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Research article

Merit functions for absolute value variational inequalities

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Abstract: This article deals with a class of variational inequalities known as absolute value variational inequalities. Some new merit functions for the absolute value variational inequalities are established. Using these merit functions, we derive the error bounds for absolute value variational inequalities. Since absolute value variational inequalities contain variational inequalities, absolute value complementarity problem and system of absolute value equations as special cases, the findings presented here recapture various known results in the related domains. The conclusions of this paper are more comprehensive and may provoke futuristic research.

Keywords: absolute value variational inequalities; fixed points; merit functions; error bounds
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1. Introduction

Variational inequalities earlier introduced and studied by Stampacchia [1] are now interestingly applied in the fields of management, finance, economics, optimization and almost in all branches of pure and applied sciences, see [2–15]. Since variational inequalities provide a natural framework to solve different mathematical and scientific problems, various techniques including projection method, auxiliary principle technique, Wiener-Hopf equations and dynamical systems have been developed for finding the solution of variational inequalities and associated optimization problems, see [2–25] the references therein.

Absolute value variational inequalities are the significant and useful generalizations of variational inequalities which were introduced and studied by Mangasarian, see [26]. It was shown by Rohn [27] that absolute value variational inequalities are equivalent to complementarity problem and further considered by Mangasarian and Meyer [28] using different methodology. Absolute value variational inequalities are more general as they contain classical variational inequalities as a special case. It has been proved through projection lemma that the absolute value variational inequality and fixed point problem are equivalent, see [2,3]. Using this equivalence between absolute value variational inequality and fixed point problem, various iterative schemes are developed for solving absolute value variational inequalities and to examine the associated optimization problems, see [3, 14].

An innovative aspect in the study of variational inequalities concerns merit functions through which the variational inequality problem can be reformulated into an optimization problem. It was Auslender [29], who introduced the first merit function in optimization theory. Merit functions play important roles in developing globally convergent iterative schemes and investigating the rate of convergence for some iterative schemes, see [20–26]. Several merit functions are being suggested and analyzed for variational inequalities and hence for the complementarity problems as a variational inequality problem can be rephrased into a complementarity problem, see [30–41] and the references therein. Error bounds also contribute significantly in the study of variational inequalities as error bounds are the functions which estimate the closeness of an arbitrary point to the solution set in an approximate computation of the iterates for solving variational inequalities, see [31–35].

In spite of the huge lift in the field of variational inequalities and optimization theory, we present and investigate some new merit functions for absolute variational inequalities in this work. We also suggest the error bounds for the solution of absolute variational inequalities under some suitable constraints. The proofs of our proposed results are easy and direct in comparison with other methods and these results also remain true for the associated problems of absolute value variational inequalities. Hence, the findings of this paper provide a substantial addition in this field.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $||.||$ and $< .,. >$ respectively. Let K be a closed and convex set in $\mathscr{H}$. For given operators $\mathcal{T}, \mathcal{B} : \mathscr{H} \to \mathscr{H}$, consider the problem of finding $u \in K$ such that

$$\langle \mathcal{T}u + \mathcal{B}|u|, v - u\rangle \geq 0, \qquad \forall v \in \mathscr{H}, \tag{2.1}$$

where $|u|$ contains the absolute values of components of $u \in \mathscr{H}$. The inequality (2.1) is called absolute value variational inequality. The absolute value variational inequality (2.1) can be viewed as a difference of two operators and includes previously known classes of variational inequalities as special cases. For the recent applications of absolute value variational inequalities, see [2,3,32,34,38] and the references therein.

In order to derive the main results of this paper, we recall some standard definitions and results.

Definition 2.1. An operator $\mathcal{T} : \mathscr{H} \to \mathscr{H}$ is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{T}u - \mathcal{T}v, u - v\rangle \geq \alpha||u - v||^2, \qquad \forall u, v \in \mathscr{H}.$$

Definition 2.2. An operator $\mathcal{T} : \mathscr{H} \to \mathscr{H}$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$||\mathcal{T}u - \mathcal{T}v|| \leq \beta||u - v||, \qquad \forall u, v \in \mathscr{H}.$$

If $\mathcal{T}$ is strongly monotone and Lipschitz continuous operator, then from definitions (2.1) and (2.2), we have $\alpha \leq \beta$.

Definition 2.3. An operator $\mathcal{T} : \mathscr{H} \to \mathscr{H}$ is said to be monotone, if

$$\langle \mathcal{T}u - \mathcal{T}v, u - v \rangle \geq 0, \qquad \forall u, v \in \mathscr{H}.$$

Definition 2.4. An operator $\mathcal{T} : \mathscr{H} \to \mathscr{H}$ is said to be pseudomonotone, if

$$\langle \mathcal{T}u, v - u \rangle \geq 0,$$

implies

$$\langle \mathcal{T}v, v - u \rangle \geq 0 \qquad \forall u, v \in \mathscr{H}.$$

Definition 2.5. [36] A function $\mathscr{M} : \mathscr{H} \to \mathbb{R}U\{+\infty\}$ is called a merit (gap) function for the inequality 2.1, if and only if

(i) $\mathscr{M}(u) \geq 0, \forall u \in \mathscr{H}$.

(ii) $\mathscr{M}(\bar{u}) = 0$, if and only if, $\bar{u} \in \mathscr{H}$ solves inequality (2.1).

