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Research article

A remark on the existence of positive radial solutions to a Hessian system

Dragos-Patru Covei*

Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana, 1st district, postal code: 010374, postal office: 22, Romania

* **Correspondence:** Email: dragos.covei@csie.ase.ro; Tel: +0400766224814.

Abstract: We give new conditions for the study of existence of positive radial solutions for a system involving the Hessian operator. The solutions to be obtained are given by successive-approximation. Our interest is to improve the works that deal with such systems at the present and to give future directions of research related to this work for researchers.

Keywords: existence; system with k-Hessian; radial symmetry
Mathematics Subject Classification: 35A01, 35A09, 35A24, 35A35

1. Introduction

This paper is devoted to develop the mathematical theory for the study of existence of positive radial solutions of a system of partial differential equations (PDE) of the form

$$\begin{cases} S_{k_1}\left(\lambda\left(D^2u\right)\right) - \alpha S_{k_2}\left(\lambda\left(D^2u\right)\right) = p\left(|x|\right) f\left(v\right), \ x \in \mathbb{R}^N, \ (N \geq 3), \\ S_{k_3}\left(\lambda\left(D^2v\right)\right) - \beta S_{k_4}\left(\lambda\left(D^2v\right)\right) = q\left(|x|\right) g\left(u\right), \ x \in \mathbb{R}^N, \ (N \geq 3), \end{cases} \tag{1.1}$$

where $\alpha, \beta \in (0, \infty)$, $k_1, k_2, k_3, k_4 \in \{1, 2, ..., N\}$ with $k_1 > k_2$ and $k_3 > k_4$, $S_{k_i}\left(\lambda\left(D^2\left(\circ\right)\right)\right)$ $(i = 1, 2, 3, 4)$ stands for the k_i-Hessian operator defined as the sum of all $k_i \times k_i$ principal minors of the Hessian matrix $D^2\left(\circ\right)$ and the functions p, q, f and g satisfy some suitable conditions.

In the case $\alpha = \beta = 0$ and $k_1 = k_3 = 1$, there are several works that deals with the existence of radially symmetric solution for (1.1), in which situation the system become

$$\begin{cases} \Delta u = p\left(|x|\right) f\left(v\right), \ x \in \mathbb{R}^N, \ (N \geq 3), \\ \Delta v = q\left(|x|\right) g\left(u\right), \ x \in \mathbb{R}^N, \ (N \geq 3). \end{cases} \tag{1.2}$$

Some of these are analyzed in the following. For example, [5] considered the existence of entire large solutions for the system (1.2) in the case $f\left(v\right) = v^a$ and $g\left(u\right) = u^b$ with $0 < ab \leq 1$ and noticed that

(1.2) has a positive entire large solution if and only if the nonnegative spherically symmetric continuous functions p and q satisfy

$$\int_0^\infty tp(t)\left(t^{2-N}\int_0^t s^{N-3}\int_0^s zq(z)\,dz\right)^a dt = \infty, \tag{1.3}$$

$$\int_0^\infty tq(t)\left(t^{2-N}\int_0^t s^{N-3}\int_0^s zp(z)\,dz\right)^b dt = \infty. \tag{1.4}$$

Moreover, if $a\cdot b>1$ he showed that the system (1.2) has a positive entire large solution if the radial functions p and q satisfy one of the two inequalities

$$\int_0^\infty tp(t)\left(t^{2-N}\int_0^t s^{N-3}\int_0^s zq(z)\,dz\right)^a dt < \infty, \tag{1.5}$$

$$\int_0^\infty tq(t)\left(t^{2-N}\int_0^t s^{N-3}\int_0^s zp(z)\,dz\right)^b dt < \infty. \tag{1.6}$$

