Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12011–12027.
DOI:10.3934/math.2021696
Received: 21 May 2021
Accepted: 10 August 2021
Published: 17 August 2021

Research article

Finite-time stability of q-fractional damped difference systems with time delay

Jingfeng Wang and Chuanzhi Bai*

Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

* **Correspondence:** Email : czbai@hytc.edu.cn.

Abstract: In this paper, we investigate and obtain a new discrete q-fractional version of the Gronwall inequality. As applications, we consider the existence and uniqueness of the solution of q-fractional damped difference systems with time delay. Moreover, we formulate the novel sufficient conditions such that the q-fractional damped difference delayed systems is finite time stable. Our result extend the main results of the paper by Abdeljawad et al. [A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems, J.Inequal. Appl. 2016, 240].

Keywords: q-fractional difference system; q-fractional Gronwall inequality; delay; finite-time stability
Mathematics Subject Classification: 26A33, 39A11

1. Introduction

In the last two decades, the fractional difference equations have recently received considerable attention in many fields of science and engineering, see [1–4] and the references therein. On the other hand, the q-difference equations have numerous applications in diverse fields in recent years and has gained intensive interest [5–9]. It is well know that the q-fractional difference equations can be used as a bridge between fractional difference equations and q-difference equations, many papers have been published on this research direction, see [10–15] for examples. We recommend the monograph [16] and the papers cited therein.

For $0 < q < 1$, we define the time scale $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$ and $n_0 \in \mathbb{Z}$, we denote $\mathbb{T}_a = [a, \infty)_q = \{q^{-i}a : i = 0, 1, 2, ...\}$.

In [17], Abdeljawad et.al generalized the q-fractional Gronwall-type inequality in [18], they obtained the following q-fractional Gronwall-type inequality.

Theorem 1.1 ([17]). *Let $\alpha > 0$, u and v be nonnegative functions and $w(t)$ be nonnegative and* *nondecreasing function for* $t \in [a,\infty)_q$ *such that* $w(t) \le M$ *where* M *is a constant. If*

$$u(t) \le v(t) + w(t)_q\nabla_a^{-\alpha}u(t),$$

then

$$u(t) \le v(t) + \sum_{k=1}^{\infty}(w(t)\Gamma_q(\alpha))^k{}_q\nabla_a^{-k\alpha}v(t). \tag{1.1}$$

Based on the above result, Abdeljawad et al. investigated the following nonlinear delay q-fractional difference system:

$$\begin{cases} {}_qC_a^{\alpha}x(t) = A_0x(t) + A_1x(\tau t) + f(t,x(t),x(\tau t)), & t \in [a,\infty)_q, \\ x(t) = \phi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{1.2}$$

where ${}_qC_a^{\alpha}$ means the Caputo fractional difference of order $\alpha \in (0,1)$, $\bar{\mathbb{I}}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, ..., a\}$, $\tau = q^d \in \mathbb{T}_q$ with $d \in \mathbb{N}_0 = \{0,1,2,...\}$.

Remark 1.1. The domain of t in (1.2) is inaccurate, please see the reference [19].

In [20], Sheng and Jiang gave the following extended form of the fractional Gronwall inequality :

Theorem 1.2 ([20]). *Suppose* $\alpha > 0$, $\beta > 0$, $a(t)$ *is a nonnegative function locally integrable on* $[0,T)$, $\tilde{g}(t)$, *and* $\bar{g}(t)$ *are nonnegative, nondecreasing, continuous functions defined on* $[0,T)$*;* $\tilde{g}(t) \le \tilde{M}$, $\bar{g}(t) \le \bar{M}$, *where* $\tilde{M}$ *and* $\bar{M}$ *are constants. Suppose* $x(t)$ *is a nonnegative and locally integrable on* $[0,T)$ *with*

$$x(t) \le a(t) + \tilde{g}(t)\int_0^t (t-s)^{\alpha-1}x(s)ds + \bar{g}(t)\int_0^t (t-s)^{\beta-1}x(s)ds, \quad t \in [0,T).$$

Then

$$x(t) \le a(t) + \int_0^t \sum_{n=1}^{\infty}[g(t)]^n \sum_{k=0}^{n} \frac{C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha + k\beta]}(t-s)^{(n-k)\alpha+k\beta-1}a(s)ds, \tag{1.3}$$

where $t \in [0,T)$, $g(t) = \tilde{g}(t) + \bar{g}(t)$ *and* $C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!}$.

Corollary 1.3 [20] *Under the hypothesis of Theorem 1.2, let* $a(t)$ *be a nondecreasing function on* $[0,T)$. *Then*

$$x(t) \le a(t)E_\gamma[g(t)(\Gamma(\alpha)t^\alpha + \Gamma(\beta)t^\beta)], \tag{1.4}$$

where $\gamma = \min\{\alpha,\beta\}$, E_γ *is the Mittag-Leffler function defined by* $E_\gamma(z) = \sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(k\gamma+1)}$.

Finite-time stability is a more practical method which is much valuable to analyze the transient behavior of nature of a system within a finite interval of time. It has been widely studied of integer differential systems. In recent decades, the finite-time stability analysis of fractional differential systems has received considerable attention, for instance [21–25] and the references therein. In [26], Du and Jia studied the finite-time stability of a class of nonlinear fractional delay difference systems by using a new discrete Gronwall inequality and Jensen inequality. Recently, Du and Jia in [27] obtained a criterion on finite time stability of fractional delay difference system with constant coefficients by virtue of a discrete delayed Mittag-Leffler matrix function approach. In [28], Ma and Sun investigated the finite-time stability of a class of fractional q-difference equations with time-delay by utilizing the proposed delayed q-Mittag-Leffler type matrix and generalized q-Gronwall inequality, respectively. Based on the generalized fractional (q, h)-Gronwall inequality, Du and Jia in [19] derived the finite-time stability criterion of nonlinear fractional delay (q, h)-difference systems.