We now consider the well-known projection lemma which is due to [6]. This lemma is useful to reformulate the variational inequalities into a fixed point problem.

Lemma 2.6. *[6] Let K be a closed and convex set in $\mathscr{H}$. Then for a given $z \in \mathscr{H}, u \in K$ satisfies*

$$\langle u - z, v - u \rangle \geq 0, \qquad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the projection of $\mathscr{H}$ onto a closed and convex set K in $\mathscr{H}$.

It is remarkable that the projection operator P_K is non-expansive operator, that is

$$\|P_K[u] - P_K[v]\| \leq \|u - v\|, \forall u, v \in \mathscr{H}.$$

3. Main results

In this section, we suggest some merit functions associated with absolute value variational inequalities. Using these merit functions, we attain some error bounds for absolute value variational inequalities. To obtain this, we show that the variational inequalities are equivalent to the fixed point problem.

Lemma 3.1. *[2, 14] Let K be a closed convex set in $\mathscr{H}$. The function $u \in K$ is a solution of absolute value variational inequality* (2.1)*, if and only if, $u \in K$ satisfies the relation*

$$u = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|], \tag{3.1}$$

where $\rho > 0$ is a constant.

It follows from the above lemma that the absolute value variational inequality (2.1) and the fixed point problem (3.1) are equivalent. This alternative equivalent formulation is very advantageous from the theoretical as well as from the numerical point of view and is obtained by using projection technique. The projection methods are due to Lions and Stampacchia [4] which provide several effective schemes to approximate the solution of variational inequalities. The equivalence between variational inequalities and the fixed point problem plays a significant role in establishing the various results for problem (2.1) and its related formulations.

Lemma 3.2. *For all $u, v \in \mathscr{H}$, we have*

$$\|u\|^2 + \langle u, v\rangle \geq -\frac{1}{4}\|v\|^2.$$

Now, we define the residue vector $\mathcal{R}(u)$ by the following relation

$$\mathcal{R}_\rho(u) \equiv \mathcal{R}(u) = u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]. \tag{3.2}$$

From lemma 2.6, it can also be concluded that $u \in K$ is a solution of the absolute value variational inequality (2.1), if and only if, $u \in K$ is a zero of the equation

$$\mathcal{R}_\rho(u) \equiv \mathcal{R}(u) = 0.$$

We now show that the residue vector $\mathcal{R}_\rho(u)$ is strongly monotone and Lipschitz continuous.

Lemma 3.3. *Let the operators $\mathcal{T}$ and $\mathcal{B}$ be Lipschitz continuous with constants $\beta_{\mathcal{T}} > 0$ and $\beta_{\mathcal{B}} > 0$ and $\mathcal{T}$ be strongly monotone with constant $\alpha_{\mathcal{T}} > 0$, respectively then the residue vector $\mathcal{R}_\rho(u)$, defined by (3.2) is strongly monotone on $\mathscr{H}$.*

Proof. For all $u, v \in \mathscr{H}$, consider

$$\begin{aligned}
\langle \mathcal{R}_\rho(u) - \mathcal{R}_\rho(v), u - v\rangle &= \langle u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - v + P_K[v - \rho\mathcal{T}v \\
&\quad - \rho\mathcal{B}|v|], u - v\rangle \\
&= \langle u - v - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] + P_K[v - \rho\mathcal{T}v \\
&\quad - \rho\mathcal{B}|v|], u - v\rangle \\
&= \langle u - v, u - v\rangle - \langle P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - P_K[v \\
&\quad - \rho\mathcal{T}v - \rho\mathcal{B}|v|], u - v\rangle \\
&\geq \|u - v\|^2 - \|P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u| - P_K[v \\
&\quad - \rho\mathcal{T}v - \rho\mathcal{B}|v|]\|\|u - v\| \\
&\geq \|u - v\|^2 - \|(u - v) - \rho(\mathcal{T}u - \mathcal{T}v)\|\|u - v\| \\
&\quad \|(\mathcal{B}|u| - \mathcal{B}|v|)\| - \rho\|u - v\| \\
&\geq \|u - v\|^2 - \{\sqrt{1 - 2\rho\beta_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2} + \rho\beta_{\mathcal{B}}\}\|u - v\|^2 \\
&= (1 - \sqrt{1 - 2\rho\beta_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2} - \rho\beta_{\mathcal{B}})\|u - v\|^2,
\end{aligned}$$

which implies that

$$\langle \mathcal{R}_\rho(u) - \mathcal{R}_\rho(v), u - v\rangle \geq \vartheta\|u - v\|^2,$$

where
$$\vartheta = (1 - \sqrt{1 - 2\rho\beta_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2} - \rho\beta_{\mathcal{B}}) > 0,$$
which proves that the residue vector $\mathcal{R}_\rho(u)$ is strongly monotone with constant $\vartheta > 0$. □

Lemma 3.4. *Let the operators $\mathcal{T}$ and $\mathcal{B}$ be Lipschitz continuous with constants $\beta_{\mathcal{T}} > 0$ and $\beta_{\mathcal{B}} > 0$ and $\mathcal{T}$ be strongly monotone with constant $\alpha_{\mathcal{T}} > 0$, respectively then the residue vector $\mathcal{R}_\rho(u)$, defined by ((3.2)), is Lipschitz continuous on $\mathscr{H}$.*

