Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 14054–14063.
DOI: 10.3934/math.2021813
Received: 29 August 2021
Accepted: 03 November 2021
Published: 08 November 2021

Research article

A characterization of ruled hypersurfaces in complex space forms in terms of the Lie derivative of shape operator

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Abstract: In this paper, it is proved that if a non-Hopf real hypersurface in a nonflat complex space form of complex dimension two satisfies Ki and Suh's condition (J. Korean Math. Soc., 32 (1995), 161–170), then it is locally congruent to a ruled hypersurface or a strongly 2-Hopf hypersurface. This extends Ki and Suh's theorem to real hypersurfaces of dimension greater than or equal to three.

Keywords: real hypersurface; complex space form; ruled hypersurface; Lie derivative
Mathematics Subject Classification: Primary: 53B25; Secondary: 53D15

1. Introduction

A complete and simply connected complex space form of complex dimension n, denoted by $M^n(c)$, is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where the constant c is the holomorphic sectional curvature of a complex space form. Let M be a real hypersurface immersed in a complex space form. On M there exists a natural almost contact metric structure (see Section 2) induced from the complex structure on $M^n(c)$ and the normal vector field, respectively. Let ξ be the Reeb (or structure) vector field of the almost contact metric structure on M. If ξ is an eigenvector of the shape operator of a real hypersurface at each point, then the hypersurface is said to be Hopf. A real hypersurface is said to be non-Hopf if there exists at least one point on which ξ is not an eigenvector of the shape operator. For a real hypersurface, an eigenfunction of an eigenvector field of the shape operator is said to be a principal curvature.

The classification of real hypersurfaces in a nonflat complex space form $M^n(c)$, $c \neq 0$, having constant principal curvatures is one of the most challenging problems in geometry of real hypersurfaces and is still an open question till now (the case of $n = 2$ has been settled completely in [20, 22] for $\mathbb{C}P^2$ and in [3] for $\mathbb{C}H^2$), we refer the reader to [2, 5, 7] for some recent progress. Under certain additional geometric conditions, the above problem was considered a long time ago.

Theorem 1.1. *[11] Let M be a connected Hopf hypersurface of $\mathbb{C}P^n(c)$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*

- (A_1) *a geodesic sphere of radius r with $0 < r < \pi/\sqrt{c}$;*
- (A_2) *a tube of radius r around a totally geodesic $\mathbb{C}P^k(c)$ $(1 \le k \le n-2)$ with $0 < r < \pi/\sqrt{c}$;*
- *(B) a tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$ with $0 < r < \pi/(2\sqrt{c})$;*
- *(C) a tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{\frac{n-1}{2}}(c)$ and $n \ge 5$ is odd with $0 < r < \pi/(2\sqrt{c})$;*
- *(D) a tube of radius r around a complex Grassmannian $\mathbb{C}G_{2,5}$ and $n = 9$ with $0 < r < \pi/(2\sqrt{c})$;*
- *(E) a tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$ with $0 < r < \pi/(2\sqrt{c})$.*

When the ambient space is the complex hyperbolic space $\mathbb{C}H^n(c)$, the corresponding version of the above theorem is given as follows:

Theorem 1.2. *[1] Let M be a connected Hopf hypersurface of $\mathbb{C}H^n(c)$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*

- (A_0) *a self-tube, that is, a horosphere;*
- $(A_{1,0})$ *a geodesic hypersphere of radius r with $0 < r < \infty$;*
- $(A_{1,1})$ *a tube of radius r around a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}(c)$ with $0 < r < \infty$;*
- (A_2) *a tube of radius r around a totally geodesic $\mathbb{C}H^k(c)$ $(1 \le k \le n-2)$ with $0 < r < \infty$;*
- *(B) a tube of radius r around a totally real totally geodesic hyperbolic space $\mathbb{R}H^n(c/4)$ with $0 < r < \infty$.*

