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Research article

On the inverse problems associated with subsequence sums of zero-sum free sequences over finite abelian groups II

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Abstract: Let G be an additive finite abelian group with exponent $\exp(G)$ and S be a sequence with elements of G. Let $\Sigma(S) \subset G$ denote the set of group elements which can be expressed as the sum of a nonempty subsequence of S. We say S is zero-sum free if $0 \notin \Sigma(S)$. In this paper, we determine the structures of the zero-sum free sequences S such that $|S| = \exp(G) + 2$ and $|\Sigma(S)| = 4\exp(G) - 1$, which partly confirms a conjecture of J. Peng et al.

Keywords: abelian groups; Davenport constant; inverse problems; subsequence sums; zero-sum free sequences
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1. Introduction

Let C_n denote the cyclic group of n elements. Every finite abelian group G can be written in the form $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$. We call $\exp(G) = n_r$ the exponent of G. Let $\mathrm{ord}(g)$ denote the order of $g \in G$. We consider sequences over G as elements in the free abelian monoid with basis G. So a sequence S over G can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N} \cup \{0\}$ denotes the *multiplicity* of g in S. We call $|S| = \ell = \sum_{g\in G} \mathsf{v}_g(S) \in \mathbb{N} \cup \{0\}$ the *length* of S, and $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$ the *sum* of S.

A sequence T is called a *subsequence* of S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$. Whenever T is a subsequence of S, let ST^{-1} denote the subsequence with T deleted from S. If S_1 and S_2 are two sequences over G, let S_1S_2 denote the sequence satisfying that $\mathsf{v}_g(S_1S_2) = \mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)$ for all $g \in G$. Let

$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \le |T| \le |S|\}.$$

The sequence S is called

- *a set* if $\mathsf{v}_g(S) \le 1$ for every $g \in G$,
- *zero-sum* if $\sigma(S) = 0 \in G$,
- *zero-sum free* if $0 \notin \Sigma(S)$,
- *minimal zero-sum* if $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every subsequence T of S with $1 \le |T| < |S|$.

For an abelian group G, let $\mathsf{D}(G)$ denote the smallest integer $\ell \in \mathbb{N}$ such that $0 \in \Sigma(S)$ for every sequence S over G of length ℓ. We call $\mathsf{D}(G)$ the *Davenport constant* of G. We remark that the maximal length of zero-sum free sequence over G is $\mathsf{D}(G) - 1$. The Davenport constant of an abelian group G is one of the starting point of Zero-sum Theory. An interesting problem associated with Davenport constant is what can we say about S over an abelian group G if $0 \notin \Sigma(S)$.

In 1972, R.B. Eggleton and P. Erdős [2] first studied the problem of determining $|\Sigma(S)|$ for zero-sum free sequences S of a finite abelian group. Since then, this problem attracts many authors including J.E. Olson [10], J.E. Olson and E.T. White [11], W. Gao et al. [6], A. Pixton [18], P. Yuan [23], P. Yuan and X. Zeng [24], Y. Qu et al. [19], J. Peng et al. [14] (see [3] and [15] for some recent progress).

Let G be a finite abelian group. For every positive integer $r \in \mathbb{N}$, let

$$\mathsf{f}_G(r) = \min\{|\Sigma(S)| \mid S \text{ is a zero-sum free sequence over } G \text{ with length } |S| = r\}.$$

If G contains no zero-sum free sequence of length r, we set $\mathsf{f}_G(r) = \infty$.

The invariant $\mathsf{f}_G(r)$ was first introduced by W. Gao and I. Leader [4]. They proved that $\mathsf{f}_G(r) = r$ if $1 \le r \le \exp(G) - 1$ and $\mathsf{f}_G(\exp(G)) = 2\exp(G) - 1$ provided that $\gcd(\exp(G), 6) = 1$. The latter result partly confirmed a case of the following conjecture stated by B. Bollobás and I. Leader in 1999 [1].

