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Research article

The Meir-Keeler type contractions in extended modular b-metric spaces with an application

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Abstract: In this paper, we introduce the notion of a modular p-metric space (an extended modular b-metric space) and establish some fixed point results for α-$\widehat{\nu}$-Meir-Keeler contractions in this new space. Using these results, we deduce some new fixed point theorems in extended modular metric spaces endowed with a graph and in partially ordered extended modular metric spaces. Also, we develop an important relation between fuzzy-Meir-Keeler and extended fuzzy p-metric with modular p-metric and get certain new fixed point results in triangular fuzzy p-metric spaces. We provide an example and an application to support our results which generalize several well known results in the literature.

Keywords: fixed point; extended modular metric space; α-$\widehat{\nu}$-Meir-Keeler contraction; integral equation; triangular fuzzy p-metric space
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1. Introduction and preliminaries

In order to generalize the celebrated Banach contraction principle, many authors obtained various types of contraction inequalities. Fixed point results in such spaces have been established in a large number of works. Some of these works are noted in [2, 9, 22, 23, 26, 27, 37].

In 1969, Meir and Keeler [25] obtained the following interesting fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \text{ implies } d(Tx, Ty) < \epsilon,$$

for all $x, y \in X$. Then T has a unique fixed point.

Meir-Keeler's fixed point theorem has been extended in many directions [6, 20, 23, 32–34].

On the other hand, the concept of modular metric spaces were introduced in [7, 8]. Here, we look at modular metric space as the nonlinear version of the classical one introduced by Nakano [30] on vector space and modular function space introduced by Musielak [29] and Orlicz [31]. For more details on modular metric spaces, we recommend [3–5, 12, 13, 28, 38].

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function, for simplicity, we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.2. [7, 8] A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$,
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$,
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \text{ for some } \lambda > 0.$$

Samet et al. [39] defined the notion of α-admissible mappings as follows:

Definition 1.3. [39] Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Finally, we recall that Karapınar et al. [23] introduced the notion of triangular α-admissible mapping as follows.

Definition 1.4. [23] Let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that a self-mapping $T : X \to X$ is triangular α-admissible if

(i) $x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1,$

(ii) $x, y, z \in X, \quad \begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \quad \Rightarrow \quad \alpha(x, y) \geq 1.$

Lemma 1.5. *[23] Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \quad \textit{for all} \quad m, n \in \mathbb{N} \textit{ with } m < n.$$

Now we deal with some notions required in b-metric, extended b-metric, modular b-metric and extended modular b-metric spaces.

Recall that a b-metric d on a set X is a generalization of standard metric [10], where the triangular inequality is replaced by

$$d(x,z) \leq s(d(x,y) + d(y,z)),$$

for all $x, y, z \in X$ and for some fixed $s \geq 1$. Parvaneh and Ghoncheh [35] introduced the following further generalization.

Definition 1.6. Let X be a nonempty set. A function $d : X \times X \to R^+$ is a p-metric if there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $t \leq \Omega(t)$ for $t \in [0,+\infty)$, such that for all $x, y, z \in X$, the following conditions hold:

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,z) \leq \Omega(d(x,y) + d(y,z))$.

In this case, the pair (X, d) is called a p-metric space or an extended b-metric space.

It should be noted that the class of p-metric spaces is considerably larger than the class of b-metric spaces since a b-metric is a p-metric with $\Omega(t) = st$, while a metric is a p-metric with $\Omega(t) = t$.

Definition 1.7. [7, 8] A function $\omega : (0,+\infty) \times X \times X \to [0,+\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$,
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$,
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Ege and Alaca [14] introduced the notion of modular b-metric space.

Definition 1.8. [14] Let X be a non-empty set and $s \geq 1$ a real number. A map $\nu : (0,\infty) \times X \times X \to [0,\infty]$ is called a modular b-metric, if the following statements hold for all $x, y, z \in X$,

(i) $\nu_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$,
(ii) $\nu_\lambda(x,y) = \nu_\lambda(y,x)$ for all $\lambda > 0$,
(iii) $\nu_{\lambda+\mu}(x,y) \leq s[\nu_\lambda(x,z) + \nu_\mu(z,y)]$ for all $\lambda, \mu > 0$.

Then (X, ν) is called a modular b-metric space.

The modular b-metric space could be seen as a generalization of the modular metric space.

Example 1.9. [14] Consider the space $l_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, $0 < p < 1$, $\lambda \in (0,\infty)$ and $\nu_\lambda(x,y) = \frac{d(x,y)}{2}$ such that

$$d(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}, \quad \text{where } \ x = (x_n),\ y = (y_n) \in l_p.$$

It could be easily seen that (X, ν) is a modular b-metric space.

Our aim in this study is to define a modular p-metric space. In the next section, we prove some fixed point theorems on α-$\widehat{\nu}$-Meir-Keeler contractions in the new space. In Sections 3 and 4, new fixed point results are obtained in extended modular metric spaces endowed with a graph and in partially ordered extended modular metric spaces. The Section 5 includes a relation between fuzzy-Meir-Keeler and extended fuzzy p-metric with modular p-metric and some fixed point theorems in triangular fuzzy p-metric spaces. The paper ends with an application on the solution of Volterra-type integral equations.

2. Main results

In this section, we define the concept of a modular p-metric space (an extended modular b-metric spaces shortly denoted by EMbM spaces) and present some fixed point results. Our results generalize the results in [16] if we take $\Omega(t) = t$.