Applying the above Theorems 1.1 and 1.2, many characterization theorems of Hopf hypersurfaces having constant principal curvatures have been obtained (for examples see a great number of references in [4, 17]). Among others, it has been proved in [21] that a real hypersurface in $M^n(c)$, $c \neq 0$, can not be totally umbilical. Applying this, the shape operator (denoted by A throughout this paper) can not be a multiple of the metric tensor at each point of the hypersurface. Generalizing this, Ki, Kim and Lee in [9] presented a characterization theorem of type (A) hypersurfaces, where by type (A) hypersurfaces we mean those real hypersurfaces of type (A_1) or (A_2) in $\mathbb{C}P^n$, or of type (A_0), $(A_{1,0})$, $(A_{1,1})$, (A_2) in $\mathbb{C}H^n$. Specifically, Ki, Kim and Lee in [9] obtained that a real hypersurface in $M^n(c)$, $c \neq 0$, satisfies

$$\mathcal{L}_\xi A = 0 \tag{1.1}$$

if and only if the hypersurface is of type (A), where $\mathcal{L}$ is the usual Lie differentiation. Furthermore, weakening condition (1.1), Ki and Suh in [10, Theorem 1] proved

Theorem 1.3. *[10] If a real hypersurface in a nonflat complex space form $M^n(c)$, $n > 2$, satisfies*

$$g((\mathcal{L}_\xi A)X, Y) = 0 \tag{1.2}$$

for any vector fields X, Y orthogonal to ξ, then it is of type (A).

Let M be a non-Hopf hypersurface in $M^n(c)$, $c \neq 0$. Assume that Ω of M is an open subset consisting those points on which the structure vector field ξ is not principal. One can set

$$A\xi = \alpha\xi + \beta U \tag{1.3}$$

on Ω, where U is a unit vector field orthogonal to ξ and $\alpha = g(A\xi, \xi)$, and β is the length of $A\xi - \alpha\xi$. Ki and Suh in [10, Theorem 2] presented a characterization theorem of ruled hypersurfaces. Where by ruled hypersurfaces we mean those real hypersurfaces having a foliation by totally geodesic complex hyperplanes (see [12]), or equivalently, the shape operator satisfies

$$g(AX, Y) = 0 \tag{1.4}$$

for any vector fields X, Y orthogonal to ξ (see [4]). In other words, Ki and Suh's result in [10, Theorem 2] can be rewritten as the following.

Theorem 1.4. *[10] Let M be a non-Hopf real hypersurface in $M^n(c)$, $n \geq 3$, $c \neq 0$. If on Ω, M satisfies*

$$g((\mathcal{L}_\xi A)X, Y) = \beta^2 g(X, \phi U)g(Y, U) \tag{1.5}$$

for any vector fields X, Y orthogonal to ξ and $\mathrm{d}\alpha(\xi) \neq 0$*, then M is locally congruent to a ruled real hypersurface.*

In this paper, we aim to extend the above three theorems to real hypersurfaces of dimension three.

Theorem 1.5. *Let M be a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ which is non-Hopf at every point. If M satisfies* (1.5) *for any vector fields X, Y orthogonal to ξ and* $\mathrm{d}\alpha(\xi) \neq 0$*, then M is locally congruent to a ruled real hypersurface.*

On a real hypersurface in a nonflat complex space form $M^n(c)$, $c \neq 0$, one can define a distribution $\mathcal{H} = \mathrm{span}\{\xi, A\xi, A^2\xi \cdots\}$ which is the smallest A-invariant distribution. A real hypersurface is said to be 2-Hopf if $\mathrm{rank}(\mathcal{H}) = 2$ and $\mathcal{H}$ is integrable (see [4,6,8]). In particular, a 2-Hopf hypersurface is said to be strongly 2-Hopf if in addition the spectrum of $A|_{\mathcal{H}}$ is constant along the integral submanifolds of $\mathcal{H}$ (see [6]). By applying such a concept and deleting condition $\mathrm{d}\alpha(\xi) \neq 0$ (it is not natural and unnecessary), we obtain a more comprehensive classification result.