Conjecture 1.1. *[1, Conjecture 6] Let $G = C_n \oplus C_n$ with $n \ge 2$ and $0 \le k \le n - 2$ be an integer. Let $\{e_1, e_2\}$ be a basis of G and $S = e_1^{n-1} e_2^{k+1}$. Then $\mathsf{f}_G(n + k) = |\Sigma(S)| = (k + 2)n - 1$.*

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \ge 2$ and $1 < n_1 \mid \ldots \mid n_r$. In 2007, F. Sun [20] proved that $\mathsf{f}_G(n_r) = 2n_r - 1$, which implies that Conjecture 1.1 holds when $k = 0$. In 2008, W. Gao et al. [6] proved that Conjecture 1.1 holds when $k = 1$ by showing that $\mathsf{f}_G(n_r + 1) = 3n_r - 1$ provided that $n_{r-1} \ge 3$. They also confirmed Conjecture 1.1 when $k = n - 2$ and generalized Conjecture 1.1 as follows.

Conjecture 1.2. *[6, Conjecture 6.2] Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \ge 2$ and $1 < n_1 \mid \ldots \mid n_r$. Let $0 \le k \le n_{r-1} - 2$ be an integer. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis of G with $\mathrm{ord}(e_i) = n_i$ for all $i \in [1, r]$ and $S = e_r^{n_r - 1} e_{r-1}^{k+1}$. Then $\mathsf{f}_G(n_r + k) = |\Sigma(S)| = (k + 2)n_r - 1$.*

In 2009, P. Yuan [23] proved that $\mathsf{f}_G(n_r + 2) = 4n_r - 1$ provided that $n_{r-1} \ge 4$, which implies that Conjecture 1.2 and also Conjecture 1.1 hold for the case when $k = 2$. Recently, J. Peng et al. [16] confirmed Conjecture 1.2 and Conjecture 1.1 for the case when $k = 3$ by showing that $\mathsf{f}_G(n_r+3) = 5n_r-1$ provided that $n_{r-1} \ge 5$.

The inverse problem associated with $|\Sigma(S)|$ is to determine the structure of the sequence S over G with the given length such that $\Sigma(S)$ archives the minimal cardinality (see [9, 12, 22] for more known results). Our main motivation is the following conjecture suggested by J. Peng et al. in 2020 [15].

Conjecture 1.3. *[15, Conjecture 2.4] Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 \mid \ldots \mid n_r$. Let $k \in [0, n_{r-1} - 2]$ be an integer and S be a zero-sum free sequence over G of length $|S| = n_r + k$. Then $|\Sigma(S)| \geq (k+2)n_r - 1$, and the equality holds if and only if S has one of the following forms.*

(1) $\langle S \rangle \cong C_{k+2} \oplus C_{n_r}$*, where* $k + 2 \mid n_r$*;*
(2) $S = g^{n_r - 1} \cdot (h + t_1 g) \cdot \ldots \cdot (h + t_{k+1} g)$*, where* $g, h \in G$ *with* $\mathrm{ord}(g) = n_r$*,* $ih \notin \langle g \rangle$ *for every* $i \in [1, k+1]$*, and* $t_1, \ldots, t_{k+1} \in [0, n_r - 1]$ *are integers.*

Conjecture 1.3 has been verified for the following cases

1. $k = 0, 1$; [15, Theorem 2.3]
2. $G = C_n \oplus C_n$ and $k = n - 3$; [21, Theorem 1.3]
3. $G = C_n \oplus C_{nm}$ and $k = n - 3$; [13, Theorem 1.5]
4. $\mathsf{v}_g(S) \geq n_r - 1$ for some elements $g \in G$; [13, Theorem 1.4]
5. $G = C_p \oplus C_p$. [17, Theorem 1.5]

In this paper we prove the following results.

Theorem 1.4. *Let $G = C_{n_1} \oplus C_{n_2}$ be a finite abelian group with $1 < n_1 \mid n_2$. Let $k \in [0, n_1 - 2]$ be a positive integer and $S = S_1 S_2$ be a zero-sum free sequence over G of length $|S| = n_2 + k$, where $N = \langle S_1 \rangle$ is a cyclic subgroup of G. Let $\varphi : G \to G/N$ denote the canonical epimorphism. Suppose*

$$|S_1| \geq k + 1, \text{ and } q = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))| > |S_2|.$$

Then $|\Sigma(S)| \geq (k+2)n_2 - 1$. Moreover, the equality holds if and only if S has one of the following forms:

(1) $\langle S \rangle \cong C_{k+2} \oplus C_{n_2}$*, where* $k + 2 \mid n_2$*;*
(2) There exist $g, h \in G$ *such that* $S = g^{n_2 - 1} \cdot (h + t_1 g) \cdot \ldots \cdot (h + t_{k+1} g)$*, where* $\mathrm{ord}(g) = n_2$*,* $ih \notin \langle g \rangle$ *for* $i \in [1, k+1]$*, and* $t_1, t_2, \ldots, t_{k+1} \in [0, n_2 - 1]$ *are integers.*

Theorem 1.5. *Conjecture 1.3 is true when $k = 2$.*

The paper is organized as follows. Section 2 provides some preliminary results. In Section 3 we prove our main results. In the last section, we give some further results.