Proof. For all $u, v \in \mathscr{H}$, consider

$$\begin{aligned}\|\mathcal{R}_\rho(u) - \mathcal{R}_\rho(v)\| &= \|u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - v + P_K[v - \rho\mathcal{T}v - \rho\mathcal{B}|v|]\| \\ &\leq \|u - v\| + \|P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - P_K[v - \rho\mathcal{T}v - \rho\mathcal{B}|v|\| \\ &\leq \|u - v\| + \|(u - v) - \rho(\mathcal{T}u - \mathcal{T}v) - \rho(\mathcal{B}|u| - \mathcal{B}|v|)\| \\ &\leq 2\|u - v\| + \sqrt{1 - 2\rho\alpha_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2}\|(u - v)\| \\ &+ \sqrt{1 - 2\rho\alpha_{\mathcal{B}} + \rho^2\beta_{\mathcal{B}}^2}\|(u - v)\| \\ &= (2 + \sqrt{1 - 2\rho\alpha_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2} + \sqrt{1 - 2\rho\alpha_{\mathcal{B}} + \rho^2\beta_{\mathcal{B}}^2})\|(u - v)\| \\ &= \varphi\|(u - v)\|,\end{aligned}$$

where
$$\varphi = 2 + \sqrt{1 - 2\rho\alpha_{\mathcal{T}} + \rho^2\beta_{\mathcal{T}}^2} + \sqrt{1 - 2\rho\alpha_{\mathcal{B}} + \rho^2\beta_{\mathcal{B}}^2} > 0.$$

For the proof of above result, we have used the Lipschitz continuity and strongly monotonicity of the operators $\mathcal{T}$ and $\mathcal{B}$ with constants $\beta_{\mathcal{T}} > 0, \beta_{\mathcal{B}} > 0$ and $\alpha_{\mathcal{T}} > 0, \alpha_{\mathcal{B}} > 0$, respectively. Thus the residue vector $\mathcal{R}_\rho(u)$ is Lipschitz continuous with constant $\varphi > 0$. This completes the proof. □

We now use the residue vector $\mathcal{R}_\rho(u)$, defined by (3.2), to derive the error bound for the solution of the problem (2.1).

Theorem 3.5. *Let $\hat{u} \in \mathscr{H}$ be a solution of the absolute value variational inequality* (2.1). *If the operators $\mathcal{T}$ and $\mathcal{B}$ are Lipschitz continuous with constants $\beta_{\mathcal{T}} > 0$ and $\beta_{\mathcal{B}} > 0$ and strongly monotone with constants $\alpha_{\mathcal{T}} > 0$ and $\alpha_{\mathcal{B}} > 0$, respectively then*

$$\frac{1}{l_1}\|\mathcal{R}_\rho(u)\| \leq \|\hat{u} - u\| \leq l_2\|\mathcal{R}_\rho(u)\| , \quad \forall u \in \mathscr{H}.$$

Proof. Let $\hat{u} \in \mathscr{H}$ solves the absolute value variational inequality (2.1). Then we have

$$\langle \rho\mathcal{T}\hat{u} + \rho\mathcal{B}|\hat{u}|, v - \hat{u}\rangle \geq 0, \qquad \forall v \in \mathscr{H}. \tag{3.3}$$

Take $v = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]$ in (3.3), to have

$$\langle \rho\mathcal{T}\hat{u} + \rho\mathcal{B}|\hat{u}|, P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - \hat{u}\rangle \geq 0. \tag{3.4}$$

Take $u = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]$, $z = u - \rho\mathcal{T}u - \rho\mathcal{B}|u|$ and $v = \hat{u}$ in projection lemma 2.6 to have

$$\langle P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - u + \rho\mathcal{T}u + \rho\mathcal{B}|u|, \hat{u} - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\rangle \geq 0,$$

which shows that

$$\langle -\rho\mathcal{T}u-\rho\mathcal{B}|u|+u-P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|],P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]-\hat{u}\rangle\geq 0. \tag{3.5}$$

Addition of the inequalities (3.4) and (3.5) implies

$$\langle\rho(\mathcal{T}\hat{u}-\mathcal{T}u)+\rho(\mathcal{B}|\hat{u}|-\mathcal{B}|u|)+(u-P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]),P_K[u-\rho\mathcal{T}u$$

$$-\rho\mathcal{B}|u|]-\hat{u}\rangle\geq 0,$$

Using (3.2), we obtain

$$\langle\mathcal{T}\hat{u}-\mathcal{T}u,P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]\rangle+\langle\mathcal{B}|\hat{u}|-\mathcal{B}|u|,\hat{u}-P_K[u-\rho\mathcal{T}u \tag{3.6}$$

$$-\rho\mathcal{B}|u|]\rangle\leq\frac{1}{\rho}\langle\mathcal{R}(u),P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]-\hat{u}\rangle. \tag{3.7}$$