Motivated by the above works, we will extend the q-fractional Gronwall-type inequality (Theorem 1.1) to the spreading form of the q-fractional Gronwall inequality. As applications, we consider the existence and uniqueness of the solution of the following nonlinear delay q-fractional difference damped system :

$$\begin{cases} {}_qC_a^\alpha x(t) - A_{0q}C_a^\beta x(t) = B_0 x(t) + B_1 x(\tau t) + f(t, x(t), x(\tau t)), & t \in [a, b)_q, \\ x(t) = \phi(t), \quad \nabla_q x(t) = \psi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{1.5}$$

where $[a, b)_q = [a, b) \cap \mathbb{T}_a$, $b \in \mathbb{T}_a$, $\mathbb{I}_\tau = \{q\tau a, \tau a, q^{-1}\tau a, q^{-2}\tau a, ..., a\}$, $\tau = q^d \in \mathbb{T}_q$ with $d \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, ${}_qC_a^\alpha$ and ${}_qC_a^\beta$ mean the Caputo fractional difference of order $\alpha \in (1, 2)$ and order $\beta \in (0, 1)$, respectively, and the constant matrices A_0, B_0 and B_1 are of appropriate dimensions. Moreover, a novel criterion of finite-time stability criterion of (1.5) is established. We generalized the main results of [17] in this paper.

The organization of this paper is given as follows: In Section 2, we give some notations, definitions and preliminaries. Section 3 is devoted to proving a spreading form of the q-fractional Gronwall inequality. In Section 4, the existence and uniqueness of the solution of system (1.5) are given and proved, and the finite-time stability theorem of nonlinear delay q-fractional difference damped system is obtained. In Section 5, an example is given to illustrate our theoretical result. Finally, the paper is concluded in Section 6.

2. Preliminaries

In this section, we provided some basic definitions and lemmas which are used in the sequel.

Let $f : \mathbb{T}_q \to \mathbb{R}$ ($q \in (0, 1)$), the nabla q-derivative of f is defined by Thabet et al. as follows:

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \in \mathbb{T}_q \setminus \{0\},$$

and q-derivatives of higher order by

$$\nabla_q^n f(t) = \nabla_q(\nabla_q^{n-1} f)(t), \quad n \in \mathbb{N}.$$

The nabla q-integral of f has the following form

$$\int_0^t f(s)\nabla_q s = (1 - q)t \sum_{i=0}^{\infty} q^i f(tq^i) \tag{2.1}$$

and for $0 \le a \in \mathbb{T}_q$

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s. \tag{2.2}$$

The definition of the q-factorial function for a nonpositive integer α is given by

$$(t-s)_q^{\alpha} = t^{\alpha} \prod_{i=0}^{\infty} \frac{1-\frac{s}{t}q^i}{1-\frac{s}{t}q^{i+\alpha}}. \tag{2.3}$$

For a function $f : \mathbb{T}_q \to \mathbb{R}$, the left q-fractional integral ${}_q\nabla_a^{-\alpha}$ of order $\alpha \neq 0, -1, -2, ...$ and starting at $0 < a \in \mathbb{T}_q$ is defined by

$${}_q\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s) \nabla_q s, \tag{2.4}$$

where

$$\Gamma_q(\alpha+1) = \frac{1-q^{\alpha}}{1-q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1,\ \alpha > 0. \tag{2.5}$$

The left q-fractional derivative ${}_q\nabla_a^{\beta}$ of order $\beta > 0$ and starting at $0 < a \in \mathbb{T}_q$ is defined by

$${}_q\nabla_a^{\beta} f(t) = ({}_q\nabla_a^m\, {}_q\nabla_a^{-(m-\beta)} f)(t), \tag{2.6}$$

where m is the smallest integer greater or equal than β.

Definition 2.1 ([11]). Let $0 < \alpha \notin \mathbb{N}$ and $f : \mathbb{T}_a \to \mathbb{R}$. Then the Caputo left q-fractional derivative of order α of a function f is defined by

$${}_qC_a^{\alpha} f(t) := {}_q\nabla_a^{-(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s, \quad t \in \mathbb{T}_a, \tag{2.7}$$

where $n = [\alpha] + 1$.

Let us now list some properties which are needed to obtain our results.

Lemma 2.1 ([29]). *Let $\alpha, \beta > 0$ and f be a function defined on $(0, b)$. Then the following formulas hold:*

$$({}_q\nabla_a^{-\beta}\, {}_q\nabla_a^{-\alpha} f)(t) = {}_q\nabla_a^{-(\alpha+\beta)} f(t), \quad 0 < a < t < b,$$

$$({}_q\nabla_a^{\alpha}\, {}_q\nabla_a^{-\alpha} f)(t) = f(t), \quad 0 < a < t < b.$$

Lemma 2.2 ([11]). *Let $\alpha > 0$ and f be defined in a suitable domain. Thus*

$${}_q\nabla_a^{-\alpha}\, {}_qC_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \tag{2.8}$$

and if $0 < \alpha \leq 1$ we have

$${}_q\nabla_a^{-\alpha}\, {}_qC_a^{\alpha} f(t) = f(t) - f(a). \tag{2.9}$$

The following identity plays a crucial role in solving the linear q-fractional equations:

$$ {}_q\nabla_a^{-\alpha}(x-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha}, \quad 0 < a < x < b, \tag{2.10} $$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.
Apply ${}_q\nabla_a^{\alpha}$ on both sides of (2.10), by virtue of Lemma 2.1, one can obtain

$$ {}_q\nabla_a^{\alpha}(x-a)_q^{\mu+\alpha} = \frac{\Gamma_q(\alpha+\mu+1)}{\Gamma_q(\mu+1)}(x-a)_q^{\mu}, \quad 0 < a < x < b, \tag{2.11} $$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

By Theorem 7 in [11], for any $0 < \beta < 1$, one has

$$ ({}_qC_a^{\beta}f)(t) = ({}_q\nabla_a^{\beta}f)(t) - \frac{(t-a)_q^{-\beta}}{\Gamma_q(1-\beta)}f(a). \tag{2.12} $$

For any $1 < \alpha \leq 2$, by (2.8), one has

$$ {}_q\nabla_a^{-\alpha}{}_qC_a^{\alpha}f(t) = f(t) - f(a) - (t-a)_q^1\nabla_q f(a). \tag{2.13} $$

Apply ${}_q\nabla_a^{\alpha}$ on both sides of (2.13), by Lemma 2.1 and (2.11), we get

$$ \begin{aligned} ({}_qC_a^{\alpha}f)(t) &= ({}_q\nabla_a^{\alpha}f)(t) - f(a){}_q\nabla_a^{\alpha}(t-a)_q^0 - f(a){}_q\nabla_a^{\alpha}(t-a)_q^1 \\ &= ({}_q\nabla_a^{\alpha}f)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)}f(a) - \frac{(t-a)_q^{1-\alpha}}{\Gamma_q(2-\alpha)}\nabla_q f(a). \end{aligned} \tag{2.14} $$

3. A generalized q-fractional Gronwall inequality

In this section, we give and prove the following spreading form of generalized q-fractional Gronwall inequality, which extend a q-fractional Gronwall inequality in Theorem 1.1.