Definition 2.1. Let X be a nonempty set. A function $\widehat{\nu}_\lambda : (0,\infty)\times X\times X \to [0,\infty]$ is a modular p-metric (an extended modular b-metric) if there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\Omega^{-1}(t) \le t \le \Omega(t)$ for $t \in [0,+\infty)$, such that for all $x, y, z \in X$, the following conditions hold:

(i) $\widehat{\nu}_\lambda(x, y) = 0$ if and only if $x = y$ for all $\lambda > 0$,
(ii) $\widehat{\nu}_\lambda(x, y) = \widehat{\nu}_\lambda(y, x)$ for all $\lambda > 0$,
(iii) $\widehat{\nu}_{\lambda+\mu}(x, y) \le \Omega[\widehat{\nu}_\lambda(x, z) + \widehat{\nu}_\mu(z, y)]$ for all $\lambda, \mu > 0$.

Then we say that $(X,\widehat{\nu})$ is a modular p-metric space.

It should be noted that the class of modular p-metric spaces is considerably larger than the class of modular b-metric spaces, since a modular b-metric is a modular p-metric with $\Omega(t) = st$, while a modular metric is a modular p-metric with $\Omega(t) = t$.

Example 2.2. Let (X, ν_λ) be a modular b-metric space with coefficient $s \ge 1$ and

$$\widehat{\nu}_\lambda(x, y) = \sinh(\nu_\lambda(x, y)).$$

We show that $\widehat{\nu}_\lambda$ is a modular p-metric with $\Omega(t) = \sinh(st)$ for all $t \ge 0$ (and $\Omega^{-1}(u) = \frac{1}{s}\sinh^{-1} u$ for $u \ge 0$).

Obviously, the conditions (i) and (ii) of Definition 2.1 are satisfied. For each $x, y, z \in X$ and $\lambda, \mu \ge 0$, we have

$$\begin{aligned}
\widehat{\nu}_{\lambda+\mu}(x, y) &= \sinh(\nu_{\lambda+\mu}(x, y)) \\
&\le \sinh(s\nu_\lambda(x, z) + s\nu_\mu(z, y)) \\
&\le \sinh(s\sinh(\nu_\lambda(x, z)) + s\sinh(\nu_\mu(z, y))) \\
&= \Omega(\widehat{\nu}_\lambda(x, z) + \widehat{\nu}_\mu(z, y)).
\end{aligned}$$

So, the condition (iii) of Definition 2.1 is also satisfied and $\widehat{\nu}$ is a modular p-metric.

Proposition 2.3. *Let (X, ν_λ) be a modular b-metric space with coefficient $s \ge 1$ and*

$$\widehat{\nu}_\lambda(x, y) = \xi(\nu_\lambda(x, y))$$

where $\xi : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $t \le \xi(t)$ for all $t \ge 0$ and $\xi(0) = 0$. Then $\widehat{\nu}_\lambda$ is a modular p-metric with $\Omega(t) = \xi(st)$.

Proof. For each $x, y, z \in X$ and $\lambda, \mu \geq 0$, we have

$$\begin{aligned} \widehat{\nu}_{\lambda+\mu}(x,y) &= \xi(\nu_{\lambda+\mu}(x,y)) \\ &\leq \xi(s\nu_\lambda(x,z) + s\nu_\mu(z,y)) \\ &\leq \xi(s\xi(\nu_\lambda(x,z)) + s\xi(\nu_\mu(z,y))) \\ &= \Omega(\widehat{\nu}_\lambda(x,z) + \widehat{\nu}_\mu(z,y)). \end{aligned}$$

□

Example 2.4. If $\xi(t) = e^t - 1$, we get $\widehat{\nu}_\lambda(x,y) = e^{\nu_\lambda(x,y)} - 1$ and $\Omega(t) = e^{st} - 1$. Note that

$$\Omega^{-1}(u) = \frac{1}{s}\ln(1+u).$$

Now, we present the definition of $\widehat{\nu}$-Cauchy and $\widehat{\nu}$-convergent sequences and $\widehat{\nu}$-complete spaces.

Definition 2.5. Let $(X, \widehat{\nu}_\lambda)$ be a modular p-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) $\widehat{\nu}$-Cauchy if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\epsilon)$ and $\lambda > 0$ we have $\widehat{\nu}_\lambda(x_n, x_m) < \epsilon$.

(b) $\widehat{\nu}$-convergent to $x \in X$ if $\widehat{\nu}_\lambda(x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.

(c) $\widehat{\nu}$-complete if each $\widehat{\nu}$-Cauchy sequence in X is $\widehat{\nu}$-convergent and its limit is in X.

Now, we define the notion of α-$\widehat{\nu}$-Meir-Keeler contractive mapping as follows:

Definition 2.6. Let $X_{\widehat{\nu}}$ be a modular p-metric space and T a self-mapping on $X_{\widehat{\nu}}$. Also suppose that $\alpha : X_{\widehat{\nu}} \times X_{\widehat{\nu}} \to [0, +\infty)$. We say that T is α-$\widehat{\nu}$-Meir-Keeler contractive if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \Omega^{-1}(\widehat{\nu}_\lambda(x,y)) < \Omega(\epsilon) + \Omega(\delta(\epsilon)) \text{ implies } \alpha(x,y)\widehat{\nu}_\lambda(Tx,Ty) < \Omega(\epsilon), \tag{2.1}$$

for any $x, y \in X_{\widehat{\nu}}$ and all $\lambda > 0$.

Remark 2.7. Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$-regular modular p-metric space and T an α-$\widehat{\nu}$-Meir-Keeler contractive mapping. Then

$$\widehat{\nu}_\lambda(Tx,Ty) < \widehat{\nu}_\lambda(x,y),$$

for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$, $\alpha(x,y) \geq 1$ and $\Omega^{-1}(\widehat{\nu}_\lambda(x,y)) < \infty$. Also, if $x = y$, then $\Omega^{-1}(\widehat{\nu}_\lambda(Tx,Ty)) = 0$. That is

$$\widehat{\nu}_\lambda(Tx,Ty) \leq \widehat{\nu}_\lambda(x,y),$$

for all $x, y \in X$ and $\lambda > 0$ with $\alpha(x,y) \geq 1$.