Theorem 1.6. *Let M be a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ which is non-Hopf at every point. Then M satisfies* (1.5) *for any vector fields X, Y orthogonal to ξ if and only if it is locally congruent to one of the following:*

- *a ruled real hypersurface;*
- *a strongly 2-Hopf hypersurface satisfying $g(AX, Y) = ag(X, Y)$ for any vector fields X, Y orthogonal to ξ and a certain nowhere vanishing function a.*

As an application of the proofs of the above results, we also extend Theorem 1.3 to real hypersurfaces of dimension three.

Theorem 1.7. *A real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ satisfies* (1.2) *for any vector fields X, Y orthogonal to ξ if and only if it is of type* (A).

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure, respectively. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g, respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX \tag{2.1}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$ and $\mathfrak{X}(M)$ is the set of all tangent vector fields. For any $X \in \mathfrak{X}(M)$, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi, \tag{2.2}$$

where ϕ and η are two tensor fields of type $(1, 1)$ and $(1, 0)$, respectively. Thus, on M there exits an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi) \tag{2.4}$$

for any $X, Y \in \mathfrak{X}(M)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface and α is called a Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla}J = 0$) of $M^n(c)$, and using (2.1) and (2.2) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any $X, Y \in \mathfrak{X}(M)$. Let R be the Riemannian curvature tensor of M. Because $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any $X, Y \in \mathfrak{X}(M)$.

3. Proof of Theorem 1.5

Let M be a real hypersurface in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$ which is non-Hopf at every point. In what follows, working on Ω (in this context it is M), let us put $e_1 = \xi$, $e_2 = U$ and $e_3 = \phi U$ such that $\{e_1, e_2, e_3\}$ forms a local orthonormal basis of the tangent space at each point of the hypersurface. We need the following result that can be seen in [19, 24–26].

Lemma 3.1. *The following relations hold on* Ω*:*

$$\begin{aligned}
&Ae_1 = \alpha e_1 + \beta e_2,\ Ae_2 = \beta e_1 + \gamma e_2 + \delta e_3,\ Ae_3 = \delta e_2 + \mu e_3,\\
&\nabla_{e_2} e_1 = -\delta e_2 + \gamma e_3,\ \nabla_{e_3} e_1 = -\mu e_2 + \delta e_3,\ \nabla_{e_1} e_1 = \beta e_3,\\
&\nabla_{e_2} e_2 = \delta e_1 + \kappa_1 e_3,\ \nabla_{e_3} e_2 = \mu e_1 + \kappa_2 e_3,\ \nabla_{e_1} e_2 = \kappa_3 e_3,\\
&\nabla_{e_2} e_3 = -\gamma e_1 - \kappa_1 e_2,\ \nabla_{e_3} e_3 = -\delta e_1 - \kappa_2 e_2,\ \nabla_{e_1} e_3 = -\beta e_1 - \kappa_3 e_2,
\end{aligned} \tag{3.1}$$

where γ*,* δ*,* μ*,* κ_i*,* $i = \{1, 2, 3\}$*, are smooth functions on* Ω*.*

By a direct calculation, we have

$$(\mathcal{L}_\xi A)X = (\nabla_\xi A)X - \nabla_{AX}\xi + A\nabla_X\xi.$$

for any vector field X orthogonal to ξ. Putting (2.6) into the above equation gives

$$(\mathcal{L}_\xi A)X = (\nabla_\xi A)X - \phi A^2X + A\phi AX.$$

Now in terms of the skew-symmetry of ϕ, the above equation becomes

$$g((\mathcal{L}_\xi A)X, Y) = g((\nabla_\xi A)X, Y) + g(A^2X, \phi Y) + g(\phi AX, AY) \tag{3.2}$$

for any vector fields X, Y orthogonal to ξ. Applying Lemma 3.1, working on Ω, according to (3.2), we get

$$g((\mathcal{L}_\xi A)e_2, e_2) = e_1(\gamma) - 2\kappa_3\delta + \delta\gamma + \delta\mu. \tag{3.3}$$

$$g((\mathcal{L}_\xi A)e_2, e_3) = e_1(\delta) - \kappa_3\mu + \kappa_3\gamma - \gamma^2 - 2\delta^2 + \gamma\mu. \tag{3.4}$$

$$g((\mathcal{L}_\xi A)e_3, e_2) = e_1(\delta) - \kappa_3\mu + \kappa_3\gamma + \beta^2 + 2\delta^2 + \mu^2 - \gamma\mu. \tag{3.5}$$

$$g((\mathcal{L}_\xi A)e_3, e_3) = e_1(\mu) + 2\kappa_3\delta - \delta\gamma - \delta\mu. \tag{3.6}$$

