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Research article

Some fixed point results on generalized metric spaces

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Abstract: In this paper, generalized metric spaces are introduced as a common generalization of tvs-cone metric spaces, partial metric spaces and b-metric spaces, and a unified approach is proposed to some fixed point results by using generalized metric spaces. Specifically, Banach's contraction principle and Kannan type fixed point theorem, as well as other types fixed point results on generalized metric spaces are given, respectively.

Keywords: fixed point; mapping; complete; generalized metric space; tvs-cone metric space
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1. Introduction

In the past years, many generalizations of metric spaces were introduced and discussed. These generalizations are embodies mainly in two directions: metric value-domains and metric axioms.

For metric value-domains, Du [13] generalized them from the set of all nonnegative real numbers to cones of ordered topological vector spaces. The following two definitions give well-known cone definition and partial orderings on cones respectively (for example, see [13]).

Definition 1.1. *Let E be a topological vector space with its zero vector θ. A subset P of E is called a tvs-cone in E if the following are satisfied.*

(1) P is non-empty and closed in E.

(2) $\alpha, \beta \in P$ and $a, b \in [0, +\infty)$ imply $a\alpha + b\beta \in P$.

(3) $\alpha, -\alpha \in P$ imply $\alpha = \theta$.

Definition 1.2. *Let P be a tvs-cone in a topological vector space E and P° denote the interior of P in E. Some partial orderings $\preceq$, $\prec$ and $\ll$ on E with respect to P are defined as follows, respectively. Let $\alpha, \beta \in E$.*

(1) $\alpha \preceq \beta$ *if* $\beta - \alpha \in P$.
(2) $\alpha \prec \beta$ *if* $\alpha \preceq \beta$ *and* $\alpha \neq \beta$.
(3) $\alpha \ll \beta$ *if* $\beta - \alpha \in P^\circ$.
Then the pair (E, P) *is called an ordered topological vector space.*

Definition 1.3 ([13]). *Let* X *be a non-empty set and* (E, P) *be an ordered topological vector space with its zero vector* θ. *A mapping* $d : X \times X \longrightarrow P$ *is called a tvs-cone metric and* (X, d) *is called a tvs-cone metric space if the following are satisfied for all* $x, y, z \in X$.
(1) $d(x, y) = \theta$ *if and only if* $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, y) \preceq d(x, z) + d(z, y)$.

Definition 1.4 ([12]). *Let* X *be a non-empty set. A mapping* $d : X \times X \longrightarrow [0, +\infty)$ *is called a b-metric with coefficient* $s \geq 1$ *and* (X, d) *is called a b-metric space (with coefficient* $s \geq 1$*) if the following are satisfied for all* $x, y, z \in X$.
(1) $d(x, y) = 0$ *if and only if* $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

Definition 1.5 ([9]). *Let* X *be a non-empty set. A mapping* $p : X \times X \longrightarrow [0, +\infty)$ *is called a partial metric and* (X, d) *is called a partial metric space if the following are satisfied for all* $x, y, z \in X$.
(1) $x = y$ *if and only if* $d(x, x) = d(y, y) = d(x, y)$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, x) \leq d(x, y)$.
(4) $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$.

Recently, these generalizations of metric spaces had aroused popular attentions and some classical fixed point results, including Banach's contraction principle and Kannan type fixed point theorem, as well as the other types fixed point results (e.g. see [33]), had been generalized to these spaces. In particular, many interesting results around (tvs-)cone metric spaces (for example, see [1, 3–6, 13, 15, 20–25, 27, 31]), b-metric spaces (for example, see [8, 10–12, 14, 18, 35–37]) and partial metric spaces (for example, see [2, 7, 9, 19, 28–30, 34, 36]) are obtained. Naturally, it is interesting to propose a unified approach to these fixed point results. For this purpose, the following generalized metric spaces are introduced as a common generalization of tvs-cone metric spaces, b-metric spaces and partial metric spaces.

Definition 1.6. *Let* X *be a non-empty set and* (E, P) *be an ordered topological vector space with its zero vector* θ. *A mapping* $d : X \times X \longrightarrow P$ *is called a generalized metric with coefficient* $s \geq 1$ *and* (X, d) *is called a generalized metric space with coefficient* $s \geq 1$ *if the following are satisfied for all* $x, y, z \in X$.
(1) $x = y$ *if and only if* $d(x, x) = d(y, y) = d(x, y)$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, x) \preceq d(x, y)$.
(4) $d(x, z) \preceq s(d(x, y) + d(y, z)) - d(y, y)$.

Remark 1.7. *(1) Generalized metric spaces in this paper is a common generalization of tvs-cone metric spaces, b-metric spaces and partial metric spaces, which are described as in Definition 1.6 and different from generalized metric spaces in [17].*

(2) Let (X, d) be a generalized metric space. If $x, y \in X$ and $d(x, y) = \theta$, then $x = y$. In fact, $d(x, x) \preceq d(x, y)$ by Definition 1.6(3), so $\theta \preceq d(x, x) \preceq \theta$. It follows that $d(x, x) = \theta$. Similarly, $d(y, y) = \theta$. Consequently, $d(x, x) = d(y, y) = d(x, y)$. By Definition 1.6(1), $x = y$.