Recently, for the particular case $\alpha,\beta\in[0,\infty)$, $k_1=k_3=N$ and $k_2=k_4=1$, the authors [7] obtained the existence of entire radial large solutions for the system (1.1) under hypotheses that $p,q:[0,\infty)\to[0,\infty)$ are spherically symmetric continuous functions and $f,\ g:[0,\infty)\to[0,\infty)$ are continuous, monotone non-decreasing nonlinearities such that

$$f(s)>0,\ g(t)>0 \text{ for all } s,t>0$$

and

$$\int_1^\infty \frac{1}{(1+f(t)+g(t))^{1/N}}=\infty. \tag{1.7}$$

Here, the results of [5] are included for $a,b\in(0,1]$, i.e. f and g are sublinear. Hence, it remains unknown the case $0<a\cdot b\le 1$, i.e. if an analogous result obtained by [5] holds for the more general system (1.1). In our paper, we give a new methodology for proving existence results under a class of general nonlinearities considered in other frameworks (see e.g. Orlicz Spaces Theory) including such the sublinear and superlinear class of functions discussed in [5]. This may be used in tackling other related problems.

The reminder of this paper is organized as follows. Section 2 contains our main result and some lemmas. In Section 3 we give the proof of our main result.

2. The main result and auxiliary lemmas

For the purpose of the paper, the following basic class of functions are considered

(P1) $p,q:[0,\infty)\to[0,\infty)$ continuous functions;

(C1) $f,g:[0,\infty)\to[0,\infty)$ are continuous and monotone non-decreasing such that

$$f(s)>0,\ g(t)>0 \text{ for all } s,t>0,$$

and

$$f(t\cdot s)\le f(t)\cdot f(s) \text{ and } g(t\cdot s)\le g(t)\cdot g(s) \text{ for all } s,t\ge 0;$$

(C2) $\int_1^\infty \frac{dt}{\left(1+f\left((1+g(t))^{1/k_3}\right)\right)^{1/k_1}}=\infty$ and $\int_1^\infty \frac{dt}{\left(1+g\left((1+f(t))^{1/k_1}\right)\right)^{1/k_3}}=\infty$.

Our main interest is to prove the following theorem.

Theorem 1. *If p, q satisfy (P1) and f, g satisfy (C1), (C2), then the system (1.1) has one positive entire radial solution $(u,v) \in C^2\left(\mathbb{R}^N\right) \times C^2\left(\mathbb{R}^N\right)$ such that*

$$u(x) \geq c_1 + \alpha_{N;k_1,k_2}\frac{|x|^2}{2} \text{ and } v(x) \geq c_2 + \beta_{N;k_3,k_4}\frac{|x|^2}{2}, \text{for all } x \in \mathbb{R}^N, \tag{2.1}$$

where

$$\alpha_{N;k_1,k_2} = \left(\frac{\alpha k_1 C_{N-1}^{k_2-1}}{k_2 C_{N-1}^{k_1-1}}\right)^{1/(k_1-k_2)}, \beta_{N;k_3,k_4} = \left(\frac{\beta k_3 C_{N-1}^{k_4-1}}{k_4 C_{N-1}^{k_3-1}}\right)^{1/(k_3-k_4)} \text{ and } c_1, c_2 \in (0,\infty).$$

Moreover, when p and q are non-decreasing, u and v are convex.

As we see from the paper of Zhang-Liu [7], our Theorem 1 represent a consistent generalization from the mathematical point of view. This is due to the fact that we deal with more general nonlinearities f and g that was considered by [7] and with a mixed nonlinear k_i-Hessian system of equations.

Next, let us recall the radial form of the k-Hessian operator, see for example [6] and [3].

Lemma 1. *Let $k \in \{1,2,...,N\}$. Assume $y \in C^2[0,R)$ is radially symmetric with $y'(0) = 0$. Then, the function u defined by $u(x) = y(r)$ where $r = |x| < R$ is $C^2(B_R)$, and*

$$\lambda\left(D^2 u(r)\right) = \begin{cases} \left(y''(r), \frac{y'(r)}{r}, ..., \frac{y'(r)}{r}\right) \text{ for } r \in (0,R), \\ (y''(0), y''(0), ..., y''(0)) \text{ for } r = 0 \end{cases}$$

$$S_k\left(\lambda\left(D^2 u(r)\right)\right) = \begin{cases} C_{N-1}^{k-1} y''(r)\left(\frac{y'(r)}{r}\right)^{k-1} + C_{N-1}^{k-1}\frac{N-k}{k}\left(\frac{y'(r)}{r}\right)^k \text{ for } r \in (0,R), \\ C_N^k (y''(0))^k \text{ for } r = 0, \end{cases}$$

where the prime denotes differentiation with respect to r.