Since $x \neq y$ by $\widehat{\nu}$-regularity we have, $\widehat{\nu}_\lambda(x,y) > 0$ for all $\lambda > 0$. Assume, $\delta > 0$ and

$$\Omega(\epsilon) = \widehat{\nu}_\lambda(x,y).$$

Then

$$\Omega^{-1}(\widehat{\nu}_\lambda(x,y)) < \Omega^{-1}(\widehat{\nu}_\lambda(x,y)) + \delta \leq \Omega(\epsilon) + \Omega(\delta)$$

and so from (1) we have,

$$\widehat{\nu}_\lambda(Tx,Ty) \leq \alpha(x,y)[\widehat{\nu}_\lambda(Tx,Ty) < \Omega(\epsilon) = \widehat{\nu}_\lambda(x,y).$$

Now we are ready to prove our first theorem.

Theorem 2.8. *Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$ regular $\widehat{\nu}$-complete modular p-metric space and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ be a self-mapping. Assume that there exists a function $\alpha : X_{\widehat{\nu}} \times X_{\widehat{\nu}} \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is α-$\widehat{\nu}$-Meir-Keeler mapping,
(iii) there exists $x_0 \in X_{\widehat{\nu}}$ such that $\alpha(x_0, Tx_0) \geq 1$,
(iv) T is $\widehat{\nu}$-continuous mapping.

Then T has a fixed point $z \in X$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $x_0 \in X_{\widehat{\nu}}$ be such that $\alpha(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be a Picard sequence starting at x_0, that is, $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a triangular α-admissible mapping, applying Lemma 1.5,

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then evidently T has a fixed point. Hence, we suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. So, by $\widehat{\nu}$–regularity we have,

$$\widehat{\nu}_\lambda(x_n, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, using Remark 2.7 and the condition (iii), we have

$$\widehat{\nu}_\lambda(x_n, x_{n+1}) < \widehat{\nu}_\lambda(x_{n-1}, x_n) < \ldots < \widehat{\nu}_\lambda(x_0, x_1) < \infty.$$

This implies that the sequence $\{c_n := \widehat{\nu}_\lambda(x_n, x_{n+1})\}$ is non-increasing and $c_n < \infty$ for all $n \in \mathbb{N} \cup \{0\}$. So the sequence $\{c_n\}$ is convergent to some $c \in \mathbb{R}_+$. We will show that $c = 0$. Suppose, to the contrary, implies that $c > 0$. Hence, we have

$$0 < c < \widehat{\nu}_\lambda(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Let $\epsilon = \Omega^{-1}(c) > 0$. Then by hypothesis, there exists a $\delta(\varepsilon) > 0$ such that (1) holds. On the other hand, by the definition of ϵ, there exists $n_0 \in \mathbb{N}$ such that

$$\epsilon = \Omega^{-1}(c) < \Omega^{-1}(c_{n_0}) < c_{n_0} = \widehat{\nu}_\lambda(x_{n_0}, x_{n_0+1}) < \Omega(\epsilon) + \Omega^{-1}(\delta) < \Omega(\epsilon) + \delta.$$

Now by (1), we have

$$\begin{aligned} c_{n_0+1} &= \widehat{\nu}_\lambda(x_{n_0+1}, x_{n_0+2}) \\ &\leq \alpha(x_{n_0}, x_{n_0+1})[\widehat{\nu}_\lambda(x_{n_0+1}, x_{n_0+2})] \\ &= \alpha(x_{n_0}, x_{n_0+1})[\widehat{\nu}_\lambda(Tx_{n_0}, Tx_{n_0+1})] \\ &< \Omega(\epsilon). \end{aligned}$$

That is,

$$c_{n_0+1} < \Omega(\epsilon) = c,$$

which is a contradiction. Hence, $c = 0$. That is,

$$\lim_{n\to\infty} \widehat{\nu}_\lambda(x_n, x_{n+1}) = 0.$$

For given $\epsilon > 0$, by the hypothesis, there exists $\delta = \delta(\epsilon) > 0$ such that (1) holds. Without loss of generality, we assume $\delta < \epsilon$. Since $c = 0$ then there exists $N_0 \in \mathbb{N}$ such that

$$c_n = \widehat{\nu}_\lambda(x_n, x_{n+1}) < \Omega(\delta), \text{ for all } n \geq N_0. \tag{2.3}$$

We will prove that for any fixed $k \geq N_0$,

$$\Omega^{-1}(\widehat{\nu}_\lambda(x_k, x_{k+l})) \leq \epsilon, \quad \text{for all } l \in \mathbb{N}, \tag{2.4}$$

holds. Note that by (2.3), (2.4) holds for $l = 1$. Suppose the condition (2.4) is satisfied for some $m \in \mathbb{N}$. That is,

$$\Omega^{-1}(\widehat{\nu}_\lambda(x_k, x_{k+m})) < \epsilon, \quad \text{for some } m \in \mathbb{N}. \tag{2.5}$$

For $l = m + 1$, by (2.3) and (2.5), we get

$$\begin{aligned} &\Omega^{-1}(\widehat{\nu}_\lambda(x_{k-1}, x_{k+m}) \\ &\leq (\widehat{\nu}_{\frac{\lambda}{2}}(x_{k-1}, x_k) + \widehat{\nu}_{\frac{\lambda}{2}}(x_k, x_{k+m})) \\ &\leq (\Omega(\epsilon) + \Omega(\delta)). \end{aligned} \tag{2.6}$$

Now, if

$$\Omega^{-1}(\widehat{\nu}_\lambda(x_{k-1}, x_{k+m})) \geq \epsilon,$$

then by (1) and (2.6), we get

$$\begin{aligned} &\widehat{\nu}_\lambda(x_k, x_{k+m+1}) \\ &\leq \alpha(x_{k-1}, x_{k+m})\widehat{\nu}_\lambda(x_k, x_{k+m+1}) \\ &= \alpha(x_{k-1}, x_{k+m})\widehat{\nu}_\lambda(Tx_{k-1}, Tx_{k+m}) \\ &< \Omega(\epsilon), \end{aligned}$$

and hence (2.4) holds.

