e_3 \\
&\quad +(11p\overline{x} + x^2 + p^{-1}x^5)f_3 \\
&= 5pa_4 + 10dpa_3 + (46p^2 + 5d^2p + dp)a_2 + (25dp^{3/2} + 2p^{3/2})b_2 \\
&\quad +(p^{-3/2}\overline{x^7} + 2p^{-1/2}x^5 + p^{-1/2}\overline{x^4} + 42p^{3/2}\overline{x} + 2dp^{1/2}x^2 \\
&\quad +(d^2 + 1)p^{1/2}\overline{x})c_2 \\
&\quad +(p^{-3/2}x^7 + 2p^{-1/2}\overline{x^5} + p^{-1/2}x^4 + 42p^{3/2}x + 2dp^{1/2}\overline{x^2} \\
&\quad +(d^2 + 1)p^{1/2}x)d_2 \\
&\quad +(10x^4 + (32p + d^2)\overline{x^2} + (4dp + p)x)e_2 \\
&\quad +(10\overline{x^4} + (32p + d^2)x^2 + (4dp + p)\overline{x})f_2 \\
&= 5pa_4 + 10dpa_3 + (46p^2 + 5d^2p + dp)a_2 \\
&\quad +(2p^2 + 120dp^2 + 3d^3p + d^2p + dp)a_1 \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} \\
&\quad +(129d^2 + 11d + 6)p^2 + d^4p)a_0.
\end{aligned}$$

So we complete the proof of this lemma.

Lemma 9. Let a_n is defined as above, then for any integer $n \geq 0$, we have

$$\begin{aligned}
a_{n+6} &= 5pa_{n+4} + 10dpa_{n+3} + (46p^2 + 5d^2p + dp)a_{n+2} \\
&\quad +(2p^2 + 120dp^2 + 3d^3p + d^2p + dp)a_{n+1} \\
&\quad +(-4p^5 + 2d^2p^4 + 64p^{7/2} + 381p^3 + 2d^2p^{5/2} \\
&\quad +(129d^2 + 11d + 6)p^2 + d^4p)a_n.
\end{aligned}$$

Proof. By (2.4) and Lemma 8, we have

$$G^6(m, 6; p) = 5pG^4(m, 6; p) + 10dpG^3(m, 6; p) + (46p^2 + 5d^2p + dp)G^2(m, 6; p)$$

$$+(2p^2 + 120dp^2 + 3d^3 p + d^2 p + dp)G(m, 6; p)$$
$$+(-4p^5 + 2d^2 p^4 + 64p^{7/2} + 381p^3 + 2d^2 p^{5/2}$$
$$+(129d^2 + 11d + 6)p^2 + d^4 p).$$

We multiple $G^n(m, 6; p)$ to the both side of the above formula, we have

$$\begin{aligned} G^{n+6}(m, 6; p) &= 5pG^{n+4}(m, 6; p) + 10dpG^{n+3}(m, 6; p) \\ &+(46p^2 + 5d^2 p + dp)G^{n+2}(m, 6; p) \\ &+(2p^2 + 120dp^2 + 3d^3 p + d^2 p + dp)G^{n+1}(m, 6; p) \\ &+(-4p^5 + 2d^2 p^4 + 64p^{7/2} + 381p^3 + 2d^2 p^{5/2} \\ &+(129d^2 + 11d + 6)p^2 + d^4 p)G^n(m, 6; p). \end{aligned}$$

By Lemma 5, we can compare the coefficient of the above equation, we have

$$\begin{aligned} a_{n+6} &= 5pa_{n+4} + 10dpa_{n+3} + (46p^2 + 5d^2 p + dp)a_{n+2} \\ &+(2p^2 + 120dp^2 + 3d^3 p + d^2 p + dp)a_{n+1} \\ &+(-4p^5 + 2d^2 p^4 + 64p^{7/2} + 381p^3 + 2d^2 p^{5/2} \\ &+(129d^2 + 11d + 6)p^2 + d^4 p)a_n. \end{aligned}$$

3. Proof of the theorem 1

In the formula below, we always let $k \geq 1$. By the following formula,

$$\sum_{a=0}^{p-1} e_p(ma) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} A(k, p) &= \frac{1}{p} \sum_{m=0}^{p-1} \sum_{x_1=0, x_2=0, \cdots, x_k=0}^{p-1} e_p(m(x_1^6 + x_2^6 + \cdots + x_k^6)) \\ &= \frac{1}{p} \sum_{m=0}^{p-1} G^k(m, 6; p). \end{aligned} \tag{3.1}$$

By (8), we have

$$\begin{aligned} A(k, p) &= \frac{1}{p} \sum_{m=0}^{p-1} G^k(m, 6; p) \\ &= \frac{1}{p}(\sum_{m=1}^{p-1} G^k(m, 6; p) + p^k) \\ &= \frac{1}{p}((p-1)a_k + p^k) = \frac{p-1}{p} a_k + p^{k-1}. \end{aligned} \tag{3.2}$$

So by Lemma 9, we have

$$\begin{aligned} A(k+6,p)-p^{k+5} &= 5p(A(k+4,p)-p^{k+3})+10dp(A(k+3,p)-p^{k+2}) \\ &+(46p^2+5d^2p+dp)(A(k+2,p)-p^{k+1}) \\ &+(2p^2+120dp^2+3d^3p+d^2p+dp)(A(k+1,p)-p^k) \\ &+(-4p^5+2d^2p^4+64p^{7/2}+381p^3+2d^2p^{5/2} \\ &+(129d^2+11d+6)p^2+d^4p)(A(k,p)-p^{k-1}). \end{aligned}$$

So we have

$$\begin{aligned} A(k+6,p) &= 5pA(k+4,p)+10dpA(k+3,p) \\ &+(46p^2+5d^2p+dp)A(k+2,p) \\ &+(2p^2+120dp^2+3d^3p+d^2p+dp)A(k+1,p) \\ &+(-4p^5+2d^2p^4+64p^{7/2}+381p^3+2d^2p^{5/2} \\ &+(129d^2+11d+6)p^2+d^4p)A(k,p) \\ &+p^{k+5}-p^{k+4}-(10dp+2d^2)p^{k+3}-64p^{k+5/2} \\ &-(429+121d+5d^2)p^{k+2} \\ &-2d^2p^{k+3/2}-(3d^3+130d^2+12d+6)p^{k+1}-d^4p^k. \end{aligned}$$

And by Lemma 7 and (3.2), we have the initial conditions

$$\begin{aligned} &A(1,p)=1, A(2,p)=4(p-1)+p, A(3,p)=10d(p-1)+p^2, \\ &A(4,p)=56p(p-1)+10d^2(p-1)+p^3, \\ &A(5,p)=188dp(p-1)+5d^3(p-1)+16dC(p)(p-1)+p^4. \\ &A(6,p)=p^5+1400p^2(p-1)+(388d^2+8d-576)p(p-1)+d^2p-d^2. \end{aligned}$$

So we complete the proof of the theorem.

4. Conclusion

The main purpose of this paper is using analytic methods to give a recurrence formula of the number of solutions of an equation over finite field. And we give an expression of the number of solutions of the above equation by the root of sixth degree polynomial. We use analytic methods to give a recurrence formula for the number of solutions of the above equation. And our method is based on the properties of the Gauss sum. It is worth noting that we used a novel method to simplify the steps and avoid complicated calculations.

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Conflict of interest

The author declares that there is no competing interest.

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