(3) For a generalized metric space (X, d), $x = y \in X$ need not imply $d(x, y) = \theta$. In fact, let (E, P) be an ordered topological vector space and $X = \{1, 2\}$. Pick $\alpha \in P^\circ$, then $\alpha \neq \theta$. Put $d(1, 1) = d(1, 2) = d(2, 1) = \alpha$ and $d(2, 2) = \theta$. Then (X, d) is a generalized metric space with coefficient $s = 1$ and $d(1, 1) \neq \theta$.

In this paper, we investigate generalized metric spaces and prove some fixed point theorems on generalized metric spaces. These results give Banach's contraction principle and Kannan type fixed point theorem, as well as other types fixed point results on generalized metric spaces, respectively.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^*$ denote the set of all natural numbers, the set of all real numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively.

2. Ordered topological vector spaces

Remark 2.1 ([27]). *Let (E, P) be an ordered topological vector space.*

(1) It is known that $\theta \in P - P^\circ$, and we always suppose $P^\circ \neq \emptyset$.

(2) For $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_n \in E$, we use notation $\alpha \preceq \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ to denote $\alpha \preceq \alpha_i$ for some $i = 1, 2, \cdots, n$.

(3) For the sake of conveniences, we also use notations "$\succeq$", "$\succ$" and "$\gg$" in (E, P). The meanings of these notations are clear and the following hold.

(a) $\alpha \succeq \beta$ if and only if $\alpha - \beta \succeq \theta$ if and only if $\alpha - \beta \in P$.

(b) $\alpha \succ \beta$ if and only if $\alpha - \beta \succ \theta$ if and only if $\alpha - \beta \in P - \{\theta\}$.

(c) $\alpha \gg \beta$ if and only if $\alpha - \beta \gg \theta$ if and only if $\alpha - \beta \in P^\circ$.

(d) $\alpha \gg \beta$ implies $\alpha \succ \beta$ implies $\alpha \succeq \beta$.

Lemma 2.2 ([27]). *Let (E, P) be an ordered topological vector space. Then the following hold.*

(1) If $\alpha \gg \theta$, then $r\alpha \gg \theta$ for each $r \in \mathbb{R}^+$.

(2) If $\alpha \gg \theta$, then $\alpha \gg \frac{1}{2}\alpha \gg \cdots \gg \frac{1}{n}\alpha \gg \cdots \gg \theta$.

(3) If $\alpha_1 \gg \beta_1$ and $\alpha_2 \succeq \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.

(4) If $\alpha \gg \beta \succeq \gamma$ or $\alpha \succeq \beta \gg \gamma$, then $\alpha \gg \gamma$.

(5) If $\alpha \gg \theta$ and $\beta \in E$, then there is $n \in \mathbb{N}$ such that $\frac{1}{n}\beta \ll \alpha$.

(6) If $\alpha \gg \theta$ and $\beta \gg \theta$, then there is $\gamma \gg \theta$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.

In order to investigate the convergence for sequences in generalized metric spaces, we need to introduce the convergence for sequences in ordered topological vector spaces, which is different from the convergence for sequences in topological vector spaces.

Definition 2.3. *Let (E, P) be an ordered topological vector space, $\{\alpha_n\}$ be a sequence in E and $\alpha \in E$. $\{\alpha_n\}$ is called to converges to α in (E, P) if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ for all $n > n_0$. We denote this by $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$.*

Lemma 2.4. *Let (E,P) be an ordered topological vector space, $\{\alpha_n\}$ be a sequence in E and $\alpha \in E$. If $\lim_{n\to+\infty} \alpha_n = \alpha$, then $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$.*

Proof. Assume that $\lim_{n\to+\infty} \alpha_n = \alpha$. Let $\varepsilon \gg \theta$, i.e., $\varepsilon \in P^\circ$. Then there is a neighborhood U of ε in E such that $U \subseteq P^\circ$. Put $U_1 = \alpha + \varepsilon - U$ and $U_2 = U + \alpha - \varepsilon$, then U_1 and U_2 are neighborhoods of α in E. Since $\{\alpha_n\}$ converges to α, there is $n_0 \in \mathbb{N}$ such that $\alpha_n \in U_1 \bigcap U_2$ for all $n > n_0$. Let $n > n_0$.

(1) Since $\alpha_n \in U_1$, $\alpha_n = \alpha + \varepsilon - \beta_n$ for some $\beta_n \in U$. It follows that $\alpha + \varepsilon - \alpha_n = \beta_n \in U \subseteq P^\circ$. So $\alpha + \varepsilon - \alpha_n \gg \theta$, i.e., $\alpha_n \ll \alpha + \varepsilon$.