Using our notations, Ki and Suh's condition (1.5) can be rewritten by

$$g((\mathcal{L}_\xi A)X, Y) = \beta^2 g(X, e_3)g(Y, e_2) \tag{3.7}$$

for any vector fields X, Y orthogonal to ξ. If the above equation is valid, from (3.4) and (3.5), we obtain

$$e_1(\delta) - \kappa_3\mu + \kappa_3\gamma - \gamma^2 - 2\delta^2 + \gamma\mu = 0 \tag{3.8}$$

and

$$e_1(\delta) - \kappa_3\mu + \kappa_3\gamma + 2\delta^2 + \mu^2 - \gamma\mu = 0, \tag{3.9}$$

respectively. The subtraction of (3.9) from (3.8) yields

$$4\delta^2 + (\mu - \gamma)^2 = 0. \tag{3.10}$$

From (3.10) we conclude that $\delta = 0$ and $\mu = \gamma$ hold on each point of Ω. Next, we show that $d\alpha(\xi) \neq 0$ means that $\mu = \gamma = 0$. Or equivalently, we assume that there exits a non-empty open subset Q of Ω on which $\mu = \gamma \neq 0$, and we aim to show that $\xi(\alpha) = 0$ on Q.

With the aid of $\delta = 0$, and $\mu = \gamma$, from the Codazzi Eq (2.8) for $X = e_2$ or $X = e_3$ and $Y = e_1$ we have

$$e_2(\beta) = \beta\kappa_2. \tag{3.11}$$

$$\alpha\mu + \beta\kappa_1 - \mu^2 - \beta^2 + \frac{1}{4}c = 0. \tag{3.12}$$

$$e_2(\alpha) = e_1(\beta). \tag{3.13}$$

$$e_1(\mu) = \beta\kappa_2. \tag{3.14}$$

$$e_3(\alpha) = \alpha\beta + \beta\kappa_3 - 3\beta\mu. \tag{3.15}$$

$$e_3(\beta) = 2\alpha\mu - 2\mu^2 + \beta\kappa_1 + \frac{1}{2}c. \tag{3.16}$$

Similarly, from the Codazzi equation for $X = e_2$ and $Y = e_3$ we have

$$e_3(\mu) = -3\beta\mu. \tag{3.17}$$

$$e_2(\mu) = 0. \tag{3.18}$$

Moreover, with the aid of $\delta = 0$ and $\mu = \gamma$, applying Lemma 3.1 we have

$$[e_1, e_2] = (\kappa_3 - \mu)e_3, \ [e_2, e_3] = -2\mu e_1 - \kappa_1 e_2 - \kappa_2 e_3. \tag{3.19}$$

Taking the derivative of μ along $[e_2, e_3]$, with the aid of (3.14), (3.17), (3.18) and the second equality of (3.19), we obtain $\kappa_2 = 0$ because of $\beta \neq 0$ and $\mu \neq 0$ on Q. In addition, substituting $\kappa_2 = 0$ into (3.14) we obtain $e_1(\mu) = 0$. Applying $\beta \neq 0$ and $\mu \neq 0$ on Q again, taking the derivative of μ along $[e_1, e_2]$, with the aid of $e_1(\mu) = 0$, (3.17), (3.18) and the first equality of (3.19), we obtain $\kappa_3 = \mu$. Eliminating $\beta\kappa_1$, from (3.12) and (3.16) we obtain

$$e_3(\beta) = \alpha\mu - \mu^2 + \beta^2 + \frac{1}{4}c. \tag{3.20}$$

Note that from (3.11) and $\kappa_2 = 0$ we have $e_2(\beta) = 0$. Now taking the derivative of β along $[e_2, e_3]$, with the aid of (3.20), (3.18), (3.13), $e_1(\mu) = 0$, $\kappa_2 = 0$ and the second equality of (3.19), we obtain