If $\Omega^{-1}(\widehat{\nu}_\lambda(x_{k-1}, x_{k+m})) < \epsilon$, then applying Remark 2.7, we have

$$\widehat{\nu}_\lambda(x_k, x_{k+m+1}) = \widehat{\nu}_\lambda(Tx_{k-1}, Tx_{k+m}) \leq \widehat{\nu}_\lambda(x_{k-1}, x_{k+m}) < \Omega(\epsilon).$$

Consequently (2.4) holds for $l = m + 1$. Hence

$$\widehat{\nu}_\lambda(x_k, x_{k+l}) < \Omega(\epsilon), \quad \text{for all } l \in \mathbb{N}.$$

Thus we have proved that $\{x_n\}$ is a $\widehat{\nu}$-Cauchy sequence. The hypothesis of $\widehat{\nu}$-completeness of $X_{\widehat{\nu}}$ ensures that there exists $x^* \in X_{\widehat{\nu}}$ such that $\widehat{\nu}_1(x_n, x^*) \to 0$ as $n \to +\infty$. Now, since T is a $\widehat{\nu}$-continuous mapping, $\widehat{\nu}_1(x_{n+1}, Tx^*) = \widehat{\nu}_1(Tx_n, Tx^*) \to 0$ as $n \to +\infty$. From

$$\widehat{\nu}_2(x^*, Tx^*) \leq \Omega(\widehat{\nu}_1(x^*, x_{n+1}) + \widehat{\nu}_1(x_{n+1}, Tx^*)),$$

taking limit as $n \to +\infty$, we get $\widehat{\nu}_2(x^*, Tx^*) = 0$ and hence $x^* = Tx^*$, because $\widehat{\nu}$ is regular. Thus T has a fixed point.

Let $\alpha(x, y) \geq 1$ for $x, y \in Fix(T)$. Now if $x \neq y$, then from Remark 2.7, we have

$$\widehat{\nu}_\lambda(x, y) = \widehat{\nu}_\lambda(Tx, Ty) < \widehat{\nu}_\lambda(x, y)$$

which is a contradiction. So $x = y$. That is, T has a unique fixed point when $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$. □

The notion of α-${}_{\nu^s}$-Meir-Keeler contractive is defined as follows:

Definition 2.9. Let X_{ν^s} be a modular b-metric space and T a self-mapping on X_{ν^s}. Also, suppose that $\alpha : X_{\nu^s} \times X_{\nu^s} \to [0, +\infty)$. We say that T is α-${}_{\nu^s}$-Meir-Keeler contractive if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \frac{(\nu^s{}_\lambda(x, y))}{s} < s\epsilon + s\delta(\epsilon) \text{ implies } \alpha(x, y)\nu^s{}_\lambda(Tx, Ty) < s\epsilon, \tag{2.7}$$

for any $x, y \in X_{\widehat{\nu}}$ and all $\lambda > 0$.

Using the above definition, we state new fixed point theorems as follows:

Theorem 2.10. *Let X_{ν^s} be a ν^s regular ν^s-complete modular b-metric space and $T : X_{\nu^s} \to X_{\nu^s}$ a self-mapping. Assume that there exists a function $\alpha : X_{\nu^s} \times X_{\nu^s} \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is α-ν^s-Meir-Keeler mapping,
(iii) there exists $x_0 \in X_{\nu^s}$ such that $\alpha(x_0, Tx_0) \geq 1$,
(iv) T is ν^s-continuous mapping.

Then T has a fixed point $z \in X$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, T has a unique fixed point.

Proof. It is sufficient to take $\Omega(t) = st$ where $s \geq 1$ is a real number and $t \geq 0$. □

For a self-mapping which is not $\widehat{\nu}$-continuous, we have the following result.

Theorem 2.11. *Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$ regular $\widehat{\nu}$-complete modular p-metric space and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that there exists a function $\alpha : X_{\widehat{\nu}} \times X_{\widehat{\nu}} \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is α-$\widehat{\nu}$-Meir-Keeler contractive,
(iii) there exists $x_0 \in X_{\widehat{\nu}}$ such that $\alpha(x_0, Tx_0) \geq 1$,
(iv) if $\{x_n\}$ be a sequence in $X_{\widehat{\nu}}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point $z \in X$.

Proof. As in the proof of Theorem 2.10, we deduce that there exists a Picard sequence $\{x_n\}$ starting at x_0 which is $\widehat{\nu}$–Cauchy and so $\widehat{\nu}$–converges to a point $x^* \in X_{\widehat{\nu}}$.

By Remark 2.7, we have

$$\widehat{\nu}_\lambda(x_{n+1}, Tx^*) = \widehat{\nu}_\lambda(Tx_n, Tx^*) \leq \widehat{\nu}_\lambda(x_n, x^*),$$

for all $n \geq 0$. Then $\lim_{n\to+\infty}[\widehat{\nu}_\lambda(x_{n+1}, Tx^*)] = 0$, for all $\lambda > 0$, and hence

$$\widehat{\nu}_2(x^*, Tx^*) \leq \lim_{n\to+\infty} \Omega[\widehat{\nu}_1(x^*, x_{n+1}) + \widehat{\nu}_1(x_{n+1}, Tx^*)] = 0.$$

Thus, we get $x^* = Tx^*$, since $\widehat{\nu}$ is regular. □

We now give an example to support Theorem 2.11.