(2) Since $\alpha_n \in U_2$, $\alpha_n = \gamma_n + \alpha - \varepsilon$ for some $\gamma_n \in U$. It follows that $\alpha_n - \alpha + \varepsilon = \gamma_n \in U \subseteq P^\circ$. So $\alpha_n - \alpha + \varepsilon \gg \theta$, i.e., $\alpha_n \gg \alpha - \varepsilon$.

By the above (1) and (2), $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ for all $n > n_0$. So $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$. □

Remark 2.5. *In the proof of [20, Lemma 2.4], Z. Kadelburg, S. Radenovic and V. Rakocevic showed that Lemma 2.4 can not be reverted even if (E,P) is an ordered Banach space.*

Lemma 2.6. *Let (E,P) be an ordered topological vector space, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in E, $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$ and $\widehat{\lim_{n\to+\infty}} \beta_n = \beta$. Then $\widehat{\lim_{n\to+\infty}} (\alpha_n \pm \beta_n) = \alpha \pm \beta$.*

Proof. Let $\varepsilon \gg \theta$. Since $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$ and $\widehat{\lim_{n\to+\infty}} \beta_n = \beta$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \frac{\varepsilon}{2} \ll \alpha_n \ll \alpha + \frac{\varepsilon}{2}$ and $\beta - \frac{\varepsilon}{2} \ll \beta_n \ll \beta + \frac{\varepsilon}{2}$ for all $n > n_0$. It follows that $\alpha \pm \beta - \varepsilon \ll \alpha_n \pm \beta_n \ll \alpha \pm \beta + \varepsilon$ for all $n > n_0$. So $\widehat{\lim_{n\to+\infty}} (\alpha_n \pm \beta_n) = \alpha \pm \beta$. □

Lemma 2.7. *Let (E,P) be an ordered topological vector space, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in E. Then the following hold.*

(1) Let $\alpha_n \geq \beta_n$ for all $n \in \mathbb{N}$. If $\widehat{\lim_{n\to+\infty}} \alpha_n = \alpha$ and $\widehat{\lim_{n\to+\infty}} \beta_n = \beta$, then $\alpha \geq \beta$.

(2) Let $\alpha_n \geq \beta_n \geq \gamma_n$ for all $n \in \mathbb{N}$. If $\widehat{\lim_{n\to+\infty}} \alpha_n = \widehat{\lim_{n\to+\infty}} \gamma_n = \alpha$, then $\widehat{\lim_{n\to+\infty}} \beta_n = \alpha$.

Proof. (1) For each $n \in \mathbb{N}$, put $\gamma_n = \alpha_n - \beta_n$, then $\gamma_n \geq \theta$ and $\widehat{\lim_{n\to+\infty}} \gamma_n = \alpha - \beta$ from Lemma 2.6. Put $\gamma = \alpha - \beta$. It suffices to prove that $\gamma \geq \theta$. At first, we claim that if U is a neighborhood of θ, then there is $\varepsilon \gg \theta$ such that $\varepsilon \in U$. In fact, pick $\delta \gg \theta$, then $\lim_{n\to+\infty} \frac{\delta}{n} = \theta$. So there is $n_0 \in \mathbb{N}$ such that $\frac{\delta}{n_0} \in U$. Put $\varepsilon = \frac{\delta}{n_0}$, then $\varepsilon \gg \theta$ and $\varepsilon \in U$. Now we prove that $\gamma \geq \theta$. If not, then $\gamma \notin P$, hence there is a neighborhood V of γ such that $V \bigcap P = \emptyset$ since P is closed. Note that $\widehat{\lim_{n\to+\infty}} \gamma_n = \gamma$ and $\gamma_n \geq \theta$ for all $n \in \mathbb{N}$. For any $\varepsilon \gg \theta$, $\gamma + \varepsilon \gg \gamma_n \geq \theta$ for some $n \in N$, hence $\gamma + \varepsilon \in P$. On the other hand, $V - \gamma$ is a neighborhood of θ. By the above claim, there is $\varepsilon_0 \gg \theta$ such that $\varepsilon_0 \in V - \gamma$. It follows that $\gamma + \varepsilon_0 \in V$, hence $\gamma + \varepsilon_0 \notin P$. This contradicts that $\gamma + \varepsilon \in P$ for any $\varepsilon \gg \theta$.

(2) Let $\varepsilon \gg \theta$. Since $\widehat{\lim_{n\to+\infty}} \alpha_n = \widehat{\lim_{n\to+\infty}} \gamma_n = \alpha$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon \ll \alpha_n \ll \alpha + \varepsilon$ and $\alpha - \varepsilon \ll \gamma_n \ll \alpha + \varepsilon$ for all $n > n_0$. It follows that $\alpha - \varepsilon \ll \beta_n \ll \alpha + \varepsilon$ for all $n > n_0$. So $\widehat{\lim_{n\to+\infty}} \beta_n = \alpha$. □

3. The main results

At first, we give a relation between the convergence for sequences in generalized metric spaces and the convergence for sequences in ordered topological vector spaces.