$$e_2(\alpha) = e_1(\beta) = 0$$

because of $\mu \neq 0$ on Q. Note that in this situation, the second equality of (3.19) becomes $[e_2, e_3] = -2\mu e_1 - \kappa_1 e_2$. Applying this and the above equation, taking the derivative of α along $[e_2, e_3]$, with the aid of (3.15), we obtain $e_1(\alpha) = 0$ because of $\mu \neq 0$ on Q. However, this contradicts our assumption ($d\alpha(\xi) \neq 0$ on the hypersurface) and means that when $\xi(\alpha) \neq 0$ on Ω, then $\gamma = \mu = 0$. Now, on Ω, the shape operator is given by

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

with respect to the local orthonormal basis $\{e_1, e_2, e_3\}$. In addition, it is easily seen that Eq (3.21) is equivalent to $g(AX, Y) = 0$ for any vector fields X and Y orthogonal to the structure vector field ξ. That is, the hypersurface M is locally congruent to a ruled hypersurface. This completes the proof of Theorem 1.5.

Remark 3.1. The converse of Theorem 1.5 is not necessarily true. For example, let M be a minimal homogeneous ruled real hypersurface in $\mathbb{C}H^2(c)$ (see Lohnherr and Reckziegel [14]). According to (1.4), (2.7) and (2.8), the shape operator of M is given by (see [14]):

$$A = \begin{pmatrix} 0 & \frac{\sqrt{-c}}{2} & 0 \\ \frac{\sqrt{-c}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On such a hypersurface, (1.5) holds necessarily and $\xi(\alpha)$ vanishes identically. We refer the reader to [10] for (1.5) in ruled hypersurfaces.

4. Proof of Theorem 1.6

Let M be a real hypersurface in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$ which is non-Hopf at every point satisfying Ki and Suh's condition (1.5). Working on Ω (in this context it is M), according to the proof of Theorem 1.5, $\delta = 0$ and $\lambda = \mu \neq 0$ are necessarily true. If $\mu = \gamma \neq 0$, we have proved $[e_1, e_2] = 0$ and that α, β and μ are all invariant along $\{e_1, e_2\} = \mathcal{H}$. This implies that the hypersurface M is locally congruent to a strongly 2-Hopf hypersurface (non-ruled). In addition, applying Lemma 3.1, $\delta = 0$ and $\lambda = \mu \neq 0$ are equivalent to $g(AX, Y) = \lambda g(X, Y)$ for any vector fields X, Y orthogonal to the structure vector field ξ. If $\mu = \lambda = 0$ and $\delta = 0$, as discussed before it is easily seen that in this case the hypersurface is a ruled one.

Conversely, suppose that M is a non-Hopf real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$. If M is a ruled hypersurface, applying directly (3.21), we have from (3.3)–(3.6) that (1.5) holds. Now assume that M satisfies $g(AX, Y) = ag(X, Y)$ for any vector fields X, Y orthogonal to ξ and a is a non-vanishing function. In order to prove (1.5), following (3.3)–(3.6) we need only to prove $e_1(a) = 0$. On such a hypersurface M, we construct a local orthonormal basis similar to that in Section three and adopt the same symbols. In Section three, $e_1(a) = 0$ has been confirmed due to $a = \mu = \gamma \neq 0$. This completes the proof of Theorem 1.6.

Remark 4.1. A three-dimensional real hypersurface satisfying that $g(AX, Y) = ag(X, Y)$ for any vector fields X, Y orthogonal to the structure vector field ξ and a function a has been considered in [13] which adopted Ivey and Ryan's formula in [8].