By the strong monotonicity of the operators $\mathcal{T}$ and $\mathcal{B}$ with constants $\alpha_{\mathcal{T}}>0$ and $\alpha_{\mathcal{B}}>0$, respectively, we obtain

$$\begin{aligned}\alpha_{\mathcal{T}}\|\hat{u}-u\|^2&\leq\langle\mathcal{T}\hat{u}-\mathcal{T}u,\hat{u}-u\rangle\\&\leq\langle\mathcal{T}\hat{u}-\mathcal{T}u,\hat{u}-P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]\rangle+\langle\mathcal{T}\hat{u}-\mathcal{T}u,P_K[u-\rho\mathcal{T}u\\&-\rho\mathcal{B}|u|]-u\rangle,\end{aligned}$$

and

$$\begin{aligned}\alpha_{\mathcal{B}}\|\hat{u}-u\|^2&\leq\langle\mathcal{B}\hat{u}-\mathcal{B}u,\hat{u}-u\rangle\\&\leq\langle\mathcal{B}\hat{u}-\mathcal{B}u,\hat{u}-P_{\mathcal{K}}[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]\rangle+\langle\mathcal{B}\hat{u}-\mathcal{B}u,P_{\mathcal{K}}[u-\rho\mathcal{T}u\\&-\rho\mathcal{B}|u|]-u\rangle,\end{aligned}$$

using (3.2) and (3.6), we obtain

$$\begin{aligned}(\alpha_{\mathcal{T}}+\alpha_{\mathcal{B}})\|\hat{u}-u\|^2\leq\frac{1}{\rho}\langle\mathcal{R}(u),P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]-\hat{u}\rangle+\langle\mathcal{T}\hat{u}\\-\mathcal{T}u,-\mathcal{R}(u)\rangle+\langle\mathcal{B}\hat{u}-\mathcal{B}u,-\mathcal{R}(u)\rangle,\end{aligned}$$

Using the Lipschitz continuity of the operators $\mathcal{T}$ and $\mathcal{B}$ with constants $\beta_{\mathcal{T}}>0,\beta_{\mathcal{B}}>0$, respectively, we obtain

$$\begin{aligned}\rho(\alpha_{\mathcal{T}}+\alpha_{\mathcal{B}})\|\hat{u}-u\|^2&\leq\frac{1}{\rho}\langle\mathcal{R}(u),P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]-\hat{u}\rangle+\rho\langle\mathcal{T}\hat{u}-\mathcal{T}u,-\mathcal{R}(u)\rangle\\&+\rho\langle\mathcal{B}\hat{u}-\mathcal{B}u,-\mathcal{R}(u)\rangle\\&\leq\langle\mathcal{R}(u),-\mathcal{R}(u)\rangle-\langle\mathcal{R}(u),\hat{u}-u\rangle+\rho\langle\mathcal{T}\hat{u}-\mathcal{T}u,-\mathcal{R}(u)\rangle+\\&\rho\langle\mathcal{B}\hat{u}-\mathcal{B}u,-\mathcal{R}(u)\rangle\\&\leq-\|\mathcal{R}(u)\|^2+\|\hat{u}-u\|\|\mathcal{R}(u)\|+\rho\beta_{\mathcal{T}}\|\hat{u}-u\|\|\mathcal{R}(u)\|+\\&\rho\beta_{\mathcal{B}}\|\hat{u}-u\|\|R(u)\|\\&=-\|\mathcal{R}(u)\|^2+(1+\rho(\beta_{\mathcal{T}}+\beta_{\mathcal{B}}))\|\hat{u}-u\|\|\mathcal{R}(u)\|\\&\leq(1+\rho(\beta_{\mathcal{T}}+\beta_{\mathcal{B}}))\|\hat{u}-u\|\|\mathcal{R}(u)\|,\end{aligned}$$

which implies that

$$||\hat{u} - u|| \leq \frac{(1 + \rho(\beta_{\mathcal{T}} + \beta_{\mathcal{B}}))}{\rho(\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}})}||\mathcal{R}(u)|| = l_2||\mathcal{R}(u)||, \tag{3.8}$$

where

$$l_2 = \frac{(1 + \rho(\beta_{\mathcal{T}} + \beta_{\mathcal{B}}))}{\rho(\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}})}.$$

Now, using the relation (3.2), we have

$$\begin{aligned}
||\mathcal{R}(u)|| &= ||u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]|| \\
&\leq ||\hat{u} - u|| + ||\hat{u} - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]|| \\
&\leq ||\hat{u} - u|| + ||P_K[\hat{u} - \rho\mathcal{T}\hat{u} - \rho\mathcal{B}|\hat{u}|] - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]|| \\
&\leq ||\hat{u} - u|| + ||\hat{u} - \rho\mathcal{T}\hat{u} - \rho\mathcal{B}|\hat{u}| - u + \rho\mathcal{T}u + \rho\mathcal{B}|u||| \\
&\leq ||\hat{u} - u|| + ||\hat{u} - u|| + \rho||\mathcal{T}\hat{u} - \mathcal{T}u|| + \rho||\mathcal{B}|\hat{u}| - \mathcal{B}|u||| \\
&\leq 2||\hat{u} - u|| + \rho\beta_{\mathcal{T}}||\hat{u} - u|| + \rho\beta_{\mathcal{B}}||\hat{u} - u|| \\
&= (2 + \rho(\beta_{\mathcal{T}} + \beta_{\mathcal{B}}))||\hat{u} - u|| \\
&= l_1||\hat{u} - u||,
\end{aligned}$$

which shows that

$$\frac{1}{l_1}||\mathcal{R}(u)|| \leq ||\hat{u} - u||, \tag{3.9}$$

where

$$l_1 = (2 + \rho(\beta_{\mathcal{T}} + \beta_{\mathcal{B}})).$$

Combining (3.8) and (3.9), we obtain

$$\frac{1}{l_1}||\mathcal{R}(u)|| \leq ||\hat{u} - u|| \leq l_2||\mathcal{R}(u)|| , \quad \forall u \in \mathscr{H}. \tag{3.10}$$

which is the required result. □

Now, substituting $u = 0$ in (3.10), we obtain

$$\frac{1}{l_1}||\mathcal{R}(0)|| \leq ||\hat{u} - u|| \leq l_2||\mathcal{R}(0)|| , \quad \forall u \in \mathscr{H}. \tag{3.11}$$