2. Preliminary results

We provide some preliminary results in this section, beginning with the famous Davenport constant.

Lemma 2.1. *[8, Theorem 5.8.3] Let $G = C_{n_1} \oplus C_{n_2}$ with $1 \leq n_1 \mid n_2$. Then $\mathsf{D}(G) = n_1 + n_2 - 1$.*

Next five lemmas provide a few results on $|\Sigma(S)|$ for zero-sum free sequence S.

Lemma 2.2. *[15, Lemma 3.7] Let G be a finite abelian group and let S be a zero-sum free sequence over G. Then $|\Sigma(S)| \geq |S|$ and the equality holds if and only if $S = g^{|S|}$ for some $g \in G$ with $\mathrm{ord}(g) \geq |S| + 1$.*

Lemma 2.3. *[18, Theorem 1.7] Let G be a finite abelian group and S be a zero-sum free sequence of G. Suppose the subgroup generated by S is of rank greater than 2. Then $|\Sigma(S)| \geq 4|S| - 5$.*

Lemma 2.4. *[16, Theorem 1.1] Let G be a finite abelian group and S be a zero-sum free sequence over G of length $|S| \geq 5$. Suppose*

(i) $\langle S \rangle \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ *with* $r \geq 2$, $1 < n_1 \mid n_2 \mid \ldots \mid n_r$, *and* $n_1 \cdot \ldots \cdot n_{r-1} \geq 6$.
(ii) $|\Sigma(\varphi_H(S)) \cup \{\overline{0}\}| \geq 5$ *for every cyclic subgroup H of G, where $\varphi_H : G \to G/H$ denotes the canonical epimorphism.*

Then $|\Sigma(S)| \geq 5|S| - 16$.

Lemma 2.5. *[13, Theorem 1.4] Conjecture 1.3 is true when* $\mathsf{v}_g(S) \geq n_r - 1$ *for some* $g \in G$.

Lemma 2.6. *[13, Theorem 1.5] Conjecture 1.3 is true when* $G = C_n \oplus C_{nm}$ *with* $k = n - 3$.

We also need the following technical results.

Lemma 2.7. *[6, Lemma 3.1] Let G be a finite abelian group and A be a finite nonempty subset of G. Let $r \in \mathbb{N}$, $y_1, \ldots y_r \in G$ and $k = \min\{\mathrm{ord}(y_i) \mid i \in [1, r]\}$. Then* $|\Sigma(0y_1 \cdot \ldots \cdot y_r) + A| \geq \min\{k, r + |A|\}$.

Lemma 2.8. *[15, Lemma 3.14] Let G be a finite abelian group and $S = S_1S_2$ be a zero-sum free sequence over G. Let $H = \langle S_1 \rangle$ and $\varphi : G \to G/H$ denote the canonical epimorphism. Suppose $q = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))|$. Then*

(1) $|\Sigma(S_1S_2)| \geq q|\Sigma(S_1)| + q - 1$;
(2) If $\varphi(S_2)$ is not zero-sum free, then $|\Sigma(S_1S_2)| \geq q(|\Sigma(S_1)| + 1)$.

Lemma 2.9. *[15, Lemma 4.2] Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 \mid \ldots \mid n_r$ and S be a zero-sum free sequence over G with $|S| \geq n_r$. Then*

(1) $\langle S \rangle$ *is not cyclic;*
(2) If $|S| = n_r + k$ *and* $\langle S \rangle \cong C_{m_1} \oplus C_{m_2}$ *where* $1 < m_1 \mid m_2$, *then* $m_2 = n_r$ *and* $m_1 \geq k + 2$.

The following lemma states a result on the inverse problems of $|\Sigma(S)|$ when S is not zero-sum free.