Example 2.12. Let $X = \mathbb{R}$ be endowed with the modular p-metric

$$\widehat{\nu}_\lambda(x,y) = \begin{cases} \sinh(\frac{|x|+|y|}{\lambda}), & \text{if } x \neq y, \\ \\ 0, & \text{if } x = y. \end{cases}$$

for all $x, y \in X$. Define $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$Tx = \begin{cases} 2x^2 + 1, & \text{if } x \in (-\infty, 0) \\ \\ \frac{1}{16}x^2, & \text{if } x \in [0,1] \\ \\ 3x - 1, & \text{if } x \in (1,2) \\ \\ 6x^{10} & \text{if } x \in [2, +\infty), \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1] \\ \\ 0, & \text{otherwise.} \end{cases}$$

We know that $\Omega(t) = \sinh(t)$. It is obvious that T is a triangular α-admissible mapping. If $\{x_n\}$ is a sequence in $X_{\widehat{\nu}}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to +\infty$, then $x_n \in [0,1]$ for all $n \in \mathbb{N}$ and so $x \in [0,1]$. This ensures that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq 1$.

Let $\alpha(x,y) \geq 1$, $\epsilon \leq \Omega^{-1}(\widehat{\nu}_\lambda(x,y)) < \Omega(\epsilon) + \Omega(\delta)$ where $\epsilon > 0$ is arbitrary and

$$\delta < \sinh^{-1}(16\epsilon - \sinh(\epsilon)).$$

Then $x, y \in [0,1]$. Now let $\frac{|x|+|y|}{\lambda} < \Omega(\epsilon) + \Omega(\delta)$. Hence we have

$$\begin{aligned} \widehat{\nu}_\lambda(Tx, Ty) &= \sinh(\frac{|Tx| + |Ty|}{\lambda}) \\ &= \sinh(\frac{1}{\lambda}(|Tx| + |Ty|)) \\ &= \sinh(\frac{1}{16\lambda}(|x|^2 + |y|^2)) \\ &\leq \sinh(\frac{1}{16\lambda}(|x| + |y|)) \\ &\leq \sinh(\frac{1}{16}(\Omega(\epsilon) + \Omega(\delta))) < \sinh(\epsilon). \end{aligned}$$

Otherwise, $\alpha(x,y) = 0$ and evidently

$$\alpha(x,y)\widehat{\nu}_\lambda(Tx, Ty) < \Omega(\epsilon).$$

That is, T is an α-$\widehat{\nu}$-Meir-Keeler contractive mapping. Thus all the conditions of Theorem 2.11 hold and T has a fixed point.

If we take $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.11, then we have the following result.

Corollary 2.13. *Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$-complete modular p-metric space which is $\widehat{\nu}$ regular and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq \Omega^{-1}(\widehat{\nu}_\lambda(x, y)) < \Omega(\epsilon) + \Omega(\delta(\epsilon)) \text{ implies } \widehat{\nu}_\lambda(Tx, Ty) < \Omega(\epsilon),$$

for any $x, y \in X_{\widehat{\nu}}$ and all $\lambda > 0$. Then T has a unique fixed point $z \in X$.

According to Theorem 2.11, we have the following corollary.

Corollary 2.14. *Let $X_{\widetilde{\nu}}$ be a $\widetilde{\nu}$-complete modular p-metric space which is $\widetilde{\nu}$ regular where*

$$\widetilde{\nu}_\lambda(x, y) = \nu_\lambda(x, y)e^{\nu_\lambda(x,y)}$$

and $T : X_{\widetilde{\nu}} \to X_{\widetilde{\nu}}$ a self-mapping. Assume that for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \Omega^{-1}(\widetilde{\nu}_\lambda(x, y)) < \epsilon e^{\epsilon} + \delta(\epsilon)e^{\delta(\epsilon)} \text{ implies } \widetilde{\nu}_\lambda(Tx, Ty) < \epsilon e^{\epsilon},$$

for any $x, y \in X_{\widetilde{\nu}}$ and all $\lambda > 0$. Then T has a unique fixed point $z \in X$.

Note that in Corollary 2.14, Ω is the Lambert W-function [11].

3. Some Meir-Keeler type fixed point results in EMbM spaces endowed with a graph

As in [21], let $(X_{\widehat{\nu}}, \widehat{\nu})$ be a modular metric space and Δ denotes the diagonal of the Cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

Definition 3.1. [21] Let $(X_{\widehat{\nu}}, \widehat{\nu})$ be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is,

$$\text{for all } x,\ y \in X, \quad (x, y) \in E(G) \text{ implies } (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

There exists $\alpha \in (0, 1)$ such that

$$\text{for all } x, y \in X, (x, y) \in E(G) \text{ implies } d(Tx, Ty) \leq \alpha d(x, y).$$

Definition 3.2. [21] A mapping $T : X \to X$ is called G-continuous if given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

In this section, we will show that many Meir-Keeler type fixed point results in modular metric spaces endowed with a graph G can be deduced easily from our presented theorems.

Definition 3.3. Let $X_{\widehat{\nu}}$ be a modular p-metric space endowed with a graph G and T a self-mapping on $X_{\widehat{\nu}}$. We say that T is an G-$\widehat{\nu}$-Meir-Keeler contractive if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\Omega^{-1}(\epsilon) \leq \widehat{\nu}_\lambda(x, y) < \Omega^{-1}(\epsilon) + \Omega^{-1}(\delta(\epsilon)) \quad \text{and} \quad (x, y) \in E(G)$$

imply $\widehat{\nu}_\lambda(T(x), T(y)) < \Omega^{-1}(\epsilon)$ for any $x, y \in X_w$ and all $\lambda > 0$.