Combining (3.10) and (3.11), we get a relative error bound for any $u \in \mathscr{H}$.

Theorem 3.6. *Suppose that all the conditions of Theorem 3.5 hold. If* $0 \neq u \in \mathscr{H}$ *is a solution of the absolute value variational inequality* (2.1)*, then*

$$s_1\frac{||\mathcal{R}(u)||}{||\mathcal{R}(0)||}|| \leq \frac{||u - \hat{u}||}{\hat{u}} \leq s_2\frac{||\mathcal{R}(u)||}{||\mathcal{R}(0)||}.$$

It is noted that the normal residue vector $\mathcal{R}(u)$, defined in (3.2), is nondifferentiable. To resolve the nondifferentiability which is a significant limitation of the regularized merit function, we examine another merit function associated with the absolute value variational inequality (2.1). This merit function can be regarded as a regularized merit function. For all $u \in \mathscr{H}$, consider the function, such that

$$\mathscr{M}_\rho(u) = \langle \mathcal{T}u + \mathcal{B}|u|, u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\rangle - \frac{1}{2\rho}\|u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\|^2. \tag{3.12}$$

It is clear from the above equation that $\mathscr{M}_\rho(u) \geq 0$, for all $u \in \mathscr{H}$.

Now, we prove that the function established in (3.12), is a merit function and this is the leading objective of our next results.

Theorem 3.7. *For all $u \in \mathscr{H}$, we have*

$$\mathscr{M}_\rho(u) \geq \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2.$$

In particular, we have $\mathscr{M}_\rho(u) = 0$, if and only if $u \in \mathscr{H}$ is a solution of the absolute value variational inequality (2.1).

Proof. By substituting $u = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|], z = u - \rho\mathcal{T}u - \rho\mathcal{B}|u|$ and $v = u$ in lemma 2.6, we obtain

$$\langle \rho\mathcal{T}u + \rho\mathcal{B}|u| + P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - u, u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\rangle \geq 0.$$

Using (3.12) and lemma 3.2, we obtain

$$\begin{aligned}
0 &\leq \langle \rho\mathcal{T}u + \rho\mathcal{B}|u| - (u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]), u - P_K[u - \rho\mathcal{T}u \\
&\quad - \rho\mathcal{B}|u|]\rangle \\
&= \langle \rho\mathcal{T}u + \rho\mathcal{B}|u| - \mathcal{R}_\rho(u), \mathcal{R}_\rho(u)\rangle \\
&= \langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\rho(u)\rangle - \frac{1}{\rho}\langle \mathcal{R}_\rho(u), \mathcal{R}_\rho(u)\rangle \\
&= \mathscr{M}_\rho(u) + \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 - \frac{1}{\rho}\|\mathcal{R}_\rho(u)\|^2 \\
&= \mathscr{M}_\rho(u) - \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2,
\end{aligned}$$

which shows that

$$\mathscr{M}_\rho(u) \geq \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2,$$

which is the required result. □

It is clear from the above inequality that $\mathscr{M}_\rho(u) \geq 0, \forall u \in \mathscr{H}$. Also, if $\mathscr{M}_\rho(u) = 0$, then from the above inequality, we obtain $\mathcal{R}_\rho(u) = 0$. Hence, according to lemma 3.1, it is clear that $u \in \mathscr{H}$ solves the absolute value variational inequality (2.1). On the other hand, if $u \in \mathscr{H}$ is a solution of absolute value variational inequality (2.1), then by lemma 3.1, we have $u = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]$. Therefore, from (3.12), we obtain, $\mathscr{M}_\rho(u) = 0$, which was the required result.

It is observed from Theorem 3.7 that $\mathscr{M}_\rho(u)$ defined by (3.12), is a merit function for the absolute value variational inequality (2.1). We also notice that the regularized merit function is differentiable, if the operators $\mathcal{T}$ and $\mathcal{B}$ are differentiable. Now, we obtain the error bounds for the absolute value variational inequality if both the operators $\mathcal{T}$ and $\mathcal{B}$ are not Lipschitz continuous.