Lemma 2.10. *[16, Lemma 3.1] Let G be a finite abelian group and $S = b_1 \cdot \ldots \cdot b_w$ be a sequence over G. Suppose $b_i \neq 0$ for every $i \in [1, w]$ and $|S| = w \geq 4 = |\{0\} \cup \Sigma(S)|$. Then either $\langle S \rangle \cong C_4$ or $\langle S \rangle \cong C_2 \oplus C_2$ or $S = g_1^{w-1}g_2$, where $\mathrm{ord}(g_1) = 2$ and $g_1 \neq g_2$.*

3. Proof of the main results

We prove our main results in this section.

3.1. Proof of Theorem 1.4

Proof. If S is of form (1) or of form (2), it is easy to verify that $|\Sigma(S)| = (k + 2)n_2 - 1$.

Next we assume that S is a zero-sum free sequence over G such that $|S| = n_2 + k$. Since N is a cyclic subgroup of G and S_1 is zero-sum free, we infer that $k + 1 \leq |S_1| \leq |N| - 1 \leq n_2 - 1$. It follows from Lemma 2.8 (1) and Lemma 2.2 that

$$|\Sigma(S)| \geq q|\Sigma(S_1)| + q - 1$$

$$\geq (|S_2| + 1)|S_1| + |S_2|$$
$$= (|S_2| + 1)(|S_1| + 1) - 1$$
$$= (n_2 + k - |S_1| + 1)(|S_1| + 1) - 1$$
$$\geq (k + 2)n_2 - 1.$$

This proves the inequality.

If $|\Sigma(S)| = (k + 2)n_2 - 1$, the above inequality forces that

(a1) $|\Sigma(S_1)| = |S_1|$ and $|S_2| = q - 1$.
(a2) $|S_1| = k + 1$ or $|S_1| = n_2 - 1$.

Since S_1 is a zero-sum free sequence over G, it follows from (a1) and Lemma 2.2 that $S_1 = g^{|S_1|}$ for some $g \in G$ with $\mathrm{ord}(g) \geq |S_1| + 1$. If $|S_1| = n_2 - 1$, then $\mathsf{v}_g(S) = n_2 - 1$, the result follows from Lemma 2.5. So we may assume that

$$|S_1| = k + 1 < n_2 - 1.$$

So $|S_2| = |S| - |S_1| = n_2 + k - (k + 1) = n_2 - 1$ and thus $q = n_2$.

If $\varphi(S_2)$ is not a zero-sum free sequence over G/N, then it follows from Lemma 2.8 (2) and Lemma 2.2 that

$$|\Sigma(S)| \geq q(|\Sigma(S_1) + 1) \geq (|S_2| + 1)(|S_1| + 1) = (k + 2)n_2,$$

yielding a contradiction. Therefore,

$$\varphi(S_2) \text{ is a zero-sum free sequence over } G/N.$$

Since $|\{\overline{0}\} \cup \Sigma(\varphi(S_2))| = q = |S_2| + 1$. Then $|\Sigma(\varphi(S_2))| = q - 1 = |S_2|$. It follows from Lemma 2.2 that $\varphi(S_2) = \overline{h}^{|S_2|}$, where $h \in G \setminus N$ and $\mathrm{ord}(\overline{h}) \geq |S_2| + 1 = n_2$. Since $\mathrm{ord}(\overline{h}) \mid n_2 = \exp(G)$, we have that $\mathrm{ord}(\overline{h}) = n_2$. Therefore, S_2 is of the following form

$$S_2 = (h + t_1 g) \cdot (h + t_2 g) \cdot \ldots \cdot (h + t_{|S_2|} g),$$

where $t_i \in [0, \mathrm{ord}(g) - 1]$ for $i = 1, 2, \ldots, |S_2|$ and $\mathrm{ord}(\overline{h}) = n_2$. Moreover, we have that $ih \notin \langle g \rangle$ for every $i \in [1, n_2 - 1]$.

If $\mathrm{ord}(g) = k+2$, then $k+2 \mid n_2$ and $\langle S \rangle \cong \langle g, h \rangle \cong C_{k+2} \oplus C_{n_2}$. So S is of form (1), and we are done.

Next we assume that $\mathrm{ord}(g) > k + 2$. Then $n_2 \geq \mathrm{ord}(g) > k + 2$ and $|S_2| = n_2 - 1 \geq k + 2$.