Theorem 3.4. *Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$-complete modular p-metric space endowed with a graph G with $\widehat{\nu}$ regular and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that the following assertions hold:*

(i) there exists $x_0 \in X_{\widehat{\nu}}$ such that, $(x_0, Tx_0) \in E(G)$,
(ii) T is G-continuous,
(iii) for all $x, y \in X_{\widehat{\nu}}[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$,
(iv) for all $x, y, z \in X_{\widehat{\nu}}[(x, y) \in E(G)$ and $(y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$,
(v) T is G-$\widehat{\nu}$-Meir-Keeler contractive.

Then T has a fixed point $z \in X$ such that $\lambda(z) = 0$. Further, if $(x, y) \in E(G)$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let
$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$
If we apply Theorem 2.8, then we have the required result. □

Theorem 3.5. *Let $X_{\widehat{\nu}}$ be a $\widehat{\nu}$-complete modular p-metric space endowed with a graph G with $\widehat{\nu}$ regular and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that the following assertions hold:*

(i) there exists $x_0 \in X_{\widehat{\nu}}$ such that $(x_0, Tx_0) \in E(G)$,
(ii) for all $x, y \in X_{\widehat{\nu}}[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$,
(iii) for all $x, y, z \in X_{\widehat{\nu}}[(x, y) \in E(G)$ and $(y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$,
(iv) T is G-$\widehat{\nu}$-Meir-Keeler contractive mapping,
(v) if $\{x_n\}$ is a sequence in $X_{\widehat{\nu}}$ such that, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point $z \in X$. Further, if $(x, y) \in E(G)$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Consider the following:
$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$
If we apply Theorem 2.11, the proof is completed. □

4. Some Meir-Keeler type fixed point results in EM*b*M spaces endowed with a partial order

The existence of fixed points in partially ordered sets has been considered in [1]. Let $X_{\widehat{\nu}}$ be a nonempty set. If $X_{\widehat{\nu}}$ be a modular p-metric space and $(X_{\widehat{\nu}}, \preceq)$ be a partially ordered set, then $X_{\widehat{\nu}}$ be called a partially ordered modular p-metric space. Two elements $x, y \in X_{\widehat{\nu}}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A mapping $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ is said to be non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X_{\widehat{\nu}}$.

In this section, we will show that many Meir-Keeler type fixed point results in modular metric spaces endowed with a partial order $\preceq$ can be deduced easily from our presented theorems.

Definition 4.1. Let $(X_{\widehat{\nu}}, \preceq)$ be a partially ordered modular p-metric space and T a self-mapping on $X_{\widehat{\nu}}$. We say that T is a $\widehat{\nu}$-Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\Omega^{-1}(\epsilon) \leq \widehat{\nu}_\lambda(x, y) < \Omega^{-1}(\epsilon) + \Omega^{-1}(\delta(\epsilon)) \quad \text{and} \quad x \preceq y$$

imply $\widehat{\nu}_\lambda(T(x), T(y)) < \Omega^{-1}(\epsilon)$ for any $x, y \in X_{\widehat{\nu}}$ and all $\lambda > 0$. Then by Remark 2.7, if partially ordered modular p-metric space is $\widehat{\nu}$-regular, we have $\widehat{\nu}_\lambda(Fx, Fy) \leq \widehat{\nu}_\lambda(x, y)$.

Theorem 4.2. *Let $(X_{\widehat{\nu}}, \preceq)$ be a $\widehat{\nu}$-complete partially ordered modular p-metric space which is $\widehat{\nu}$-regular and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that the following assertions hold:*

(i) there exists $x_0 \in X_\omega$ such that $x_0 \preceq Tx_0$,
(ii) T is $\widehat{\nu}$-continuous,
(iii) T is an increasing mapping,
(iv) T is a partially $\widehat{\nu}$-Meir-Keeler contractive mapping.

Then T has a fixed point $z \in X$. Moreover, if $x \preceq y$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let

$$\alpha(x, y) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise}, \end{cases}$$

and apply Theorem 2.8. □

Theorem 4.3. *Let $(X_{\widehat{\nu}}, \preceq)$ be a $\widehat{\nu}$-complete partially ordered modular p-metric space which is $\widehat{\nu}$-regular and $T : X_{\widehat{\nu}} \to X_{\widehat{\nu}}$ a self-mapping. Assume that the following assertions hold:*

(i) there exists $x_0 \in X_{\widehat{\nu}}$ such that $x_0 \preceq Tx_0$,
(ii) T is $\widehat{\nu}$-continuous,
(iii) T is an increasing mapping,
(iv) T is a partially $\widehat{\nu}$-Meir-Keeler contractive mapping.
(v) if $\{x_n\}$ be an increasing sequence in $X_{\widehat{\nu}}$ with $x_n \to x$ as $n \to \infty$, then we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point $z \in X$. Also, if $x \preceq y$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. If we consider

$$\alpha(x, y) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise}, \end{cases}$$

then the proof is completed by Theorem 2.11. □

5. Relation between extended modular b-metric spaces and extended fuzzy b-metric spaces

Fuzzy metric space was introduced by Kramosil and Michalek [24]. Subsequently, George and Veeramani gave a modified definition of fuzzy metric spaces [15].

Definition 5.1. [15] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 5.2. [15] A 3-tuple $(X, M, *)$ is called a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In [19], Hussain and Salimi presented the relationship between modular metrics and fuzzy metrics and deduced certain fixed point results in triangular partially ordered fuzzy metric spaces.

Definition 5.3. [18] A fuzzy b-metric space is an ordered triple $(X, B, \star)$ such that X is a nonempty set, $\star$ is a continuous t-norm and B is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and for all $t, s > 0$:

(F1) $B(x, y, t) > 0$,
(F2) $B(x, y, t) = 1$ if and only if $x = y$,
(F3) $B(x, y, t) = B(y, x, t)$,
(F4) $B(x, y, t) \star B(y, z, s) \le B(x, z, b(t + s))$ where $b \ge 1$,
(F5) $B(x, y, \cdot) : (0, \infty) \to (0, 1]$ is left-continuous.