Theorem 3.1. *Let $\alpha > 0$ and $\beta > 0$. Assume that $u(t)$ and $g(t)$ are nonnegative functions for $t \in [a, T)_q$. Let $w_i(t)$ $(i = 1, 2)$ be nonnegative and nondecreasing functions for $t \in [a, T)_q$ with $w_i(t) \leq M_i$, where M_i are positive constants $(i = 1, 2)$ and*

$$ [\Gamma_q(\alpha)T^{\alpha}(1-q)^{\alpha} + \Gamma_q(\beta)T^{\beta}(1-q)^{\beta}]\max\left\{\frac{M_1}{\Gamma_q(\alpha)}, \frac{M_2}{\Gamma_q(\beta)}\right\} < 1. \tag{3.1} $$

If

$$ u(t) \leq g(t) + w_1(t){}_q\nabla_a^{-\alpha}u(t) + w_2(t){}_q\nabla_a^{-\beta}u(t), \quad t \in [a, T)_q, \tag{3.2} $$

then

$$ u(t) \leq g(t) + \sum_{n=1}^{\infty} w(t)^n \sum_{k=0}^{n} C_n^k \Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k{}_q\nabla_a^{-((n-k)\alpha+k\beta)}g(t), \quad t \in [a, T)_q, \tag{3.3} $$

where $w(t) = \max\left\{\frac{w_1(t)}{\Gamma_q(\alpha)}, \frac{w_2(t)}{\Gamma_q(\beta)}\right\}$.

Proof. Define the operator

$$Au(t) = w(t)\int_a^t [(t-qs)_q^{\alpha-1} + (t-qs)_q^{\beta-1}]u(s)\nabla_q s, \quad t \in [a,T)_q. \tag{3.4}$$

According to (3.2), one has

$$u(t) \leq g(t) + Au(t). \tag{3.5}$$

By (3.5) and the monotonicity of the operator A, we obtain

$$u(t) \leq \sum_{k=0}^{n-1} A^k g(t) + A^n u(t), \quad t \in [a,T)_q. \tag{3.6}$$

In the following, we will prove that

$$A^n u(t) \leq w(t)^n \sum_{k=0}^{n} C_n^k \Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k {}_q\nabla_a^{-((n-k)\alpha+k\beta)}u(t), \quad t \in [a,T)_q, \tag{3.7}$$

and

$$\lim_{n\to\infty} A^n u(t) = 0. \tag{3.8}$$

Obviously, the inequality (3.7) holds for $n = 1$. Assume that (3.7) is true for $n = m$, that is

$$\begin{aligned} A^m u(t) &\leq w(t)^m \sum_{k=0}^{m} C_m^k \Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k {}_q\nabla_a^{-((m-k)\alpha+k\beta)}u(t) \\ &= w(t)^m \sum_{k=0}^{m} \frac{C_m^k \Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}\int_a^t (t-qs)_q^{(m-k)\alpha+k\beta-1}u(s)\nabla_q s, \quad t \in [a,T)_q. \end{aligned} \tag{3.9}$$

When $n = m+1$, by using (3.4), (3.9), (2.10) and the nondecreasing of function $w(t)$, we get

$$\begin{aligned} A^{m+1}u(t) &= A(A^m u(t)) \\ &\leq w(t)\int_a^t [(t-qs)_q^{\alpha-1} + (t-qs)_q^{\beta-1}] \\ &\quad \times\left(w(s)^m \sum_{k=0}^{m} \frac{C_m^k \Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}\int_a^s (s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\right)\nabla_q s \\ &\leq w(t)^{m+1}\int_a^t \sum_{k=0}^{m} \frac{C_m^k \Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}[(t-qs)_q^{\alpha-1} + (t-qs)_q^{\beta-1}] \\ &\quad \times\left[\int_a^s (s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\right]\nabla_q s \end{aligned}$$

$$= w(t)^{m+1}\sum_{k=0}^{m}\frac{C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}\left[\int_a^t(t-qs)_q^{\alpha-1}\int_a^s(s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\nabla_q s\right.$$

$$\left.+\int_a^t(t-qs)_q^{\beta-1}\int_a^s(s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\nabla_q s\right]$$

$$= w(t)^{m+1}\sum_{k=0}^{m}\frac{C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}\left[\int_a^t\int_{qr}^t(t-qs)_q^{\alpha-1}(s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\nabla_q s\right.$$

$$\left.+\int_a^t\int_{qr}^t(t-qs)_q^{\beta-1}(s-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\nabla_q s\right]$$

$$= w(t)^{m+1}\sum_{k=0}^{m}\frac{C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}$$

$$\times\left(\Gamma_q(\alpha)\int_a^t\left[\frac{1}{\Gamma_q(\alpha)}\int_{qr}^t(t-qs)_q^{\alpha-1}(s-qr)_q^{(m-k)\alpha+k\beta-1}\nabla_q s\right]u(r)\nabla_q r\right.$$

$$\left.+\Gamma_q(\beta)\int_a^t\left[\frac{1}{\Gamma_q(\beta)}\int_{qr}^t(t-qs)_q^{\beta-1}(s-qr)_q^{(m-k)\alpha+k\beta-1}\nabla_q s\right]u(r)\nabla_q r\right)$$

$$= w(t)^{m+1}\sum_{k=0}^{m}\frac{C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}$$

$$\times\left(\Gamma_q(\alpha)\int_a^t {}_q\nabla_{qr}^{-\alpha}(t-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\right.$$