Theorem 3.8. *Let $\hat{u} \in \mathcal{H}$ be a solution of the absolute value variational inequality* (2.1). *Let the operators $\mathcal{T}$ and $\mathcal{B}$ be strongly monotone with the constants $\alpha_{\mathcal{T}} > 0, \alpha_{\mathcal{B}} > 0$, respectively. Then*

$$\|u-\hat{u}\|^2 \leq \frac{4\rho}{\rho(\alpha_{\mathcal{T}}+\alpha_{\mathcal{B}})}[\mathcal{M}_\rho(u) + \frac{1}{\rho}\|\rho\mathcal{T}\hat{u}+\rho\mathcal{B}|\hat{u}|\|^2], \quad \forall u \in \mathcal{H}.$$

Proof. Let $\hat{u} \in \mathcal{H}$ be a solution of the absolute value variational inequality (2.1) and by taking $v = u$, we have

$$\langle \rho\mathcal{T}\hat{u} + \rho\mathcal{B}|\hat{u}|, u-\hat{u}\rangle \geq 0,$$

using lemma 3.1, we have

$$\langle \mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|, u-\hat{u}\rangle \geq \frac{1}{4\rho}\|u-\hat{u}\|^2 - \frac{1}{\rho}\|\mathcal{T}\hat{u}+\mathcal{B}|\hat{u}|\|^2.$$

Using (3.12) and strong monotonicity of the operators $\mathcal{T}$ and $\mathcal{B}$, we have

$$\begin{aligned}
\mathcal{M}_\rho(u) &= \langle \mathcal{T}u + \mathcal{B}|u|, u-\hat{u}]\rangle - \frac{1}{2\rho}\|u-\hat{u}\|^2 \\
&= \langle \mathcal{T}u - \mathcal{T}\hat{u} + \mathcal{T}\hat{u} + \mathcal{B}|u| - \mathcal{B}|\hat{u}| + \mathcal{B}|\hat{u}|, u-\hat{u}]\rangle - \frac{1}{2\rho}\|u-\hat{u}\|^2 \\
&= \langle \mathcal{T}u - \mathcal{T}\hat{u}, u-\hat{u}]\rangle + \langle \mathcal{B}|u| - \mathcal{B}|\hat{u}|, u-\hat{u}]\rangle + \langle \mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|, u-\hat{u}\rangle \\
&- \frac{1}{2\rho}\|u-\hat{u}\|^2 \\
&\geq \alpha_{\mathcal{T}}\|u-\hat{u}\|^2 + \alpha_{\mathcal{B}}\|u-\hat{u}\|^2 + \langle \mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|, u-\hat{u}\rangle - \frac{1}{2\rho}\|u-\hat{u}\|^2 \\
&\geq (\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}} - \frac{1}{2\rho})\|u-\hat{u}\|^2 + \frac{1}{4\rho}\|u-\hat{u}\|^2 - \frac{1}{\rho}\|\mathcal{T}\hat{u} + B|\hat{u}|\|^2 \\
&= (\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}} - \frac{1}{2\rho} + \frac{1}{4\rho})\|u-\hat{u}\|^2 - \frac{1}{\rho}\|\mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|\|^2,
\end{aligned}$$

which shows that

$$\|u-\hat{u}\|^2 \leq \frac{4\rho}{4\rho(\alpha_{\mathcal{T}}+\alpha_{\mathcal{B}})-1}[\mathcal{M}_\rho(u) + \frac{1}{\rho}\|\rho\mathcal{T}\hat{u}+\rho\mathcal{B}|\hat{u}|\|^2],$$

which is the required result. □

Now, we study one more merit function associated to the absolute value variational inequality. This merit function is the difference between two regularized merit functions associated with (2.1). Many authors used to study such type of merit functions to find the solution of variational inequalities and complementarity problems, see [2, 3, 38–41]. We define the D-merit function for absolute value variational inequality, which is the difference of regularized merit function defined by (3.12). We consider the following function

$$\begin{aligned}
\mathcal{D}_{\rho,\zeta}(u) &= \mathscr{M}_\rho(u) - \mathscr{M}_\zeta(u) \\
&= \langle \mathcal{T}u + \mathcal{B}|u|, u - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\rangle - \frac{1}{2\rho}\|u - P_K[u - \rho\mathcal{T}u \\
&- \rho\mathcal{B}|u|]\|^2 - \langle \mathcal{T}u + \mathcal{B}|u|, u - P_K[u - \zeta\mathcal{T}u - \zeta\mathcal{B}|u|]\rangle + \frac{1}{2\zeta}\|u \\
&- P_K[u - \zeta\mathcal{T}u - \zeta\mathcal{B}|u|]\|^2\rangle \\
&= \langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\rho(u)\rangle - \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 - \langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\zeta(u)\rangle + \\
&\frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2\rangle \\
&= \langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\rho(u) - \mathcal{R}_\zeta(u)\rangle - \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 + \frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2, \quad \forall u \in \mathscr{H}.
\end{aligned} \tag{3.13}$$

It is clear from (3.13) that $\mathcal{D}_{\rho,\zeta}(u)$ is finite everywhere. We will now prove that $\mathcal{D}_{\rho,\zeta}(u)$ is in fact a merit function for the absolute value variational inequality which is the prime inspiration for the following result.

Theorem 3.9. *For all $u \in \mathscr{H}$ and $\rho \geq \zeta$, we have*

$$(\rho - \zeta)\|\mathcal{R}_\rho(u)\|^2 \geq 2\rho\zeta\mathcal{D}_{\rho,\zeta}(u) \geq (\rho - \zeta)\|\mathcal{R}_\rho(u)\|^2\|\mathcal{R}_\zeta(u)\|^2.$$

Particularly, $\mathcal{D}_{\rho,\zeta}(u) = 0$, if and only if $u \in \mathscr{H}$ is the solution of the absolute value variational inequality (2.1).