Definition 5.4. [19] An extended fuzzy b-metric space is an ordered quadruple $(X, B, \star, \Omega)$ such that X is a nonempty set, $\star$ is a continuous t-norm and B is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and for all $t, s > 0$:

(F1) $B(x, y, t) > 0$,
(F2) $B(x, y, t) = 1$ if and only if $x = y$,
(F3) $B(x, y, t) = B(y, x, t)$,
(F4) $B(x, y, t) \star B(y, z, s) \le B(x, z, \Omega(t + s))$,
(F5) $B(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is left continuous.

Definition 5.5. [19] The extended fuzzy b-metric space $(X, B, *, \Omega)$ is called triangular whenever

$$\frac{1}{B(x, y, t)} - 1 \le \Omega[\frac{1}{B(x, z, t)} - 1 + \frac{1}{B(z, y, t)} - 1]$$

for all $x, y, z \in X$ and for all $t > 0$.

Definition 5.6. [36] Let X be a nonempty set and $b \geq 1$. A mapping $\upsilon : (0,\infty) \times X \times X \to [0,\infty)$ is called a modular b-metric, if for all $x, y, z \in X$ and $\lambda, \mu > 0$, we have the following assertions:

(1) $\upsilon_\lambda(x,y) = 0 \ \ iff \ \ x = y$,

(2) $\upsilon_\lambda(x,y) = \upsilon_\lambda(y,x)$,

(3) $\upsilon_{\lambda+\mu}(x,y) \leq b[\upsilon_\lambda(x,z) + \upsilon_\mu(z,y)]$.

Remark 5.7. [36] Let $(X, B, *)$ be a triangular fuzzy b-metric space. Define $\upsilon : X \times X \times (0,\infty) \to [0,\infty)$ by $\upsilon(x,y,t) = b\Big[\frac{1}{B(x,y,t)} - 1\Big]$. Then υ is a modular b-metric.

Remark 5.8. [17] Let $(X, B, *, \Omega)$ be a triangular extended fuzzy b-metric space. Define the mapping $\nu : X \times X \times (0,\infty) \to [0,\infty)$ by $\nu(x,y,t) = \Omega\Big[\frac{1}{B\big(x,y,t\big)} - 1\Big]$. Then ν is an extended modular b-metric.

Motivated by Remark 2 of [36], we present the following Lemma.

Lemma 5.9. *Let X be a nonempty set and $\upsilon : (0,\infty) \times X \times X \to [0,\infty)$ a modular b-metric for all $x, y \in X$ and $t > 0$. Let $a * c = ac$ for all $a, c \in [0,1]$ and B the fuzzy set on $X \times X \times (0,+\infty)$ defined by*

$$B(x,y,t) = \exp^{-\frac{\upsilon_t(x,y)}{t}}$$

where υ is modular b-metric on set X. Then $(X, B, *)$ is a fuzzy b-metric space.

Proof. It is clear from the definition that $B(x,y,t)$ is well defined for each $x, y \in X$ and $t > 0$.

(i) $B(x,y,t) > 0$ for all $x, y \in X$ and $t > 0$ is trivial.

(ii) $B(x,y,t) = 1 \Leftrightarrow \upsilon_t(x,y) = 0$ for all $t > 0 \Leftrightarrow x = y$.

(iii) $B(x,y,t) = \exp^{-\frac{\upsilon_t(x,y)}{t}} = \exp^{-\frac{\upsilon_t(y,x)}{t}} = B(y,x,t)$.

(iv) Since the function $\lambda \to \upsilon_\lambda(x,y)$ is nonincreasing on $(0,\infty)$, we have

$$\begin{aligned}
B(x,y,b(t+s)) &= \exp^{-\frac{\upsilon_{b(t+s)}(x,y)}{b(t+s)}} \\
&\geq \exp^{-\frac{b[\upsilon_{bt}(x,z)+\upsilon_{bs}(z,y)]}{b(t+s)}} = \exp^{-\frac{\upsilon_{bt}(x,z)+\upsilon_{bs}(z,y)}{t+s}} \\
&\geq \exp^{-\frac{\upsilon_t(x,z)+\upsilon_s(z,y)}{t+s}} \\
&= \exp^{-\frac{\upsilon_t(x,z)}{t+s}} . \exp^{-\frac{\upsilon_s(z,y)}{t+s}} \\
&\geq \exp^{-\frac{\upsilon_t(x,z)}{t}} . \exp^{-\frac{\upsilon_s(z,y)}{s}} \\
&= B(x,z,t) * B(z,y,s)
\end{aligned}$$

This proves that B is a fuzzy b-metric on X. □

Now we define the notion of a $\widehat{B}$-Meir-Keeler contractive mapping as follows.

Definition 5.10. Let $(X_{\widehat{B}}, B, *, \Omega)$ be a triangular extended fuzzy b-metric space and T a self-mapping on $X_{\widehat{B}}$. We say that T is $\widehat{B}$-Meir-Keeler contractive if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \frac{1}{B(x,y,t)} - 1 < \Omega(\epsilon) + \Omega(\delta(\epsilon)) \text{ implies } \frac{1}{B(Tx,Ty,t)} - 1 < \epsilon \tag{5.1}$$

for any $x, y \in X_{\widehat{B}}$ and all $t > 0$.

A fuzzy metric $\widehat{B}$ on X is called regular if

$$x = y \quad \text{if and only if} \quad \widehat{B}(x, y, t) = 1 \quad \text{for some} \quad t > 0.$$

Now it is easy to prove the following theorems for $\widehat{B}$-Meir-Keeler contractive.