Definition 3.1. *Let (X, d) be a generalized metric space. A sequence $\{x_n\}$ in X is said to converge to x in (X, d) if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x, x_n) \ll d(x, x) + \varepsilon$ for all $n > n_0$, which is denoted by $\lim\limits_{n\to+\infty} x_n = x$.*

Proposition 3.2. *Let (X, d) be a generalized metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then the following are equivalent.*

(1) $\lim\limits_{n\to+\infty} x_n = x$.

(2) $\widehat{\lim\limits_{n\to+\infty}} d(x, x_n) = d(x, x)$.

Proof. (1) $\Longrightarrow$ (2): Assume that $\lim\limits_{n\to+\infty} x_n = x$. Let $\varepsilon \gg \theta$. Then there is $n_0 \in \mathbb{N}$ such that $d(x, x_n) \ll d(x, x)+\varepsilon$ for all $n > n_0$. It follows that $d(x, x)-\varepsilon \ll d(x, x) \preceq d(x, x_n) \ll d(x, x)+\varepsilon$. So $\widehat{\lim\limits_{n\to+\infty}} d(x, x_n) = d(x, x)$.

(2) $\Longrightarrow$ (1): Assume that $\widehat{\lim\limits_{n\to+\infty}} d(x, x_n) = d(x, x)$. Let $\varepsilon \gg \theta$. Then there is $n_0 \in \mathbb{N}$ such that $d(x, x) - \varepsilon \ll d(x, x_n) \ll d(x, x) + \varepsilon$ for all $n > n_0$. So $\lim\limits_{n\to+\infty} x_n = x$. □

Definition 3.3 ([36]). *Let (X, d) be a generalized metric space and $\{x_n\}$ be a sequence in X.*

(1) $\{x_n\}$ is called a Cauchy sequence in (X, d) if there is $\alpha \in E$, such that $\widehat{\lim\limits_{n,m\to+\infty}} d(x_n, x_m) = \alpha$, i.e., for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\alpha - \varepsilon \ll d(x_n, x_m) \ll \alpha + \varepsilon$ for all $n, m > n_0$.

(2) (X, d) is called to be complete if for each Cauchy sequence $\{x_n\}$, there is $x \in X$ such that $d(x, x) = \widehat{\lim\limits_{n\to+\infty}} d(x, x_n) = \widehat{\lim\limits_{n,m\to+\infty}} d(x_n, x_m)$.

Definition 3.4. *Let (X, d) be a generalized metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping. $x \in X$ is called a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $Fix(T)$ and cardinal of $Fix(T)$ by $|Fix(T)|$.*

Now we give Banach's contraction principleon generalized metric spaces.

Theorem 3.5. *Let (X, d) be a complete generalized metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that $d(Tx, Ty) \preceq \lambda d(x, y)$ for all $x, y \in X$, where $\lambda \in [0, 1)$. Then T has a unique fixed point $x \in X$ and $d(x, x) = \theta$.*

Proof. Pick $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} \leq \lambda/s < 1$. Write $k = \lambda/s$ and put $F = T^{n_0}$. It is clear that $d(Fx, Fy) = d(T^{n_0}x, T^{n_0}y) \preceq \lambda^{n_0} d(x, y) \preceq kd(x, y)$ for all $x, y \in X$.

Claim 1: If $Fix(F) \neq \emptyset$, then $|Fix(F)| = 1$.

Let $Fix(F) \neq \emptyset$. If $x, y \in Fix(F)$, i.e., $x, y \in X$, $Fx = x$ and $Fy = y$, then $d(x, y) = d(Fx, Fy) \preceq kd(x, y)$. If $d(x, y) \neq \theta$, then $d(x, y) \succ \theta$, hence $d(x, y) \preceq kd(x, y) \prec d(x, y)$. This is a contradiction. So $d(x, y) = \theta$. It follows that $x = y$ from Remark 1.7(2). This shows that $|Fix(F)| = 1$.

Claim 2: There is $x \in Fix(F)$ such that $d(x, x) = \theta$.

Pick $x_0 \in X$ and put $x_n = Fx_{n-1}$ for each $n \in \mathbb{N}$. Without loss of generality, we assume that for all $i, j \in \mathbb{N}$ and $i \neq j$, $x_i \neq x_j$, and so $d(x_i, x_j) \succ \theta$. Note that $d(x_1, x_2) = d(Fx_0, Fx_1) \preceq kd(x_0, x_1)$

and $d(x_2, x_3) = d(Fx_1, Fx_2) \preceq kd(x_1, x_2) \preceq k^2 d(x_0, x_1)$. By induction, $d(x_n, x_{n+1}) \preceq k^n d(x_0, x_1)$ for each $n \in \mathbb{N}$. Let $m \in \mathbb{N}$, then