5. Proof of Theorem 1.7

First of all, we prove that a real hypersurface M in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$ satisfying (1.2) for any vector fields X and Y orthogonal to ξ must be Hopf. Suppose that such a hypersurface M is non-Hopf, then Ω defined in Section one is non-empty. Working on Ω, by Lemma 3.1, both (3.4) and (3.5) in Section one are necessarily true in this context. If Eq (1.2) is valid for any vector fields X, Y orthogonal to ξ, it follows from (3.4) and (3.5) that

$$e_1(\delta) - \kappa_3\mu + \kappa_3\gamma - \gamma^2 - 2\delta^2 + \gamma\mu = 0$$

and

$$e_1(\delta) - \kappa_3\mu + \kappa_3\gamma + \beta^2 + 2\delta^2 + \mu^2 - \gamma\mu = 0,$$

respectively. The subtraction of the above equation from the previous one gives

$$\beta^2 + 4\delta^2 + (\mu - \gamma)^2 = 0.$$

This reduces to $\beta = 0$, and contradicts our assumption. Therefore, Ω is empty and M is Hopf. Recall that the Hopf principal curvature ($\alpha = g(A\xi, \xi)$) of any Hopf hypersurface in a nonflat complex space form is a constant (see [4, 17]).

Considering $Y = \xi$ in the Codazzi Eq (2.8) and using (2.6), for any vector field X, we have

$$(\nabla_\xi A)X = \alpha\phi AX - A\phi AX + \frac{1}{4}c\phi X.$$

Putting this into (3.2) we have

$$g((\mathcal{L}_\xi A)X, Y) = \frac{1}{4}cg(\phi X, Y) + g(A^2X, \phi Y) + \alpha g(\phi AX, Y)$$

for any vector fields X, Y orthogonal to ξ. If (1.2) is valid, it follows from the above equation that

$$A^2X - \alpha AX + \frac{1}{4}c\phi^2 X = 0 \tag{5.1}$$

for any vector field X orthogonal to ξ. Let X be an eigenvector field of the shape operator orthogonal to ξ with eigenfunction λ. Since the dimension of the real hypersurface is three, then ϕX is also an eigenvector field of the shape operator whose eigenfunction is denoted by μ. It follows from (5.1) that

$$\lambda^2 - \alpha\lambda - \frac{1}{4}c = 0, \tag{5.2}$$

and μ is also a root of the quadratic Eq (5.2). From (5.2), we observe that all principal curvatures of the hypersurface M are constant. On the other hand, from [17, Corollary 2.3], we have

$$\lambda\mu = \frac{1}{2}(\lambda + \mu)\alpha + \frac{1}{4}c. \tag{5.3}$$

Eliminating c, from (5.2) and (5.3) we get

$$(\lambda - \frac{1}{2}\alpha)(\lambda - \mu) = 0.$$

It follows immediately that $\lambda = \mu$. In fact, if $\lambda \neq \mu$ and hence $\lambda = \frac{1}{2}\alpha$, applying the Vieta theorem, from (5.2) we obtain $\mu = \frac{1}{2}\alpha$ and we arrive at a contradiction. For any Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, the two principal curvatures on the holomorphic distribution $\ker\eta$ being the same is equivalent to $A\phi = \phi A$. Thus, the hypersurface is of type (A) (see [16, 18]).

Conversely, if the hypersurface is of type (A), applying the equivalent condition $\phi A = A\phi$, from (3.2) we see that (1.2) is valid if and only if $g((\nabla_\xi A)X, Y) = 0$ for any vector fields X, Y orthogonal to ξ. Such an equation holds on any real hypersurfaces of type (A) (see [4, Theorem 8.120]). This completes the proof of Theorem 1.7.

6. Conclusions

The geometry of real hypersurfaces in nonflat complex space forms is determined completely by the shape operator. There exist a great number literature in the study of real hypersurfaces in nonflat complex space forms in terms of the shape operator. The present paper give some new characterizations for type (A) hypersurfaces, ruled hypersurfaces and strongly 2-Hopf hypersurfaces of dimension three by means of the Lie derivative of the shape operator. This can be regarded as extensions for real hypersurfaces of dimension greater than three which were obtained by Ki and Suh. In view of reuslts in this paper, ones are helpful to understand better the geometry of real hypersurfaces of dimension three.

Acknowledgments

This paper was supported by the Basic Research Projects of Key Scientific Research Projects Plan in Henan Higher Education Institutions (No. 20zx003). The author would like to thank referees for their many useful comments which improves the original paper.

Conflict of interest

The author declares no conflict of interest.

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