We will show that $t_1 = t_2 = \ldots = t_{n_2-1}$. Suppose $t_1 \neq t_2$. Then

$$ih + (\sum_{j=3}^{i+1} t_j)g + \{t_1 g, t_2 g\} \subset \Sigma(S_2) \cap (ih + \langle g \rangle),$$

for every $i \in [1, n_2 - 2]$. It follows from Lemma 2.7 that

$$|\Sigma(S) \cap (ih + N)| \geq |(\Sigma(S_2) \cap (ih + N)) + \Sigma(0S_1)|$$
$$\geq |ih + (\sum_{j=3}^{i+1} t_j)g + \{t_1 g, t_2 g\} + \{0, g, 2g, \ldots, (k + 1)g\}|$$

$$\geq k+3,$$

for every $1 \leq i \leq n_2 - 2$. Similarly,

$$\begin{aligned}
&|\Sigma(S) \cap ((n_2-1)h + N)| \\
\geq &|(\Sigma(S_2) \cap ((n_2-1)h+N)) + \Sigma(0S_1)| \\
\geq &|(n_2-1)h + (\sum_{j=1}^{n_2-1} t_j)g + \{0, g, 2g, \ldots, (k+1)g\}| \geq k+2.
\end{aligned}$$

Note that $|\Sigma(S) \cap N| \geq |\{g, 2g, \ldots, (k+1)g\}| = k+1$ and

$$\Sigma(S) = \cup_{i=0}^{n_2-1} \Sigma(S) \cap (ih + N).$$

Therefore,

$$\begin{aligned}
|\Sigma(S)| \geq &\sum_{i=0}^{n_2-1} |\Sigma(S) \cap (ih+N)| \\
\geq &(k+1) + (k+3)(n_2-2) + (k+2) \\
= &(k+3)n_2 - 3 > (k+2)n_2 - 1,
\end{aligned}$$

yielding a contradiction. So $t_1 = t_2$. Moreover we obtain that $t_1 = t_2 = \ldots = t_{n_2-1}$ and thus $S_2 = (h + t_1 g)^{n_2-1}$.

Let $g_1 = h + t_1 g$ and $h_1 = g$. Then $S = g_1^{n_2-1} h_1^{k+1}$ is of form (2), and we are done. □

3.2. Proof of Theorem 1.5

Proof. Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 \mid \ldots \mid n_r$. Let S be a zero-sum free sequence over G of length $|S| = n_r + 2$. By Lemma 2.9 (1) we obtain that $\langle S \rangle$ is not cyclic and thus $r \geq 2$.

If S is of form (1) or of form (2) in Conjecture 1.3, it is easy to check that $|\Sigma(S)| = 4n_r - 1$.

Next we assume that $|\Sigma(S)| = 4n_r - 1$, we will show that S is of form (1) or (2).

We first show that $r = 2$. If $r \geq 3$, by Lemma 2.3, we infer that $|\Sigma(S)| \geq 4|S| - 5 = 4(n_r + 2) - 5 = 4n_r + 3 > 4n_r - 1$, yielding a contradiction. Hence $r = 2$.

Suppose $\langle S \rangle \cong C_{m_1} \oplus C_{m_2}$ with $1 < m_1 \mid m_2$. Since $|S| = n_r + 2$, by Lemma 2.9 (3) we infer that $m_2 = n_r$ and $m_1 \geq 4$. Therefore, $|S| = n_r + 2 = m_2 + 2 \geq m_1 + 2 \geq 6$.

If $m_1 = 4$, we obtain that S is of form (1), and we are done.

If $m_1 = 5$, by Lemma 2.1 we infer that $|S| = n_r + 2 = m_1 + m_2 - 3 = \mathsf{D}(C_{m_1} \oplus C_{m_2}) - 2$. It follows from Lemma 2.6 that S is of form (2), and we are done.

Next we assume that $m_1 \geq 6$. Then $n_r = m_2 \geq m_1 \geq 6$. We will show that

$$|\Sigma(\varphi_H(S)) \cup \{\overline{0}\}| \leq 4$$

for some cyclic subgroup H of G, where $\varphi_H : G \to G/H$ denotes the canonical epimorphism. Assume to the contrary that $|\Sigma(\varphi_H(S)) \cup \{\overline{0}\}| \geq 5$ for every cyclic subgroup H of G. By Lemma 2.4, we infer that

$4n_r - 1 = |\Sigma(S)| \geq 5|S| - 16 = 5(n_r + 2) - 16 = 5n_r - 6$. It follows that $n_r \leq 5$, yielding a contradiction. Hence $|\Sigma(\varphi_H(S)) \cup \{\overline{0}\}| \leq 4$ for some cyclic subgroup H of G.