$$\begin{aligned}
&\theta \preceq d(x_n, x_{n+m})\\
&\preceq s(d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} d(x_{n+m-2}, x_{n+m-1} + s^{m-1} d(x_{n+m-1}, x_{n+m})\\
&\preceq sk^n(d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + \cdots + s^{m-1} k^{n+m-2} d(x_0, x_1) + s^{m-1} k^{n+m-1} d(x_0, x_1)\\
&\preceq \frac{\lambda^n}{s^{n-1}} d(x_0, x_1) + \frac{\lambda^{n+1}}{s^{n-1}} d(x_0, x_1) + \cdots + \frac{\lambda^{n+m-2}}{s^{n-1}} d(x_0, x_1) + \frac{\lambda^{n+m-1}}{s^n} d(x_0, x_1)\\
&\preceq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{n+m-1} d(x_0, x_1)\\
&= (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+m-1}) d(x_0, x_1)\\
&\preceq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).
\end{aligned}$$

Since $\lambda \in [0, 1)$, $\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} = 0$, and hence $\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} d(x_0, x_1) = \theta$. By Lemma 2.4 and Lemma 2.7(2), $\widehat{\lim_{n,m\to+\infty}} d(x_n, x_m) = \theta$. So $\{x_n\}$ is a Cauchy sequence in (X, d). It follows that there is $x \in X$ such that $d(x, x) = \widehat{\lim_{n\to+\infty}} d(x, x_n) = \widehat{\lim_{n\to+\infty}} d(x_n, x_n) = \theta$ by the completeness of (X, d). Furthermore, $\theta \preceq d(x_n, Fx) = d(Fx_{n-1}, Fx) \preceq kd(x_{n-1}, x)$. By Lemma 2.7(2), $\widehat{\lim_{n\to+\infty}} d(x_n, Fx) = \theta$. It follows that $\theta \preceq d(x, Fx) \preceq s(d(x, x_n) + d(x_n, Fx)) - d(x_n, x_n)$. By Lemma 2.6 and Lemma 2.7(2), $d(x, Fx) = \theta$. By Remark 1.7(2), $x = Fx$, i.e., x is a fixed point for F. This proves that $x \in Fix(F)$ and $d(x, x) = \theta$.

Claim 3: $x \in Fix(T)$ and $|Fix(T)| = 1$.

It is clear that $F(Tx) = T(Fx) = Tx$. So Tx is also a fixed point of F, i.e., $Tx \in Fix(F)$. By Claim 1 and Claim 2, $Tx = x$. This proves that x is the fixed point of T, i.e., $x \in Fix(T)$. Note that $Fix(T) \subseteq Fix(F)$. $|Fix(T)| = 1$ from Claim 1.

By Claim 2 and Claim 3, T has a unique fixed point $x \in X$ and $d(x, x) = \theta$. □

The following theorem gives a Kannan type [26] fixed point result on generalized metric spaces.

Theorem 3.6. *Let (X, d) be a complete generalized metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that $d(Tx, Ty) \preceq \lambda(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$ and $\lambda s < 1$. Then T has a unique fixed point $x \in X$ and $d(x, x) = \theta$.*

Proof. We complete the proof by the following three claims.

Claim 1: If $x \in Fix(T)$, then $d(x, x) = \theta$.

Let x be a fixed point of T, i.e., $x \in X$ and $Tx = x$. If $d(x, x) \neq \theta$, then $d(x, x) \succ \theta$. Since $2\lambda < 1$, $d(x, x) = d(Tx, Tx) \preceq \lambda(d(x, Tx) + d(x, Tx)) = 2\lambda d(x, Tx) = 2\lambda d(x, x) \prec d(x, x)$. This is a contradiction. So $d(x, x) = \theta$.

Claim 2: If $Fix(T) \neq \emptyset$, then $|Fix(T)| = 1$.

Let $Fix(T) \neq \emptyset$. If $x, y \in Fix(T)$, i.e., $x, y \in X$, $Tx = x$ and $Ty = y$. By Claim 1, $d(x, x) = d(y, y) = \theta$. It follows that $d(x, y) = d(Tx, Ty) \preceq \lambda(d(x, Tx) + d(y, Ty)) = \lambda(d(x, x) + d(y, y)) = \theta$. So $x = y$ from Remark 1.7(2). This shows that $|Fix(T)| = 1$.

Claim 3: There is $x \in Fix(T)$.

Pick $x_0 \in X$ and put $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Without loss of generality, we assume that for all $i, j \in \mathbb{N}$ and $i \neq j$, $x_i \neq x_j$, and so $d(x_i, x_j) \succ \theta$. For each $n \in \mathbb{N}$, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \lambda(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)) = \lambda(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$, and hence