$$\left.+\Gamma_q(\beta)\int_a^t {}_q\nabla_{qr}^{-\beta}(t-qr)_q^{(m-k)\alpha+k\beta-1}u(r)\nabla_q r\right)$$

$$= w(t)^{m+1}\sum_{k=0}^{m}\frac{C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k}{\Gamma_q((m-k)\alpha+k\beta)}$$

$$\times\left(\frac{\Gamma_q(\alpha)\Gamma_q((m-k)\alpha+k\beta)}{\Gamma_q((m-k+1)\alpha+k\beta)}\int_a^t(t-qr)_q^{(m-k+1)\alpha+k\beta-1}u(r)\nabla_q r\right.$$

$$\left.+\frac{\Gamma_q(\beta)\Gamma_q((m-k)\alpha+k\beta)}{\Gamma_q((m-k)\alpha+(k+1)\beta)}\int_a^t(t-qr)_q^{(m-k)\alpha+(k+1)\beta-1}u(r)\nabla_q r\right)$$

$$= w(t)^{m+1}\sum_{k=0}^{m}C_m^k\Gamma_q(\alpha)^{m-k}\Gamma_q(\beta)^k$$

$$\times\left(\Gamma_q(\alpha){}_q\nabla_a^{-((m-k+1)\alpha+k\beta)}u(t)+\Gamma_q(\beta){}_q\nabla_a^{-((m-k)\alpha+(k+1)\beta)}u(t)\right)$$

$$= w(t)^{m+1}\sum_{k=0}^{m}C_m^k\Gamma_q(\alpha)^{m+1-k}\Gamma_q(\beta)^k{}_q\nabla_a^{-((m-k+1)\alpha+k\beta)}u(t)$$

$$+w(t)^{m+1}\sum_{k=1}^{m+1}C_m^{k-1}\Gamma_q(\alpha)^{m+1-k}\Gamma_q(\beta)^k{}_q\nabla_a^{-((m+1-k)\alpha+k\beta)}u(t)$$

$$= w(t)^{m+1}\Big[C_m^0\Gamma_q(\alpha)^{m+1}{}_q\nabla_a^{-((m+1)\alpha)}u(t)$$

$$+\sum_{k=1}^{m}(C_m^k+C_m^{k-1})\Gamma_q(\alpha)^{m+1-k}\Gamma_q(\beta)^k{}_q\nabla_a^{-((m-k+1)\alpha+k\beta)}u(t)$$

$$+C_m^m\Gamma_q(\beta)^{m+1}{}_q\nabla_a^{-((m+1)\beta)}u(t)\Big]$$

$$= w(t)^{m+1}\sum_{k=0}^{m+1}C_{m+1}^k\Gamma_q(\alpha)^{m+1-k}\Gamma_q(\beta)^k{}_q\nabla_a^{-((m+1-k)\alpha+k\beta)}u(t).$$

Thus, (3.7) is proved.

Using Stirling's formula of the q-gamma function [30], yields that

$$\Gamma_q(x) = [2]_q^{1/2}\Gamma_{q^2}(1/2)(1-q)^{\frac{1}{2}-x}e^{\frac{\theta q^x}{(1-q)-q^x}},\quad 0<\theta<1,$$

that is

$$\Gamma_q(x)\sim D(1-q)^{\frac{1}{2}-x},\quad x\to\infty, \tag{3.10}$$

where $D = [2]_q^{1/2}\Gamma_{q^2}(1/2)$. Moreover, if $t>a>0$ and $\gamma>0$ (γ is not a positive integer), then $1-\frac{a}{t}q^j < 1-\frac{a}{t}q^{\gamma+j}$ for each $j=0,1,...,$ and

$$(t-a)_q^\gamma = t^\gamma\prod_{j=0}^{\infty}\frac{1-\frac{a}{t}q^j}{1-\frac{a}{t}q^{\gamma+j}} < t^\gamma. \tag{3.11}$$

By $w_1(t)<M_1$ and $w_2(t)<M_2$, one has that $w(t)<\max\left\{\frac{M_1}{\Gamma_q(\alpha)},\ \frac{M_2}{\Gamma_q(\beta)}\right\} := M$. Applying the first mean value theorem for definite integrals [31], (3.10) and (3.11), there exists a $\xi\in[a,t]_q$ such that

$$\begin{aligned}
\lim_{n\to\infty}A^nu(t) &\le \lim_{n\to\infty}u(\xi)\sum_{k=0}^{n}\frac{M^nC_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k}{\Gamma_q((n-k)\alpha+k\beta)}\int_a^t(t-qr)_q^{(n-k)\alpha+k\beta-1}\nabla_q s\\
&= \lim_{n\to\infty}u(\xi)\sum_{k=0}^{n}\frac{M^nC_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k}{\Gamma_q((n-k)\alpha+k\beta+1)}(t-a)_q^{(n-k)\alpha+k\beta}\\
&\le \lim_{n\to\infty}u(\xi)\sum_{k=0}^{n}\frac{M^nC_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k}{\Gamma_q((n-k)\alpha+k\beta+1)}t^{(n-k)\alpha+k\beta}\\
&= \lim_{n\to\infty}u(\xi)\sum_{k=0}^{n}\frac{M^nC_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\beta)^k}{D(1-q)^{\frac{1}{2}-((n-k)\alpha+k\beta+1)}}t^{(n-k)\alpha+k\beta}\\
&= \lim_{n\to\infty}\frac{u(\xi)\sqrt{1-q}}{D}\sum_{k=0}^{n}M^nC_n^k[\Gamma_q(\alpha)t^\alpha(1-q)^\alpha]^{n-k}[\Gamma_q(\beta)t^\beta(1-q)^\beta]^k\\
&= \lim_{n\to\infty}\frac{u(\xi)\sqrt{1-q}}{D}[M(\Gamma_q(\alpha)(1-q)^\alpha t^\alpha+\Gamma_q(\beta)(1-q)^\beta t^\beta)]^n.
\end{aligned}$$

From (3.1), for each $t \in [a, T)_q$, we have

$$[M(\Gamma_q(\alpha)(1-q)^\alpha t^\alpha + \Gamma_q(\beta)(1-q)^\beta t^\beta)]^n \to 0, \quad \text{as } n \to \infty.$$

Thus, $A^n u(t) \to 0$ as $n \to \infty$. Let $n \to \infty$ in (3.6), by (3.8) we get

$$u(t) \leq g(t) + \sum_{k=1}^{\infty} A^k g(t). \tag{3.12}$$

From (3.7) and (3.12), we obtain (3.3). This completes the proof.