Proof. Take $u = P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]$, $v = P_K[u - \zeta\mathcal{T}u - \zeta\mathcal{B}|u|$ and $z = u - \rho\mathcal{T}u - \rho\mathcal{B}|u|$ in lemma 2.6, to have

$$\langle P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|] - u + \rho\mathcal{T}u + \rho\mathcal{B}|u|, P_K[u - \zeta\mathcal{T}u - \zeta\mathcal{B}|u|] - P_K[u - \rho\mathcal{T}u - \rho\mathcal{B}|u|]\rangle \geq 0,$$

which shows that

$$\langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\rho(u) - \mathcal{R}_\zeta(u)\rangle \geq \frac{1}{\rho}\langle \mathcal{R}_\rho(u), \mathcal{R}_\rho(u) - \mathcal{R}_\zeta(u)\rangle. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\begin{aligned}
\mathcal{D}_{\rho,\zeta}(u) &\geq \frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2 - \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 + \frac{1}{\rho}\|\mathcal{R}_\rho(u)\|^2 - \frac{1}{\rho}\langle \mathcal{R}_\rho(u), \mathcal{R}_\zeta(u)\rangle \\
&= \frac{1}{2}(\frac{1}{\zeta} - \frac{1}{\rho})\|\mathcal{R}_\zeta(u)\|^2 + \frac{1}{\rho}\|\mathcal{R}_\rho(u)\|^2 - \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 + \frac{1}{2\rho}\|\mathcal{R}_\zeta(u)\|^2 \\
&- \frac{1}{\rho}\langle \mathcal{R}_\rho(u), \mathcal{R}_\zeta(u)\rangle \\
&= \frac{1}{2}(\frac{1}{\zeta} - \frac{1}{\rho})\|\mathcal{R}_\zeta(u)\|^2 + \frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2 + \frac{1}{2\rho}\|\mathcal{R}_\zeta(u)\|^2 - \frac{1}{\rho}\langle \mathcal{R}_\rho(u), \mathcal{R}_\zeta(u)\rangle \\
&= \frac{1}{2}(\frac{1}{\zeta} - \frac{1}{\rho})\|\mathcal{R}_\zeta(u)\|^2 + \frac{1}{2\rho}\|\mathcal{R}_\zeta(u) - \mathcal{R}_\rho(u)\|^2 \\
&\geq \frac{1}{2}(\frac{1}{\zeta} - \frac{1}{\rho})\|\mathcal{R}_\zeta(u)\|^2,
\end{aligned}$$

which clearly shows that

$$2\rho\zeta\mathcal{D}_{\rho,\zeta}(u) \geq (\rho-\zeta)\|\mathcal{R}_\zeta(u)\|^2. \tag{3.15}$$

Similarly, by substituting $u = P_K[u-\zeta\mathcal{T}u-\zeta\mathcal{B}|u|]$, $v = P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]$ and $z = u-\zeta\mathcal{T}u-\zeta\mathcal{B}|u|$ in lemma 2.6 , we obtain

$$\langle P_K[u-\zeta\mathcal{T}u-\zeta\mathcal{B}|u|]-u+\zeta\mathcal{T}u+\zeta\mathcal{B}|u|, P_K[u-\rho\mathcal{T}u-\rho\mathcal{B}|u|]-P_v[u-\zeta\mathcal{T}u-\zeta\mathcal{B}|u|]\rangle \geq 0,$$

which shows that

$$\langle \mathcal{T}u+\mathcal{B}|u|, \mathcal{R}_\rho(u)-\mathcal{R}_\zeta(u)\rangle \leq \frac{1}{\zeta}\langle \mathcal{R}_\zeta(u), \mathcal{R}_\rho(u)-\mathcal{R}_\zeta(u)\rangle. \tag{3.16}$$

From (3.13) and (3.16), we obtain

$$\begin{aligned}
\mathcal{D}_{\rho,\zeta}(u) &\leq -\frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2+\frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2+\frac{1}{\zeta}\langle \mathcal{R}_\zeta(u), \mathcal{R}_\rho(u)-\mathcal{R}_\zeta(u)\rangle \\
&= \frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2-\frac{1}{2\rho}\|\mathcal{R}_\rho(u)\|^2-\frac{1}{\zeta}\|\mathcal{R}_\zeta(u)\|^2+\frac{1}{\zeta}\langle \mathcal{R}_\zeta(u), \mathcal{R}_\rho(u)\rangle \\
&= \frac{1}{2}(\frac{1}{\zeta}-\frac{1}{\rho})\|\mathcal{R}_\rho(u)\|^2-\frac{1}{2\zeta}\|\mathcal{R}_\rho(u)\|^2-\frac{1}{2\zeta}\|\mathcal{R}_\zeta(u)\|^2+\frac{1}{\zeta}\langle \mathcal{R}_\zeta(u), \mathcal{R}_\rho(u)\rangle \\
&= \frac{1}{2}(\frac{1}{\zeta}-\frac{1}{\rho})\|\mathcal{R}_\rho(u)\|^2-\frac{1}{2\zeta}\|\mathcal{R}_\rho(u)-\mathcal{R}_\zeta(u)\|^2 \\
&\leq \frac{1}{2}(\frac{1}{\zeta}-\frac{1}{\rho})\|\mathcal{R}_\rho(u)\|^2,
\end{aligned}$$

which proves the left most inequality of the required result, that is,

$$(\rho-\zeta)\|\mathcal{R}_\rho(u)\|^2 \geq 2\rho\zeta\mathcal{D}_{\rho,\zeta}(u). \tag{3.17}$$