Suppose $q = |\Sigma(\varphi_H(S)) \cup \{\overline{0}\}|$ and $\Sigma(\varphi_H(S)) \cup \{\overline{0}\} = \{\overline{0}, \overline{a_1}, \ldots, \overline{a_{q-1}}\}$ for some $a_1, a_2, \ldots, a_{q-1} \in G$. It is clear that

$$\Sigma(S) \subset H \cup (a_1 + H) \cup \ldots \cup (a_{q-1} + H).$$

Since H is a cyclic subgroup of G, we infer that $|H| \leq n_r$. It follows from $|\Sigma(S)| = 4n_r - 1$ and $q \leq 4$ that $q = 4$ and $|H| = n_r$. Moreover, since S is zero-sum free, we infer that

$$\Sigma(S) = (H \setminus \{0\}) \cup (a_1 + H) \cup (a_2 + H) \cup (a_3 + H).$$

Now we can write $S = S_1 S_2$, where S_1 is the subsequence of S that consisting all elements of H, and none elements of S_2 is in H. If

$$|S_1| \geq 3, \text{ and } 4 = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))| > |S_2|,$$

applying Theorem 1.4 with $k = 2$, we infer that S is of form (1) or (2), and we are done. So we may assume that

$$\text{either } |S_1| \leq 2, \text{ or } 4 = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))| \leq |S_2|.$$

Noting that $|S| = n_r + 2 \geq 6 + 2 = 8$. If $|S_1| \leq 2$, then $|S_2| \geq 6 > 4 = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))|$. Hence, it remains to consider the case

$$4 = |\{\overline{0}\} \cup \Sigma(\varphi(S_2))| \leq |S_2|.$$

By Lemma 2.10, we infer that $\langle\varphi(S_2)\rangle \cong C_4$, or $\langle\varphi(S_2)\rangle \cong C_2 \oplus C_2$, or $\varphi(S_2) = \overline{b_1}^{|S_2|-1}\overline{b_2}$, for some terms b_1, b_2 from S with $\mathrm{ord}(\overline{b_1}) = 2$ and $\overline{b_1} \neq \overline{b_2}$.

If $\langle\varphi(S_2)\rangle \cong C_4$, we obtain that $\langle S\rangle \cong C_4 \oplus C_{n_r}$, yielding a contradiction to that $\langle S\rangle \cong C_{m_1} \oplus C_{m_2}$ with $m_1 \geq 6$. If $\langle\varphi(S_2)\rangle \cong C_2 \oplus C_2$, we obtain that $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{n_r}$, yielding a contradiction to that $\langle S\rangle$ is of rank 2. If $\varphi(S_2) = \overline{b_1}^{|S_2|-1}\overline{b_2}$, for some terms b_1, b_2 from S with $\mathrm{ord}(\overline{b_1}) = 2$ and $\overline{b_1} \neq \overline{b_2}$, then $\langle S b_2^{-1}\rangle \cong C_2 \oplus C_{n_r}$. So $S b_2^{-1}$ is a zero-sum free sequence of length $|S b_2^{-1}| = n_r + 1$ over $C_2 \oplus C_{n_r}$. By Lemma 2.1, we infer that $\mathsf{D}(C_2 \oplus C_{n_r}) = 2 + n_r - 1 = n_r + 1 = |S b_2^{-1}|$, yielding a contradiction to that $S b_2^{-1}$ is zero-sum free.

This completes the proof. □

4. Concluding remarks

In 2008, W. Gao et al. [6] confirmed Conjecture 1.1 when $k = n - 2$ by using the following result.

Lemma 4.1. *[8, Proposition 5.1.4] Let G be a finite abelian group and S be a zero-sum free sequence over G with $|S| = \mathsf{D}(G) - 1$. Then $|\Sigma(S)| = |G| - 1$.*

Moreover, we have the following results.

Theorem 4.2. *Both Conjecture 1.2 and Conjecture 1.3 are true when $G = C_n \oplus C_{nm}$ with $k = n - 2$.*

Proof. It follows from Lemma 2.1 and Lemma 4.1 immediately. □

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Conflict of interest

The authors declare no conflict of interest in this paper.

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