Before to consider the proof of our main result, we give an useful lemma that can be easy proved as in the papers of Zhang-Liu [7] and Kusano-Swanson [4].

Lemma 2. *Setting*

$$\varphi_i(t) = t^{k_i} - t^{k_{i+1}} \text{ for } t \in \mathbb{R},\ i = 1,3,\ t_0^i = (k_{i+1}/k_i)^{1/(k_i-k_{i+1})}$$

the following hold:

1. $\varphi_i\left(t_0^i\right) = \frac{k_{i+1}-k_i}{k_i}\left(\frac{k_{i+1}}{k_i}\right)^{k_{i+1}/(k_i-k_{i+1})} < 0$, *$\varphi_i(1) = 0$ and $\varphi_i(\infty) := \lim_{t\to\infty}\varphi_i(t) = \infty$;*
2. $\varphi_i : \left[t_0^i, \infty\right) \to \left[\varphi_i\left(t_0^i\right), \infty\right)$ *is strictly increasing for $t > t_0^i$ and in fact has a uniquely defined inverse function $\phi_i : \left[\varphi_i\left(t_0^i\right), \infty\right) \to \left[t_0^i, \infty\right)$ with $\phi_i(0) = 1$;*
3. $\phi_i : \left[\varphi_i\left(t_0^i\right), \infty\right) \to \left[t_0^i, \infty\right)$ *is analytic, strictly increasing for $t > \varphi_i\left(t_0^i\right)$ and concave. In particular, $\phi_i(t) \geq 1$ for all $t \geq 0$, $\phi_i(\infty) := \lim_{t\to\infty}\phi_i(t) = \infty$ and for $t > \varphi_i\left(t_0^i\right)$ it hold*

$$\phi_i'(t) = \frac{1}{\left(\varphi_i' \circ \phi_i\right)(t)} = \frac{1}{k_i(\phi_i(t))^{k_i-1} - k_{i+1}(\phi_i(t))^{k_{i+1}-1}} > 0,$$

$$\phi_i''(t) = -\frac{k_i(k_i-1)(\phi_i(t))^{k_i-2} - k_{i+1}(k_{i+1}-1)(\phi_i(t))^{k_{i+1}-2}}{\left[k_i(\phi_i(t))^{k_i-1} - k_{i+1}(\phi_i(t))^{k_{i+1}-1}\right]^3} < 0;$$

4. $\phi_i(s\xi) \leq \xi^{1/k_i}\phi_i(s)$ *for all $s \geq 0$ and $\xi \geq 1$.*

3. The proof of Theorem 1

The main references for proving Theorem 1 are the works of [7] and [2]. In the next, r is referred for the Euclidean norm

$$|x| = \sqrt{x_1^2 + ... + x_N^2}$$

of a vector

$$x = (x_1, ..., x_N) \in \mathbb{R}^N.$$

We are ready to prove the existence of a radial solution

$$(u(r), v(r)) \in C^2([0,\infty)) \times C^2([0,\infty)),$$

to the problem (1.1). For beginning, we observe that we can rewrite (1.1) as follows

$$\begin{cases} C_{N-1}^{k_1-1}\left[\frac{r^{N-k_1}}{k_1}\left(u'(r)\right)^{k_1}\right]' - \alpha C_{N-1}^{k_2-1}\left[\frac{r^{N-k_2}}{k_2}\left(u'(r)\right)^{k_2}\right]' = r^{N-1}p(r) f(v(r)), \\ C_{N-1}^{k_3-1}\left[\frac{r^{N-k_3}}{k_3}\left(v'(r)\right)^{k_3}\right]' - \beta C_{N-1}^{k_4-1}\left[\frac{r^{N-k_4}}{k_4}\left(v'(r)\right)^{k_4}\right]' = r^{N-1}q(r) g(u(r)), \end{cases} \tag{3.1}$$

and that, the radial solution of (3.1) is a solution (u, v) of (3.1) with the initial conditions