$d(x_n, x_{n+1}) \preceq \mu d(x_{n-1}, x_n)$, where $\mu = \dfrac{\lambda}{1-\lambda} < 1$. It is easy to see that $\theta \preceq d(x_n, x_{n+1}) \preceq \mu^n d(x_0, x_1)$ for each $n \in \mathbb{N}$. Since $\lim_{n\to+\infty} \mu^n = 0$, $\lim_{n\to+\infty} \mu^n d(x_0, x_1) = \theta$. By Lemma 2.4 and Lemma 2.7(2), $\widehat{\lim_{n\to+\infty}} d(x_n, x_{n+1}) = \theta$. Let $n, m \in \mathbb{N}$. Then $\theta \preceq d(x_n, x_m) = d(Tx_{n-1}, Tx_{m-1}) \preceq \lambda(d(x_{n-1}, Tx_{n-1}) + d(x_{m-1}, Tx_{m-1})) = \lambda(d(x_{n-1}, x_n) + d(x_{m-1}, x_m))$. Since $\widehat{\lim_{n,m\to+\infty}} \lambda(d(x_{n-1}, x_n) + d(x_{m-1}, x_m)) = \theta$, $\widehat{\lim_{n,m\to+\infty}} d(x_n, x_m) = \theta$ from Lemma 2.7(2). So $\{x_n\}$ is a Cauchy sequence in (X, d). It follows that there is $x \in X$ such that $d(x, x) = \widehat{\lim_{n\to+\infty}} d(x, x_n) = \widehat{\lim_{n\to+\infty}} d(x_n, x_n) = \theta$ from the completeness of (X, d). By $d(x, Tx) \preceq sd(x, x_n) + sd(x_n, Tx)$ and $\theta \preceq d(x_n, Tx) = d(Tx_{n-1}, Tx) \preceq \lambda(d(x_{n-1}, Tx_{n-1}) + d(x, Tx)) = \lambda(d(x_{n-1}, x_n) + d(x, Tx))$, we have $d(x, Tx) \preceq sd(x, x_n) + s\lambda(d(x_{n-1}, x_n) + d(x, Tx))$. By Lemma 2.7(1), $\widehat{\lim_{n\to+\infty}} d(x, Tx) \preceq \widehat{\lim_{n\to+\infty}} (sd(x, x_n) + s\lambda(d(x_{n-1}, x_n) + d(x, Tx)))$, i.e., $d(x, Tx) \preceq s\lambda d(x, Tx)$. If $d(x, Tx) \neq \theta$, then $d(x, Tx) \succ \theta$. Note that $s\lambda < 1$. So $d(x, Tx) \prec s\lambda d(x, Tx)$. This is a contradiction. So $d(x, Tx) = \theta$. It follows that $Tx = x$ from Remark 1.7(2), i.e., x is the fixed point of T. □

The following theorem gives a fixed point result on generalized metric spaces, which generalizes [36, Theorem 3] from partial b-metric spaces to generalized metric spaces.

Theorem 3.7. *Let (X, d) be a complete generalized metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that $d(Tx, Ty) \preceq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s})$. Then T has a unique fixed point $x \in X$ and $d(x, x) = \theta$.*

Proof. We complete the proof by the following two claims.

Claim 1: If $Fix(T) \neq \emptyset$, then $|Fix(T)| = 1$.

Let $Fix(T) \neq \emptyset$. If $x, y \in Fix(T)$, i.e., $x, y \in X$, $Tx = x$ and $Ty = y$, then $d(x, y) = d(Tx, Ty) \preceq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty)\} = \lambda \max\{d(x, y), d(x, x), d(y, y)\} = \lambda d(x, y)$. It follows that $d(x, y) = 0$ since $\lambda < 1$. By Remark 1.7(2), $x = y$. So $|Fix(T)| = 1$.

Claim 2: There is $x \in Fix(T)$ such that $d(x, x) = \theta$.

Pick $x_0 \in X$ and put $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Without loss of generality, we assume that for all $i, j \in \mathbb{N}$ and $i \neq j$, $x_i \neq x_j$, and so $d(x_i, x_j) \succ \theta$. For each $n \in \mathbb{N}$, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \lambda \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} = \lambda \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$. It follows that $d(x_n, x_{n+1}) \preceq \lambda d(x_n, x_{n+1})$ or $d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n)$. If $d(x_n, x_{n+1}) \preceq \lambda d(x_n, x_{n+1})$, then $d(x_n, x_{n+1}) \prec d(x_n, x_{n+1})$. This is a contradiction. So $d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) \preceq \lambda^n d(x_0, x_1)$. Note that $0 \leq s\lambda < 1$. Let $n, m \in \mathbb{N}$, then

$$
\begin{aligned}
&\theta \preceq d(x_n, x_{n+m}) \\
&\preceq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^{m-1} d(x_{n+m-1}, x_{n+m}) \\
&\preceq s\lambda^n (d(x_0, x_1) + s^2\lambda^{n+1} d(x_0, x_1) + \cdots + s^{m-1}\lambda^{n+m-2} d(x_0, x_1) + s^m \lambda^{n+m-1} d(x_0, x_1) \\
&\preceq (s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-1}\lambda^{n+m-2} + s^m\lambda^{n+m-1}) d(x_0, x_1). \\
&\preceq \frac{s\lambda^n}{1 - s\lambda} d(x_0, x_1).
\end{aligned}
$$