Corollary 3.2. *Under the hypothesis of Theorem 3.1, let $g(t)$ be a nondecreasing function on $t \in [a, T)_q$. Then*

$$u(t) \leq g(t) \sum_{n=0}^{\infty} w(t)^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k} \Gamma_q(\beta)^k}{\Gamma_q((n-k)\alpha + k\beta + 1)} (t-a)_q^{(n-k)\alpha + k\beta} \tag{3.13}$$

Proof. By (3.3), (2.10) and the assumption that $g(t)$ is nondecreasing function for $t \in [a, T)_q$, we have

$$\begin{aligned} u(t) &\leq g(t) \left[1 + \sum_{n=1}^{\infty} w(t)^n \sum_{k=0}^{n} C_n^k \Gamma_q(\alpha)^{n-k} \Gamma_q(\beta)^k {}_q\nabla_a^{-((n-k)\alpha + k\beta)} 1 \right] \\ &= g(t) \left[1 + \sum_{n=1}^{\infty} w(t)^n \sum_{k=0}^{n} C_n^k \Gamma_q(\alpha)^{n-k} \Gamma_q(\beta)^k \frac{1}{\Gamma_q((n-k)\alpha + k\beta + 1)} (t-a)_q^{(n-k)\alpha + k\beta} \right] \\ &= g(t) \sum_{n=0}^{\infty} w(t)^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k} \Gamma_q(\beta)^k}{\Gamma_q((n-k)\alpha + k\beta + 1)} (t-a)_q^{(n-k)\alpha + k\beta}. \end{aligned}$$

4. Main results

Throughout this paper, we make the following assumptions:

(H1) $f \in D(\mathbb{T}_q \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ is a Lipschitz-type function. That is, for any $x, y : \mathbb{T}_{\tau a} \to \mathbb{R}^n$, there exists a positive constant $L > 0$ such that

$$\|f(t, y(t), y(\tau t)) - f(t, x(t), x(\tau t))\| \leq L(\|y(t) - x(t)\| + \|y(\tau t) - x(\tau t)\|), \tag{4.1}$$

for $t \in [a, T)_q$.

(H2)

$$f(t, 0, 0) = \underbrace{[0, 0, ..., 0]}_{n}{}^T. \tag{4.2}$$

(H3)

$$[\Gamma_q(\alpha) T^\alpha (1-q)^\alpha + \Gamma_q(\alpha - \beta) T^{\alpha-\beta} (1-q)^{\alpha-\beta}] \max \left\{ \frac{\|B_0\| + \|B_1\| + 2L}{\Gamma_q(\alpha)}, \frac{\|A_0\|}{\Gamma_q(\alpha - \beta)} \right\} < 1. \tag{4.3}$$

Definition 4.1. The system (1.5) is finite-time stable w.r.t.$\{\delta, \epsilon, T_e\}$, with $\delta < \epsilon$, if and only if $\max\{\|\phi\|, \|\psi\|\} < \delta$ implies $\|x(t)\| < \epsilon$, $\forall t \in [a, T_e]_q = [a, T_e] \cap [a, T)_q$.

Theorem 4.1. *Assume that (H1) and (H3) hold. Then the problem (1.5) has a unique solution.*

Proof. First we have to prove that $x : \mathbb{T}_{\tau a} \to \mathbb{R}^m$ is a solution of system (1.5) if and only if

$$\begin{aligned} x(t) &= \phi(a) + \psi(a)(t-a) - A_0 \frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\phi(a) \\ &\quad + \frac{A_0}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} x(s) \nabla_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 x(s) + B_1 x(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s, \quad t \in [a, T)_q, \\ x(t) &= \phi(t), \quad \nabla_q x(t) = \psi(t), \qquad t \in \mathbb{I}_\tau. \end{aligned} \tag{4.4}$$

For $t \in \mathbb{I}_\tau$, it is clear that $x(t) = \phi(t)$ with $\nabla_q x(t) = \psi(t)$ is the solution of (1.5). For $t \in [a, T)_q$, we apply ${}_q\nabla_a^\alpha$ on both sides of (4.4) to obtain

$$\begin{aligned} {}_q\nabla_a^\alpha x(t) = \phi(a) \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} + \psi(a) \frac{(t-a)_q^{1-\alpha}}{\Gamma_q(2-\alpha)} - \phi(a) A_0 \frac{(t-a)_q^{-\beta}}{\Gamma_q(1-\beta)} \\ + A_{0q}\nabla_a^\beta x(t) + B_0 x(t) + B_1 x(\tau t) + f(t, x(t), x(\tau t)), \end{aligned} \tag{4.5}$$

where $({}_q\nabla_a^\alpha {}_q\nabla_a^{-\alpha} x)(t) = x(t)$ and $({}_q\nabla_a^\alpha {}_q\nabla_a^{-(\alpha-\beta)} x)(t) = {}_q\nabla_a^\beta x(t)$ (by Lemma 2.1) have been used. By using (2.12) and (2.14), we get

$${}_qC_a^\alpha x(t) - A_{0q}C_a^\beta x(t) = B_0 x(t) + B_1 x(\tau t) + f(t, x(t), x(\tau t)), \quad t \in [a, T)_q.$$

Conversely, from system (1.5), we can see that $x(t) = \phi(t)$ and $\nabla_q x(t) = \psi(t)$ for $t \in \mathbb{I}_\tau$. For $t \in [a, T)_q$, we apply ${}_q\nabla_a^{-\alpha}$ on both sides of (1.5) to get

$$\begin{aligned} &{}_q\nabla_a^{-\alpha}[{}_qC_a^\alpha x(t) - A_{0q}C_a^\beta x(t)] \\ &\quad = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 x(s) + B_1 x(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s. \end{aligned}$$

According to Lemma 2.2, we obtain

$$\begin{aligned} x(t) &= \phi(a) + \psi(a)(t-a) - A_0 \frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\phi(a) \\ &\quad + \frac{A_0}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} x(s) \nabla_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 x(s) + B_1 x(\tau s) + f(s, x(s), x(\tau s))] \nabla_q s, \quad t \in [a, T)_q. \end{aligned}$$

Secondly, we will prove the uniqueness of solution to system (1.5). Let x and y be two solutions of system (1.5). Denote z by $z(t) = x(t) - y(t)$. Obviously, $z(t) = 0$ for $t \in \mathbb{I}_\tau$, which implies that system (1.5) has a unique solution for $t \in \mathbb{I}_\tau$.