Theorem 5.11. *Let $(X_{\widehat{B}}, B, *, \Omega)$ be a $\widehat{B}$-regular $\widehat{B}$-complete fuzzy p-metric space and $T : X_{\widehat{B}} \to X_{\widehat{B}}$ a self-mapping. Assume that the following assertions hold:*

(i) T is a $\widehat{B}$-Meir-Keeler contraction,
(ii) T is a $\widehat{B}$-continuous mapping.

Then T has a unique fixed point $z \in X_{\widehat{B}}$.

Proof. We define $\widehat{\nu}_t(x, y) = \Omega\Big[\frac{1}{B(x,y,t)} - 1\Big]$ for every $x, y \in X_{\widehat{B}}$ where $t > 0$. Then by Remark 5.8, ν is an extended modular b-metric and $X_{\widehat{\nu}}$ is a $\widehat{\nu}$ regular $\widehat{\nu}$-complete modular p-metric space. Hence all of the conditions of Theorem 2.8 hold and T has a unique fixed point $z \in X_{\widehat{B}}$. □

In the next, we define the concept of $\widehat{B}^s$-Meir-Keeler contractive map as follows:

Definition 5.12. Let $X_{\widehat{B}^s}$ be a fuzzy b-metric space and T a self-mapping on $X_{\widehat{B}^s}$. We say that T is $\widehat{B}^s$-Meir-Keeler contractive if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \frac{1}{B(x, y, t)} - 1 < s\epsilon + s\delta(\epsilon) \text{ implies } \frac{1}{B(Tx, Ty, t)} - 1 < \epsilon, \tag{5.2}$$

for any $x,\ y \in X_{\widehat{B}}$ and all $t > 0$.

If we set $\Omega(t) = t$ in Theorem 5.11, we have the following Theorem.

Theorem 5.13. *Let $X_{\widehat{B}^s}$ be a $\widehat{B}^s$-regular $\widehat{B}^s$-complete fuzzy b-metric space and $T : X_{\widehat{B}^s} \to X_{\widehat{B}^s}$ a self-mapping. Assume that the following assertions hold:*

(i) T is $\widehat{B}^s$-Meir-Keeler contractive,
(iv) T is a $\widehat{B}^s$-continuous mapping.

Then T has a unique fixed point $z \in X_{\widehat{B}}$.

6. Existence theorem for solutions of Volterra-type integral equations

Consider the integral equation

$$x(t) = \int_a^b f(t, r, x(r))\, dr, \quad t \in I = [a, b], \tag{6.1}$$

where $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is a given function. The purpose of this section is to provide an existence theorem for solutions of the Eq (6.1) that belongs to $X = C(I, \mathbb{R})$ (the set of continuous real functions defined on I), via the result obtained in Theorem 4.2. With this application, we develop a new and effective approach instead of the classical fixed point viewpoint to the solution of Volterra equations.

We endow X with the partial order $\preceq$ given by

$$x \preceq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for all } t \in I.$$

For $x \in X$ define

$$\|x\|_\infty = \max_{t \in I} |x(t)|.$$

Note that $(X, \|\cdot\|_\infty)$ is a Banach space. The modular metric induced by this norm is given by

$$\nu_\lambda(x, y) = \frac{\|x - y\|_\infty}{\lambda} = \max_{t \in I} \frac{|x(t) - y(t)|}{\lambda},$$

for all $x, y \in X$.

Define $F : X \to X$ by

$$F(x(t)) = \int_a^b f(t, r, x(r))\, dr, \quad x \in X,\ t \in I.$$

Clearly, a function $u \in X$ is a solution of (1.3) if and only if it is a fixed point of F.

We will consider the Eq (1.3) under the following assumptions:

(i) if $x \preceq y$, then

$$f(t, r, x(r)) \leq f(t, r, y(r)), \text{ for all } t, r \in I.$$

(ii) For all $x, y \in X$ with $x \preceq y$, and for all $t \in I$,

$$\int_a^b \Big| f(t, r, x(r)) - f(t, r, y(r)) \Big|\, dr) \leq \frac{\|x - y\|_\infty}{2}.$$

(iii) There exists a continuous function $x_0 : I \to \mathbb{R}$ such that

$$x_0(t) \leq \int_a^b f(t, r, x_0(r))\, dr, \quad t \in I.$$

Theorem 6.1. *Under assumptions (i)–(iii), the Eq* (1.3) *has a solution in X, where* $X = C(I, \mathbb{R})$.

Proof. It follows from (ii) that the mapping F is non-decreasing. Now, let $\epsilon > 0$ be arbitrary and choose $\delta < \frac{\epsilon}{2}$. In this case, if $\widehat{\nu_\lambda}(x, y) < \epsilon + \delta$, then for all $t \in I$,

$$\frac{\big| Fx(t) - Fy(t) \big|}{\lambda} \leq \frac{1}{\lambda} \int_a^b \Big| f(t, r, x(r)) - f(t, r, y(r)) \Big|\, dr \leq \frac{\|x - y\|_\infty}{2\lambda} \leq \frac{\widehat{\nu_\lambda}(x, y)}{2} < \epsilon.$$

Hence, we get that

$$\widehat{\nu_\lambda}(Fx, Fy) \leq \epsilon.$$

Let x_0 be the function appearing in assumption (iii). Then we get $x_0 \preceq F(x_0)$. Thus, all the assumptions of Theorem 4.2 are fulfilled and we deduce the existence of $u \in X$ such that $u = F(u)$. $\square$

7. Conclusions and future works

In this paper, we introduced the concept of extended modular b-metric spaces which induced the notion of extended fuzzy b-metric space. The authors encourage the readers to work on cone versions of these new structures. There are many contractive conditions which can be investigated in these new spaces. The properties of the set $Fix(T)$ also can be considered.

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Conflicts of interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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