Since $0 \leq \lambda \leq s\lambda < 1$, $\lim_{n\to+\infty} \dfrac{s\lambda^n}{1 - s\lambda} = 0$, and hence $\lim_{n\to+\infty} \dfrac{s\lambda^n}{1 - s\lambda} d(x_0, x_1) = \theta$. By Lemma 2.4 and

Lemma 2.7(2), $\widehat{\lim_{n,m\to+\infty}} d(x_n, x_m) = \theta$. So $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there is $x \in X$ such that $d(x, x) = \widehat{\lim_{n\to+\infty}} d(x, x_n) = \widehat{\lim_{n\to+\infty}} d(x_n, x_n) = \theta$. It is clear that $d(x_n, Tx) = d(Tx_{n-1}, Tx) \leq \lambda \max\{d(x_{n-1}, x), d(x_{n-1}, Tx_{n-1}), d(x, Tx)\} = \lambda \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, Tx)\}$. Therefore, $d(x, Tx) \leq s(d(x, x_n) + d(x_n, Tx)) \leq s(d(x, x_n) + \lambda \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, Tx)\})$. Thus, $\lim_{n\to+\infty} d(x, Tx) \leq \lim_{n\to+\infty} s(d(x, x_n) + \lambda \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, Tx)\})$ by Lemma 2.7(1), and so $d(x, Tx) \leq s\lambda d(x, Tx)$. Since $s\lambda < 1$, $d(x, Tx) = \theta$. By Remark 1.7(2), $x = Tx$. This proves that $x \in Fix(T)$ and $d(x, x) = \theta$. □

As an application of Theorem 3.7, the following corollary generalizes a fixed point result in [32] from metric spaces to generalized metric spaces.

Corollary 3.8. *Let (X, d) be a complete generalized metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty)$ for all $x, y \in X$, where $\lambda_1 + \lambda_2 + \lambda_3 \in [0, \frac{1}{s})$. Then T has a unique fixed point $x \in X$ and $d(x, x) = \theta$.*

Proof. Put $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, then $\lambda \in [0, \frac{1}{s})$. For all $x, y \in X$, $d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) \leq \lambda_1 \max\{d(x, y), d(x, Tx), d(y, Ty)\} + \lambda_2 \max\{d(x, y), d(x, Tx), d(y, Ty)\} + \lambda_3 \max\{d(x, y), d(x, Tx), d(y, Ty)\} = (\lambda_1 + \lambda_2 + \lambda_3) \max\{d(x, y), d(x, Tx), d(y, Ty)\} = \lambda \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. By Theorem 3.7, T has a unique fixed point $x \in X$ and $d(x, x) = \theta$. □

4. Some examples

In this section, we give some examples to verify our results. The following Lemma is similar to [36, Example 1], we omit its proof.

Lemma 4.1. *Let (E, P) be a ordered topological vector space and $\varepsilon \in P^\circ$. Put $X = \mathbb{R}^*$, where $\mathbb{R}^*$ is the set of all nonnegative real numbers. For $n \in \mathbb{N}$, define $d_n : X \times X \longrightarrow P$ by $d_n(x, y) = ((\max\{x, y\})^n + |x - y|^n)\varepsilon$. Then (X, d_n) is a generalized metric space with coefficient $s = 2^{n-1}$.*

The following example verifies Theorem 3.5.

Example 4.2. *Let $E = \{(x, y) : x, y \in \mathbb{R}\}$ and $P = \{(x, y) : x, y \in \mathbb{R}^*\}$. Then (E, P) is an ordered topological vector space. Put $X = \{0, 1, 2\}$. Define $d : X \times X \longrightarrow P$ by $d(x, y) = (\max\{x, y\} + |x - y|)\varepsilon$, where $\varepsilon = (1, 1) \in P^\circ$. Then*

$$d(0,0) = \theta,\ \ d(1,1) = \varepsilon,\ \ d(2,2) = 2\varepsilon,$$

$$d(0,1) = 2\varepsilon,\ d(0,2) = 4\varepsilon,\ d(1,2) = 3\varepsilon.$$

Put a mapping $T : X \longrightarrow X$ by $T0 = T1 = 0$ and $T2 = 1$. Then

$$d(T0, T0) = d(0,0) = \theta,\ \ d(T1, T1) = d(0,0) = \theta,\ \ d(T2, T2) = d(1,1) = \varepsilon,$$

$$d(T0, T1) = d(0,0) = \theta,\ \ d(T0, T2) = d(0,1) = 2\varepsilon,\ d(T1, T2) = d(0,1) = 2\varepsilon.$$

(1) By Lemma 4.1, (X, d) is a generalized metric space with coefficient $s = 2^{1-1} = 1$, which is a partial tvs-cone metric space in the sense of [16]. Obviously, (X, d) is complete.

(2) It is not difficult to check that $d(Tx, Ty) \preceq \frac{2}{3} d(x, y)$ for all $x, y \in X$.