For $t \in [a, T)_q$, one has

$$z(t) = \frac{A_0}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} z(s) \nabla_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 z(s) + B_1 z(\tau s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s. \tag{4.6}$$

If $t \in \mathbb{J}_\tau = \{a, q^{-1}a, ..., \tau^{-1}a\}$, then $\tau t \in \mathbb{I}_\tau$ and $z(\tau t) = 0$. Hence,

$$z(t) = \frac{A_0}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} z(s) \nabla_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 z(s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s,$$

which implies that

$$\begin{aligned} \|z(t)\| \leq & \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} \|z(s)\| \nabla_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [\|B_0\| \|z(s)\| + \|f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))\|] \nabla_q s \\ \leq & \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} \|z(s)\| \nabla_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [\|B_0\| \|z(s)\| + L(\|z(s)\| + \|z(\tau s)\|)] \nabla_q s \quad \text{(by (H1))} \\ = & \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} \|z(s)\| \nabla_q s + \frac{\|B_0\| + L}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \|z(s)\| \nabla_q s. \end{aligned} \tag{4.7}$$

By applying Corollary 3.2 and (H3), we get

$$\|z(t)\| \leq 0 \cdot \sum_{n=0}^{\infty} w_1^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k} \Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha + k(\alpha-\beta)+1)} (t-a)_q^{(n-k)\alpha + k(\alpha-\beta)} = 0, \tag{4.8}$$

where $w_1 = \max\left\{\frac{\|A_0\|}{\Gamma(\alpha-\beta)}, \frac{\|B_0\|+L}{\Gamma(\alpha)}\right\}$. This implies $x(t) = y(t)$ for $t \in \mathbb{J}_\tau$.

For $t \in [\tau^{-1}a, T)_q$, we obtain

$$\begin{aligned} z(t) = & \frac{A_0}{\Gamma_q(\alpha-\beta)} \int_a^t (t-qs)_q^{\alpha-\beta-1} z(s) \nabla_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} [B_0 z(s) + f(s, x(s), x(\tau s)) - f(s, y(s), y(\tau s))] \nabla_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} B_1 z(\tau s) \nabla_q s. \end{aligned} \tag{4.9}$$

Therefore,

$$
\begin{aligned}
\|z(t)\| &= \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}\|z(s)\|\nabla_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}[\|B_0\|\|z(s)\| + \|f(s,x(s),x(\tau s)) - f(s,y(s),y(\tau s))\|]\nabla_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\|B_1\|\|z(\tau s)\|\nabla_q s \\
&\le \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}\|z(s)\|\nabla_q s \\
&\quad + \frac{\|B_0\|+L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\|z(s)\|\nabla_q s + \frac{\|B_1\|+L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\|z(\tau s)\|\nabla_q s.
\end{aligned}
\tag{4.10}
$$

Let $z^*(t) = \max_{\theta\in[a,t]_q}\{\|z(\theta)\|, \|z(\tau\theta)\|\}$ for $t \in [\tau^{-1}a, T)_q$, where $[a,t]_q = [a,t] \cap \mathbb{T}_a$, it is obvious that $z^*(t)$ is a increasing function. From (4.10), we obtain that

$$
\begin{aligned}
z^*(t) &\le \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}z^*(s)\nabla_q s \\
&\quad + \frac{\|B_0\|+L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}z^*(s)\nabla_q s + \frac{\|B_1\|+L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}z^*(s)\nabla_q s \\
&= \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}z^*(s)\nabla_q s \\
&\quad + \frac{\|B_0\|+\|B_1\|+2L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}z^*(s)\nabla_q s.
\end{aligned}
\tag{4.11}
$$

By applying Corollary 3.2 and (H3) again, we get

$$
\|z(t)\| \le z^*(t) \le 0 \cdot \sum_{n=0}^{\infty} w_2^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k}\Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha + k(\alpha-\beta)+1)}(t-a)_q^{(n-k)\alpha+k(\alpha-\beta)} = 0,
$$

where $w_2 = \max\left\{\frac{\|A_0\|}{\Gamma(\alpha-\beta)}, \frac{\|B_0\|+\|B_1\|+2L}{\Gamma(\alpha)}\right\}$. Thus, we end up with $x(t) = y(t)$ for $t \in [\tau^{-1}a, T)_q$. The proof is completed.

Theorem 4.2. *Assume that the conditions (H1), (H2) and (H3) hold. Then the system (1.5) is finite-time stable if the following condition is satisfied:*

$$
\left(1 + (t-a) + \frac{\|A_0\|(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right)\sum_{n=0}^{\infty} w_2^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k}\Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha + k(\alpha-\beta)+1)}(t-a)_q^{(n-k)\alpha+k(\alpha-\beta)} < \frac{\varepsilon}{\delta}, \tag{4.12}
$$

where $w_2 = \max\left\{\frac{\|B_0\|+\|B_1\|+2L}{\Gamma_q(\alpha)}, \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\right\}$.