Combining (3.15) and (3.17), we obtain

$$(\rho-\zeta)\|\mathcal{R}_\rho(u)\|^2 \geq 2\rho\zeta\mathcal{D}_{\rho,\zeta}(u) \geq (\rho-\zeta)\|\mathcal{R}_\zeta(u)\|^2,$$

which is the required result. □

Theorem 3.10. *Let $\hat{u} \in \mathscr{H}$ be a solution of the absolute value variational inequality* (2.1). *If the operators $\mathcal{T}$ and $\mathcal{B}$ are strongly monotone with constants $\alpha_\mathcal{T} > 0$ and $\alpha_\mathcal{B} > 0$, respectively then*

$$\|u-\hat{u}\|^2 \leq \frac{4\rho\zeta}{4(\alpha_\mathcal{T}+\alpha_\mathcal{B})-3\zeta+2\rho}[\mathcal{D}_{\rho,\zeta}(u)+\frac{1}{\rho}\|\mathcal{T}\hat{u}+\mathcal{B}|\hat{u}|\|^2], \quad \forall u \in \mathscr{H}.$$

Proof. Since $\hat{u} \in \mathscr{H}$ is a solution of the absolute value variational inequality (2.1) and by substituting $v = u$ in (2.1), we obtain

$$\langle \rho\mathcal{T}\hat{u}+\rho\mathcal{B}|\hat{u}|, u-\hat{u}\rangle \geq 0,$$

using lemma 3.2, we obtain

$$\langle \mathcal{T}\hat{u}+\mathcal{B}|\hat{u}|, u-\hat{u}\rangle \geq \frac{-1}{\rho}\|\mathcal{T}\hat{u}+\mathcal{B}|\hat{u}|\|^2-\frac{1}{4\rho}\|u-\hat{u}\|^2. \tag{3.18}$$

From (3.13), using the strong monotonicity of the operators $\mathcal{T}$ and $\mathcal{B}$ with constants $\alpha_{\mathcal{T}} > 0$ and $\alpha_{\mathcal{B}} > 0$ respectively and (3.18), we obtain

$$\begin{aligned}
\mathcal{D}_{\rho,\zeta}(u) &= \langle \mathcal{T}u + \mathcal{B}|u|, \mathcal{R}_\rho(u) - \mathcal{R}_\zeta(u)\rangle - \frac{1}{2\rho}||\mathcal{R}_\rho(u)||^2 + \frac{1}{2\zeta}||\mathcal{R}_\zeta(u)||^2 \\
&= \langle \mathcal{T}u + \mathcal{B}|u|, u - \hat{u}\rangle - \frac{1}{2\rho}||u - \hat{u}||^2 + \frac{1}{2\zeta}||u - \hat{u}||^2 \\
&= \langle \mathcal{T}u - \mathcal{T}\hat{u}, u - \hat{u}\rangle + \langle \mathcal{B}|u| - \mathcal{B}|\hat{u}|, u - \hat{u}\rangle + \langle \mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|, u - \hat{u}\rangle \\
&- \frac{1}{2\rho}||u - \hat{u}||^2 + \frac{1}{2\zeta}||u - \hat{u}||^2\rangle \\
&\geq \alpha_{\mathcal{T}}||u - \hat{u}||^2 + \alpha_{\mathcal{B}}||u - \hat{u}||^2 - \frac{1}{\rho}||\mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|||^2 - \frac{1}{4\rho}||u - \hat{u}||^2 \\
&- \frac{1}{2\rho}||u - \hat{u}||^2 + \frac{1}{2\zeta}||u - \hat{u}||^2\rangle \\
&= (\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}} - \frac{3}{4\rho} + \frac{1}{2\zeta})||u - \hat{u}||^2 - \frac{1}{\rho}||\mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|||^2 \\
&= \frac{4(\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}}) - 3\zeta + 2\rho}{4\rho\zeta}||u - \hat{u}||^2 - \frac{1}{\rho}||\mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|||^2,
\end{aligned}$$

which shows that

$$||u - \hat{u}||^2 \leq \frac{4\rho\zeta}{4(\alpha_{\mathcal{T}} + \alpha_{\mathcal{B}}) - 3\zeta + 2\rho}[\mathcal{D}_{\rho,\zeta}(u) + \frac{1}{\rho}||\mathcal{T}\hat{u} + \mathcal{B}|\hat{u}|||^2],$$

the required result. □

4. Conclusions

In this paper, we have proposed and investigated various merit functions for a new type of variational inequalities, namely absolute value variational inequalities. These merit functions are utilized to obtain error bounds for the estimated solution of absolute value variational inequalities and the associated optimization problems. The results proved in this paper may be considered as primary contribution in this alluring domain. Interested researchers are urged to discover the uses of absolute value variational inequalities in a variety of pure and applied disciplines. The suggestions made in this paper may be used in further research work.

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Conflict of interest

The authors declare that they have no competing interests.

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