$$(u(0), v(0)) = (c_1, c_2) \text{ and } (u'(0), v'(0)) = (0, 0). \tag{3.2}$$

Integrating from 0 to $r > 0$ in (3.1) we obtain

$$\begin{cases} C_{N-1}^{k_1-1}\frac{r^{N-k_1}}{k_1}\left(u'(r)\right)^{k_1} - \alpha C_{N-1}^{k_2-1}\frac{r^{N-k_2}}{k_2}\left(u'(r)\right)^{k_2} = \int_0^r s^{N-1}p(s) f(v(s))\,ds, \\ C_{N-1}^{k_3-1}\frac{r^{N-k_3}}{k_3}\left(v'(r)\right)^{k_3} - \beta C_{N-1}^{k_4-1}\frac{r^{N-k_4}}{k_4}\left(v'(r)\right)^{k_4} = \int_0^r s^{N-1}q(s) g(u(s))\,ds, \end{cases}$$

or, equivalently

$$\begin{cases} \left(\frac{u'(r)}{\alpha_{N;k_1,k_2} r}\right)^{k_1} - \left(\frac{u'(r)}{\alpha_{N;k_1,k_2} r}\right)^{k_2} = \frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s) f(v(s))\,ds, \\ \left(\frac{v'(r)}{\beta_{N;k_3,k_4} r}\right)^{k_3} - \left(\frac{v'(r)}{\beta_{N;k_3,k_4} r}\right)^{k_4} = \frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s) g(u(s))\,ds. \end{cases} \tag{3.3}$$

Using, the definition of ϕ_i given in Lemma 2, we rewrite (3.3) in an equivalent form

$$\begin{cases} \frac{u'(r)}{\alpha_{N;k_1,k_2} r} = \phi_1\left(\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s) f(v(s))\,ds\right), r > 0, \\ \frac{v'(r)}{\beta_{N;k_3,k_4} r} = \phi_3\left(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s) g(u(s))\,ds\right), r > 0, \end{cases}$$

which yields

$$\begin{cases} u'(r) = \alpha_{N;k_1,k_2} r\phi_1\left(\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s) f(v(s))\,ds\right), r > 0, \\ v'(r) = \beta_{N;k_3,k_4} r\phi_3\left(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s) g(u(s))\,ds\right), r > 0. \end{cases} \tag{3.4}$$

Since

$$\lim_{r\to 0+} u^{'}(r) = \lim_{r\to 0+} v^{'}(r) = 0 = u^{'}(0) = v^{'}(0),$$

via L'Hôpital's rule and (3.2), the equations in (3.4) can be extended by continuity at $r = 0$. Then, the system (3.1) with the initial conditions (3.2) can be equivalently written as an integral system of equations

$$\begin{cases} u(r) = c_1 + \alpha_{N;k_1,k_2} \int_0^r t\phi_1 \left(\frac{k_1 t^{-N}}{C_{N-1}^{k_1-1} \alpha_{N;k_1,k_2}^{k_1}} \int_0^t s^{N-1} p(s) f(v(s))\, ds \right) dt, \ r \geq 0, \\ v(r) = c_2 + \beta_{N;k_3,k_4} \int_0^r t\phi_3 \left(\frac{k_3 t^{-N}}{C_{N-1}^{k_3-1} \beta_{N;k_3,k_4}^{k_3}} \int_0^t s^{N-1} q(s) g(u(s))\, ds \right) dt, \ r \geq 0. \end{cases}$$

Let us now construct a sequence

$$\{(u_n(r), v_n(r))\}_{n\geq 0} \text{ on } [0,\infty)\times[0,\infty),$$

in such a way

$$\begin{cases} u_0(r) = u_0(0) = c_1,\ v_0(r) = v_0(0) = c_2, \\ u_n(r) = c_1 + \alpha_{N;k_1,k_2} \int_0^r t\phi_1 \left(\frac{k_1 t^{-N}}{C_{N-1}^{k_1-1} \alpha_{N;k_1,k_2}^{k_1}} \int_0^t s^{N-1} p(s) f(v_{n-1}(s))\, ds \right) dt, \\ v_n(r) = c_2 + \beta_{N;k_3,k_4} \int_0^r t\phi_3 \left(\frac{k_3 t^{-N}}{C_{N-1}^{k_3-1} \beta_{N;k_3,k_4}^{k_3}} \int_0^t s^{N-1} q(s) g(u_{n-1}(s))\, ds \right) dt. \end{cases} \tag{3.5}$$