Proof. Applying left q-fractional integral on both sides of (1.5), we obtain

$$
{}_q\nabla_a^{-\alpha}({}_qC_a^{\alpha}x(t)) - A_{0q}\nabla_a^{-\alpha}({}_qC_a^{\beta}x(t)) = {}_q\Delta_a^{-\alpha}(B_0x(t) + B_1x(\tau t) + f(t,x(t),x(\tau t))). \tag{4.13}
$$

By (4.12) and utilizing Lemma 2.2 we have

$$\begin{aligned}x(t) &= \phi(a) + \psi(a)(t-a) - A_0\frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\phi(a)\\ &+ \frac{A_0}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}x(s)\nabla_q s\\ &+ \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}[B_0x(s)+B_1x(\tau s)+f(s,x(s),x(\tau s))]\nabla_q s.\end{aligned}$$

Thus, by (H1) and (H2), we get

$$\begin{aligned}\|x(t)\| &\le \|\phi\| + \|\psi\|(t-a) + \|A_0\|\|\phi\|\frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\\ &+ \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}\|x(s)\|\nabla_q s\\ &+ \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}[\|B_0\|\|x(s)\| + \|B_1\|\|x(\tau s)\| + \|f(s,x(s),x(\tau s))\|]\nabla_q s\\ &\le \|\phi\| + \|\psi\|(t-a) + \|A_0\|\|\phi\|\frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\\ &+ \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}\|x(s)\|\nabla_q s\\ &+ \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}[(\|B_0\|+L)\|x(s)\| + (\|B_1\|+L)\|x(\tau s)\|]\nabla_q s.\end{aligned} \tag{4.14}$$

Let $g(t) = \|\phi\| + \|\psi\|(t-a) + \|A_0\|\|\phi\|\frac{(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}$, then g is a nondecreasing function.
Set $\bar{x}(t) = \max_{\theta\in[a,t]_q}\{\|x(\theta)\|, \|x(\tau\theta)\|\}$, then by (4.14) we get

$$\begin{aligned}\bar{x}(t) &\le g(t) + \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\int_a^t (t-qs)_q^{\alpha-\beta-1}\bar{x}(s)\nabla_q s\\ &+ \frac{\|B_0\|+\|B_1\|+2L}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\bar{x}(s)\nabla_q s\\ &= g(t) + (\|B_0\|+\|B_1\|+2L)_q\nabla_a^{-\alpha}\bar{x}(t) + \|A_0\|_q\nabla_a^{-(\alpha-\beta)}\bar{x}(t).\end{aligned} \tag{4.15}$$

Applying the result of Corollary 3.2, we have

$$\begin{aligned}\|x(t)\| \le \bar{x}(t) &\le g(t)\sum_{n=0}^{\infty} w_2^n \sum_{k=0}^{n}\frac{C_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha+k(\alpha-\beta)+1)}(t-a)_q^{(n-k)\alpha+k(\alpha-\beta)}\\ &\le \delta\left(1+(t-a)+\frac{\|A_0\|(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right)\sum_{n=0}^{\infty} w_2^n \sum_{k=0}^{n}\frac{C_n^k\Gamma_q(\alpha)^{n-k}\Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha+k(\alpha-\beta)+1)}(t-a)_q^{(n-k)\alpha+k(\alpha-\beta)}\\ &< \varepsilon.\end{aligned} \tag{4.16}$$

Therefore, the system (1.5) is finite-time stable. The proof is completed.

5. Example

If $x \in \mathbb{R}^n$, then $\|x\| = \sum_{i=1}^n |x_i|$. If $A \in \mathbb{R}^{n\times n}$, then the induced norm $\|\cdot\|$ is defined as $\|A\| = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$.

Example 5.1. Consider the nonlinear delay q-fractional differential difference system

$$\begin{cases} {}_qC_a^{1.8}x(t) - \begin{pmatrix} 0 & 0.62 \\ 0.56 & 0 \end{pmatrix} {}_qC_a^{0.8}x(t) \\ \qquad = \begin{pmatrix} 0 & 0.08 \\ 0.109 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0.15 & 0 \\ 0 & 0.12 \end{pmatrix} x(\tau t) + f(t, x(t), x(\tau t)), & t \in [a, T)_q, \\ x(t) = \phi(t), \quad \nabla_q x(t) = \psi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{5.1}$$

where $\alpha = 1.8, \beta = 0.8, q = 0.6, a = q^5 = 0.6^5, T = q^{-1} = 0.6^{-1}, \tau = q^3 = 0.6^3, x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$,

$$f(t, x(t), x(\tau t)) = \frac{1}{4}[\sin x_1(t), \sin x_2(\tau t)]^T - \frac{1}{5}[\arctan x_1(\tau t), \arctan x_2(\tau t)]^T,$$

and

$$\phi(t) = [0.05, 0.035]^T, \quad \psi(t) = [0.04, 0.045]^T, \quad t \in \mathbb{I}_\tau = \{0.6^9, 0.6^8, 0.6^7, 0.6^6, 0.6^5\}.$$

Obviously, $\|\phi\| = \|\psi\| = 0.0085 < 0.1 = \delta$, $\epsilon = 1$. We can see that f satisfies conditions (H1) ($L = \frac{1}{4}$) and (H2). We can calculate $\|A_0\| = 0.62$, $\|B_0\| = 0.109$, $\|B_1\| = 0.15$.

When $T = 0.6^{-1}$, it is easy to check that

$$[\Gamma_q(\alpha)T^\alpha(1-q)^\alpha + \Gamma_q(\alpha-\beta)T^{\alpha-\beta}(1-q)^{\alpha-\beta}] \max\left\{\frac{\|B_0\| + \|B_1\| + 2L}{\Gamma_q(\alpha)}, \frac{\|A_0\|}{\Gamma_q(\alpha-\beta)}\right\} = 0.8992 < 1,$$

that is, (H3) holds. By using Matlab (the pseudo-code to compute different values of $\Gamma_q(\sigma)$, see [32]), when $t = 1 \in [a, T)_q$,

$$\begin{aligned} &\left(1 + (t-a) + \frac{\|A_0\|(t-a)_q^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right) \sum_{n=0}^{\infty} w_2^n \sum_{k=0}^{n} \frac{C_n^k \Gamma_q(\alpha)^{n-k}\Gamma_q(\alpha-\beta)^k}{\Gamma_q((n-k)\alpha + k(\alpha-\beta)+1)} (t-a)_q^{(n-k)\alpha+k(\alpha-\beta)} \\ &\quad \approx 8.4593 < 10 = \frac{\epsilon}{\delta}. \end{aligned}$$

Thus, we obtain $T_e = 1$.