(3) By the above (1), (2) and Theorem 3.5, T has a unique fixed point $x \in X$ with $d(x, x) = \theta$. In fact, $T0 = 0$ and $d(0, 0) = \theta$. In addition, $T1 \neq 1$, $T2 \neq 2$.

However, the mapping T in Example 4.2 does not satisfy condition in Theorem 3.6. We give the following example to verify Theorem 3.6.

Example 4.3. *Let (E, P) be the ordered topological vector space described in Example 4.2. Put $X = \{0, 1, 2\}$. Define $d : X \times X \longrightarrow P$ by $d(x, y) = ((\max\{x, y\})^2 + |x - y|^2)\varepsilon$, where $\varepsilon = (1, 1) \in P^\circ$. Then*

$$d(0, 0) = \theta, \quad d(1, 1) = \varepsilon, \quad d(2, 2) = 4\varepsilon,$$

$$d(0, 1) = 2\varepsilon, \; d(0, 2) = 8\varepsilon, \; d(1, 2) = 5\varepsilon.$$

Put $T : X \longrightarrow X$ is the mapping described in Example 4.2. Then

$$d(T0, T0) = d(0, 0) = \theta, \quad d(T1, T1) = d(0, 0) = \theta, \quad d(T2, T2) = d(1, 1) = \varepsilon,$$

$$d(T0, T1) = d(0, 0) = \theta, \quad d(T0, T2) = d(0, 1) = 2\varepsilon, \; d(T1, T2) = d(0, 1) = 2\varepsilon,$$

$$d(0, T0) = d(0, 0) = \theta, \quad d(1, T1) = d(1, 0) = 2\varepsilon, \; d(2, T2) = d(2, 1) = 5\varepsilon.$$

In addition, we have

$$d(0, T0) + d(0, T0) = \theta, \; d(1, T1) + d(1, T1) = 4\varepsilon, \; d(2, T2) + d(2, T2) = 10\varepsilon,$$

$$d(0, T0) + d(1, T1) = 2\varepsilon, \; d(0, T0) + d(2, T2) = 5\varepsilon, \; d(1, T1) + d(2, T2) = 7\varepsilon,$$

(1) By Lemma 4.1, (X, d) is a generalized metric space with coefficient $s = 2^{2-1} = 2$. Obviously, (X, d) is complete.

(2) It is not difficult to check that $d(Tx, Ty) \preceq \frac{2}{5}(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$. In addition, $\frac{2}{5} \in [0, \frac{1}{2})$ and $\frac{2}{5}s < 1$ since $s = 2$.

(3) By the above (1), (2) and Theorem 3.6, T has a unique fixed point $x \in X$ with $d(x, x) = \theta$. In fact, $T0 = 0$ and $d(0, 0) = \theta$. In addition, $T1 \neq 1$, $T2 \neq 2$.

Remark 4.4. *In Example 4.3,*

$$\max\{d(0, 0), d(0, T0), d(0, T0)\} = \theta, \; \max\{d(1, 1), d(1, T1), d(1, T1)\} = 2\varepsilon,$$

$$\max\{d(2, 2), d(2, T2), d(2, T2)\} = 5\varepsilon, \; \max\{d(0, 1), d(0, T0), d(1, T1)\} = 2\varepsilon,$$

$$\max\{d(0, 2), d(0, T0), d(2, T2)\} = 8\varepsilon, \; \max\{d(1, 2), d(1, T1), d(2, T2)\} = 5\varepsilon.$$

It is not difficult to check that $d(Tx, Ty) \preceq \frac{2}{5} \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$. In addition, $\frac{2}{5} \in [0, \frac{1}{s})$ since $s = 2$. So Example 4.3 also verifies Theorem 3.7.

Remark 4.5. *Theorems 3.5, 3.6 and 3.7 give some generalizations of Banachs contraction principle, Kannan type fixed point theorem [26] and a fixed point result of Shukla [36] on generalized metric spaces, respectively.*

(1) In Example 4.2, the mapping $T : X \longrightarrow X$ satisfies the condition of Theorems 3.5. By applying Theorem 3.5, we can gets that T has a fixed point. However, T does not satisfy the condition of classical Banachs contraction principle. So, in this case, Banachs contraction principle can not be applied in Example 4.2 to get that T has a fixed point.

(2) In Example 4.3, the mapping $T : X \longrightarrow X$ satisfies the condition of Theorems 3.6. By applying Theorem 3.6, we can gets that T has a fixed point. However, T does not satisfy the condition of important Kannan type fixed point theorem. So, in this case, Kannan type fixed point theorem can not be applied in Example 4.3 to get that T has a fixed point.

(2) Remark 4.4 illustrates that the mapping $T : X \longrightarrow X$ in Example 4.3 satisfies the condition of Theorems 3.7. By applying Theorem 3.7, we can gets that T has a fixed point. However, T does not satisfy the condition of [36, Theorem 3]. So, in this case, [36, Theorem 3] can not be applied in Remark 4.4 to get that T has a fixed point.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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