By construction, for all $r \geq 0$ and $n \in \mathbb{N}$ we have

$$u_n(r) \geq c_1 \text{ and } v_n(r) \geq c_2.$$

Moreover, proceeding by induction we conclude

$$\{(u_n(r), v_n(r))\}_{n\geq 0}$$

is a non-decreasing sequence on

$$[0,\infty)\times[0,\infty).$$

We note that, for all $r > 0$ the sequence

$$\{(u_n(r), v_n(r))\}_{n\geq 0}$$

satisfies

$$\begin{cases} u_n^{'}(r) = \alpha_{N;k_1,k_2} r\phi_1 \left(\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1} \alpha_{N;k_1,k_2}^{k_1}} \int_0^r s^{N-1} p(s) f(v_{n-1}(s))\, ds \right) > \alpha_{N;k_1,k_2} r, \\ v_n^{'}(r) = \beta_{N;k_3,k_4} r\phi_3 \left(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1} \beta_{N;k_3,k_4}^{k_3}} \int_0^r s^{N-1} q(s) g(u_{n-1}(s))\, ds \right) > \beta_{N;k_3,k_4} r. \end{cases} \tag{3.6}$$

Integrating (3.6) from 0 to $r > 0$ we get (3.5). We now briefly, (3.5) imply

$$u_n(r) = c_1 + \alpha_{N;k_1,k_2} \int_0^r t\phi_1 \left(\frac{k_1 t^{-N}}{C_{N-1}^{k_1-1} \alpha_{N;k_1,k_2}^{k_1}} \int_0^t s^{N-1} p(s) f(v_{n-1}(s))\, ds \right) dt$$

$$
\begin{aligned}
&\leq c_1+\alpha_{N;k_1,k_2}\int_0^r t\phi_1\Big(\frac{k_1t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)f(v_n(s))\,ds\Big)dt\\
&\leq c_1+\alpha_{N;k_1,k_2}\left[1+f(v_n(r))\right]^{1/k_1}\int_0^r t\phi_1\Big(\frac{k_1t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)\,ds\Big)dt\\
&= \left[1+f(v_n(r))\right]^{1/k_1}\Big\{\frac{c_1}{\left[1+f(v_n(r))\right]^{1/k_1}}+\alpha_{N;k_1,k_2}\int_0^r t\phi_1\Big(\frac{k_1t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)\,ds\Big)dt\Big\}\\
&\leq \left[1+f(v_n(r))\right]^{1/k_1}\Big\{\frac{c_1}{\left[1+f(c_2)\right]^{1/k_1}}+\alpha_{N;k_1,k_2}\int_0^r t\phi_1\Big(\frac{k_1t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)\,ds\Big)dt\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
v_n(r) &= c_2+\beta_{N;k_3,k_4}\int_0^r t\phi_3\Big(\frac{k_3t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}q(s)g(u_{n-1}(s))\,ds\Big)dt\\
&\leq c_2+\beta_{N;k_1,k_2}\int_0^r t\phi_3\Big(\frac{k_3t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}q(s)g(u_n(s))\,ds\Big)dt\\
&\leq \left[1+g(u_n(r))\right]^{1/k_3}\Big\{\frac{c_2}{\left[1+g(u_n(r))\right]^{1/k_3}}+\beta_{N;k_3,k_4}\int_0^r t\phi_3\Big(\frac{k_3t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}p(s)\,ds\Big)dt\Big\}\\
&\leq \left[1+g(u_n(r))\right]^{1/k_3}\Big\{\frac{c_2}{\left[1+g(c_1)\right]^{1/k_3}}+\beta_{N;k_3,k_4}\int_0^r t\phi_3\Big(\frac{k_3t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}q(s)\,ds\Big)dt\Big\}.
\end{aligned}
$$