6. Conclusions

In this paper, we introduced and proved new generalizations for q-fractional Gronwall inequality. We examined the validity and applicability of our results by considering the existence and uniqueness of solutions of nonlinear delay q-fractional difference damped system. Moreover, a novel and easy to verify sufficient conditions have been provided in this paper which are easy to determine the finite-time stability of the solutions for the considered system. Finally, an example is given to illustrate the effectiveness and feasibility of our criterion. Motivated by previous works [33, 34], the possible applications of fractional q-difference in the field of stability theory will be considered in the future.

Acknowledgments

The authors are grateful to the anonymous referees for valuable comments and suggestions that helped to improve the quality of the paper. This work is supported by Natural Science Foundation of China (11571136).

Conflict of interest

The authors declare that there is no conflicts of interest.

References

1. F. M. Atici, P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Am. Math. Soc.,* **137** (2009), 981–989.
2. G. A. Anastassiou, Nabla discrete fractional calculus and nabla inequalities, *Math. Comput. Model.,* **51** (2010), 562–571.
3. C. S. Goodrich, Continuity of solutions to discrete fractional initial value problems, *Comput. Math. Appl.,* **59** (2010), 3489–3499.
4. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst., Ser. A,* **29** (2011), 417–437.
5. R. Floreanini, L. Vinet, Quantum symmetries of q-difference equations, *J. Math. Phys.,* **36** (1995), 3134–3156.
6. M. Marin, On existence and uniqueness in thermoelasticity of micropolar bodies, *C. R. Acad. Sci. Paris, Ser. II,* **321** (1995), 475–480.
7. R. Finkelstein, E. Marcus, Transformation theory of the q-oscillator, *J. Math. Phys.,* **36** (1995), 2652–2672.
8. M. Marin, Lagrange identity method for microstretch thermoelastic materials, *J. Math. Anal. Appl.,* **363** (2010), 275–286.
9. T. Ernst, *A Comprehensive Treatment of q-Calculus*, Birkhäuser, Basel, 2012.
10. F. Jarad, T. Abdeljawad, D. Baleanu, Stability of q-fractional non-autonomous systems, *Nonlinear Anal., Real World Appl.,* **14** (2013), 780–784.
11. T. Abdeljawad, D. Baleanu, Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function, *Commun. Nonlinear Sci. Numer. Simul.,* **16** (2011), 4682–4688.
12. Z. S. I. Mansour, Linear sequential q-difference equations of fractional order, *Fract. Calc. Appl. Anal.,* **12** (2009), 159–178.

13. X. Li, Z. Han, X. Li, Boundary value problems of fractional q-difference Schrödinger equations, *Appl. Math. Lett.,* **46** (2015), 100–105.
14. J. Mao, Z. Zhao, C. Wang, The unique iterative positive solution of fractional boundary value problem with q-difference, *Appl. Math. Lett.,* **100** (2020), 106002.
15. Y. Liang, H. Yang, H. Li, Existence of positive solutions for the fractional q-difference boundary value problem, *Adv. Differ. Equ.,* **2020** (2020), 1–11.
16. M. H. Annaby, Z. S. Mansour, *q-fractional Calculus and Equations. Lecture Notes in Mathematics,* vol. 2056, Springer, Heidelberg, 2012.
17. T. Abdeljawad, J. Alzabut, D. Baleanu, A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems, *J. Inequal. Appl.,* **2016** (2016), 240.
18. T. Abdeljawad, J. Alzabut, The q-fractional analogue for Gronwall-type inequality, *J. Funct. Spaces,* **2013** (2013), 543839.
19. F. Du, B. Jia, A generalized fractional (q,h)-Gronwall inequality and its applications to nonlinear fractional delay (q,h)-difference systems, *Math. Methods Appl. Sci.,* **44** (2021), 10513–10529.
20. J. Sheng, W. Jiang, Existence and uniqueness of the solution of fractional damped dynamical systems, *Adv. Differ. Equ.,* **2017** (2017), 16.
21. M. P. Lazarevic, A. M. Spasic, Finite-time stability analysis of fractional order time-delay systems: Gronwall approach, *Math. Comput. Model.,* **49** (2009), 475–481.
22. V. N. Phat, N. T. Thanh, New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach, *Appl. Math. Lett.,* **83** (2018), 169–175.
23. R. Wu, Y. Lu, L. Chen, Finite-time stability of fractional delayed neural networks, *Neurocomputing,* **149** (2015), 700–707.
24. M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, *Appl. Math. Comput.,* **24** (2018), 254–265.
25. F. Du, J. G. Lu, Finite-time stability of neutral fractional order time delay systems with Lipschitz nonlinearities, *Appl. Math. Comput.,* **375** (2020), 125079.
26. F. Du, B. Jia, Finite-time stability of a class of nonlinear fractional delay difference systems, *Appl. Math. Lett.,* **98** (2019), 233–239.
27. F. Du, B. Jia, Finite time stability of fractional delay difference systems: A discrete delayed Mittag-Leffler matrix function approach, *Chaos, Solitons and Fractals,* **141** (2020), 110430.
28. K. Ma, S. Sun, Finite-time stability of linear fractional time-delay q-difference dynamical system, *J. Appl. Math. Comput.,* **57** (2018), 591–604.
29. P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q-calculus, *Appl. Anal. Discr. Math.,* **1** (2007), 311–323.
30. M. Mansour, An asymptotic expansion of the q-gamma function $\Gamma_q(x)$, *J. Nonlinear Math. Phys.,* **13** (2006), 479–483.
31. R. A. Adams, C. Essex, *Calculus A Complete Course*, Seventh Edition, Pearson Canada, Toronto, 2010.

32. N. Phuong, F. Sakar, S. Etemad, S. Rezapour, A novel fractional structure of a multi-order quantum multi-integro-differential problem, *Adv. Differ. Equ.,* **2020** (2020), 633.

33. R.Butt, T. Abdeljawad, M. Alqudah, M. Rehman, Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach, *J. Inequal. Appl.,* **2019** (2019), 305.

34. L. Franco-Péreza, G. Fernández-Anaya, L. A. Quezada-Téllez, On stability of nonlinear nonautonomous discrete fractional Caputo systems, *J. Math. Anal. Appl.,* **487** (2020), 124021.

AIMS Press