Setting

$$
\begin{cases}
\Lambda_p(r)=\frac{c_1}{\left[1+f(c_2)\right]^{1/k_1}}+\alpha_{N;k_1,k_2}\int_0^r t\phi_1\Big(\frac{k_1t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)\,ds\Big)dt,\\
\Lambda_q(r)=\frac{c_2}{\left[1+g(c_1)\right]^{1/k_3}}+\beta_{N;k_3,k_4}\int_0^r t\phi_3\Big(\frac{k_3t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}q(s)\,ds\Big)dt,
\end{cases}
$$

we have

$$
\begin{cases}
u_n(r)\leq\left[1+f(v_n(r))\right]^{1/k_1}\Lambda_p(r),\ r\geq 0,\\
v_n(r)\leq\left[1+g(u_n(r))\right]^{1/k_3}\Lambda_q(r),\ r\geq 0.
\end{cases}\tag{3.7}
$$

By the monotonicity of the sequence $\{(u_n,v_n)\}_{n\geq 0}$ respectively of f and g, the inequalities in (3.7) and with the use of Lemma 2 for

$$
\xi=1+f(\left[1+g(u_n(r))\right]^{1/k_3})\text{ and }s=\frac{k_1r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f\left(\Lambda_q(s)\right)ds,
$$

we have

$$
\begin{aligned}
u_n'(r) &= \alpha_{N;k_1,k_2}r\phi_1\Big(\frac{k_1r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f(v_{n-1}(s))\,ds\Big)\\
&\leq \alpha_{N;k_1,k_2}r\phi_1\Big(\frac{k_1r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f(v_n(s))\,ds\Big)\\
&\leq \alpha_{N;k_1,k_2}r\phi_1\Big(\frac{k_1r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f\left(\left[1+g(u_n(s))\right]^{1/k_3}\Lambda_q(s)\right)ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_{N;k_1,k_2} r\phi_1(\left(1+f\left([1+g(u_n(r))]^{1/k_3}\right)\right)\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f\left(\Lambda_q(s)\right)ds) \\
&\leq \alpha_{N;k_1,k_2} r\left(1+f\left([1+g(u_n(r))]^{1/k_3}\right)\right)^{1/k_1}\phi_1(\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f\left(\Lambda_q(s)\right)ds),
\end{aligned}
$$

and, similarly

$$
\begin{aligned}
v_n'(r) &= \beta_{N;k_3,k_4} r\phi_3(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s)g(u_{n-1}(s))ds) \\
&\leq \beta_{N;k_3,k_4} r\phi_3(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s)g(u_n(s))ds) \\
&\leq \beta_{N;k_3,k_4} r\phi_3(\left(1+g\left([1+f(v_n(r))]^{1/k_1}\right)\right)\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s)g\left(\Lambda_p(s)\right)ds) \\
&\leq \beta_{N;k_3,k_4} r\left(1+g\left([1+f(v_n(r))]^{1/k_1}\right)\right)^{1/k_3}\phi_3(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s)g\left(\Lambda_p(s)\right)ds).
\end{aligned}
$$

Finally

$$
\begin{cases}
\dfrac{u_n'(r)}{\left(1+f\left(\left[1+g(u_n(r))\right]^{1/k_3}\right)\right)^{1/k_1}} \leq \alpha_{N;k_1,k_2} r\phi_1(\frac{k_1 r^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^r s^{N-1}p(s)f\left(\Lambda_q(s)\right)ds), \\
\dfrac{v_n'(r)}{\left(1+g\left(\left[1+f(v_n(r))\right]^{1/k_1}\right)\right)^{1/k_3}} \leq \beta_{N;k_3,k_4} r\phi_3(\frac{k_3 r^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^r s^{N-1}q(s)g\left(\Lambda_q(s)\right)ds).
\end{cases}
\tag{3.8}
$$

Integrating (3.8) from 0 to $r > 0$ we get

$$
H_{1,c_1}(u_n(r)) \leq \Lambda_{p,\alpha}(r) \text{ and } H_{2,c_2}(v_n(r)) \leq \Lambda_{q,\beta}(r),
$$

where

$$
\begin{cases}
H_{1,c_1}(s) = \int_{c_1}^s \frac{dt}{\left(1+f\left((1+g(t))^{1/k_3}\right)\right)^{1/k_1}}, \\
\Lambda_{p,\alpha}(r) = \alpha_{N;k_1,k_2}\int_0^r t\phi_1\left(\frac{k_1 t^{-N}}{C_{N-1}^{k_1-1}\alpha_{N;k_1,k_2}^{k_1}}\int_0^t s^{N-1}p(s)f\left(\Lambda_q(s)\right)ds\right)dt, \\
H_{2,c_2}(s) = \int_{c_2}^s \frac{dt}{\left(1+g\left((1+f(t))^{1/k_1}\right)\right)^{1/k_3}}, \\
\Lambda_{q,\beta}(r) = \beta_{N;k_3,k_4}\int_0^r t\phi_3\left(\frac{k_3 t^{-N}}{C_{N-1}^{k_3-1}\beta_{N;k_3,k_4}^{k_3}}\int_0^t s^{N-1}q(s)g\left(\Lambda_q(s)\right)ds\right)dt.
\end{cases}
$$

Choose $R > 0$. We are now ready to show that

$$
\{(u_n(r), v_n(r))\}_{n\geq 0} \text{ and } \{(u_n'(r), v_n'(r))\}_{n\geq 0}, \text{ for } r \in [0, R],
$$

both of which are non-negative, are bounded above independent of n. To solve this problem, we observe that

$$
\begin{cases}
H_{1,c_1}(u_n(r)) \leq \Lambda_{p,\alpha}(r) \leq \Lambda_{p,\alpha}(R) \text{ for all } r \in [0, R], \\
H_{2,c_2}(v_n(r)) \leq \Lambda_{q,\beta}(r) \leq \Lambda_{q,\beta}(R) \text{ for all } r \in [0, R].
\end{cases}
$$

On the other hand, since

$$s \to H_{i,c_i}(s), i = 1, 2$$

is a bijection map for all $s > c_i$ with the inverse denoted by $H_{i,c_i}^{-1}(s)$ on $[0, \infty)$ such that

$$H_{i,c_i}^{-1}(\infty) = \infty \text{ and } H_{i,c_i}^{-1}(s) \text{ is increasing on } [c_i, \infty),$$

we see that

$$\begin{cases} u_n(r) \le H_{1,c_1}^{-1}\left(\Lambda_{p,\alpha}(R)\right) \text{ for all } r \in [0, R], \\ v_n(r) \le H_{2,c_2}^{-1}\left(\Lambda_{q,\beta}(R)\right) \text{ for all } r \in [0, R], \end{cases}$$

which proved that

$$\{(u_n(r), v_n(r))\}_{n\ge 0},$$

is an uniformly bounded independent of n sequence on

$$[0, R] \times [0, R],$$

for arbitrary $R > 0$. On the other hand, using this result in (3.8) the same is true for

$$\{(u_n'(r), v_n'(r))\}_{n\ge 0}.$$

We finished the proof that the sequences

$$\{(u_n(r), v_n(r))\}_{n\ge 0} \text{ and } \{(u_n'(r), v_n'(r))\}_{n\ge 0},$$

are bounded above independent of n which coupled with the fact that

$$(u_n(r), v_n(r)),$$

is non-decreasing on $[0, \infty) \times [0, \infty)$ we see that

$$\{u_n(r), v_n(r)\}_{n\ge 0}$$

itself converges to a function

$$(u(r), v(r)) \text{ as } n \to \infty,$$

and the limit $(u(r), v(r))$ is a positive entire radial solution of equation (1.1). Clearly, the arguments in Zhang and Liu [7] (see also [2]) guarantees that the solution

$$(u(x), v(x)) := (u(|x|), v(|x|))$$

is in the space

$$C^2\left(\mathbb{R}^N\right) \times C^2\left(\mathbb{R}^N\right)$$

and moreover is convex for any $x \in \mathbb{R}^N$. This is the end of the proof of the theorem.

4. Conclusions

We have obtained new conditions for the study of existence of positive radial solutions for a system involving the Hessian operator.

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Conflict of interest

All author declare no conflicts of interest in this paper.

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