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Research article

Quantum Hermite-Hadamard type integral inequalities for convex stochastic processes

Thanin Sitthiwirattham[1], Muhammad Aamir Ali[2,*], Hüseyin Budak[3] and Saowaluck Chasreechai[4]

[1] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, 10300, Thailand

[2] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

[4] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand

* **Correspondence:** Email: mahr.muhammad.aamir@gmail.com.

Abstract: In this paper, we introduce the notions of q-mean square integral for stochastic processes and co-ordinated stochastic processes. Furthermore, we establish some new quantum Hermite-Hadamard type inequalities for convex stochastic processes and co-ordinated stochastic processes via newly defined integrals. It is also revealed that the results presented in this research transformed into some already proved results by considering the limits as $q,\ q_1,\ q_2 \to 1^-$ in the newly obtained results.

Keywords: Hermite-Hadamard inequality; Jensen inequality; convex stochastic processes
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1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention with an interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalizing, refining and extending it for different classes of functions, such as convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see [26,49]). These inequalities state that if $F : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} F(x)dx \leq \frac{F(a)+F(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if F is concave. For more recent developments of inequality (1.1), one can consult [2, 3, 24, 25, 27, 48, 50, 60, 61].

The stochastic processes may be described in a general sense, and it has piqued the interest of many researchers due to its numerous applications in disciplines, such as physics, mathematics, economics, and engineering, therefore in 1980, K. Nikodem [44] introduced the notion of convex stochastic processes and discussed their regularity properties. In [54], A. Skowroński discussed some more results for convex stochastic processes which generalize to some results about the classical convex functions. After that, D. Kotrys established Hermite-Hadamard inequality for the convex stochastic processes in [38]. The inequality states that if a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is a Jensen-convex, mean-square continuous in the interval I. Then for any $u, v \in I$ we have

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{v-u} \int_{u}^{v} X(t, \cdot)\, dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad \text{(a.e.).} \tag{1.2}$$

For more results regarding the inequality (1.2), one can read [14, 18, 19, 30, 36, 39, 42, 43, 51–53, 57, 58].

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications, such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [10, 29, 35]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [15, 17]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by F. H. Jackson [28, 33]. In 1966, W. Al-Salam [13] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, J. Tariboon and S. K. Ntouyas introduced ${}_aD_q$-difference operator and q_a-integral in [56]. In 2020, S. Bermudo et al. introduced the notion of ${}^bD_q$ derivative and q^b-integral in [16].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [5, 8, 11, 12, 16, 21, 22, 34, 41, 45], the authors used ${}_aD_q, {}^bD_q$-derivatives and q_a, q^b-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [46], M. A. Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, E. R. Nwaeze et al. proved certain parameterized quantum integral inequalities in [47]. M. A. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [37]. H. Budak et al. [20], M. A. Ali et al. [4, 6] and M. Vivas-Cortez et al. [59] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions on can consult [7, 9, 23].

Recently, in [32], W. U. Haq introduced the notions about the q_a-mean square integral and gave the following quantum version of the inequality (1.2).

Theorem 1.1. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval I, then for any $u, v \in I$ we have*

$$X\left(\frac{qu+v}{[2]_q},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\ {}_u d_q t \leq \frac{qX(u,\cdot)+X(v,\cdot)}{[2]_q} \qquad (a.e.).$$

Despite the fact that stochastic processes theory and applications have advanced significantly in recent years, there are still many new and challenging problems in the areas of theory, analysis, and application, which include fields such as stochastic control, Markov chains, renewal processes, actuarial science, and so on.

Inspired by these ongoing studies, we introduce the notions of q-mean square integrals with respect to b and prove some new Hermite-Hadamard type inequality for convex stochastic processes. Moreover, we introduce four different variants of q_1q_2-mean square integrals for co-ordinated stochastic processes and prove some new Hermite-Hadamard type inequalities for co-ordinated stochastic processes.

This paper's organization is as follows: We summarize the convex stochastic processes in Section 2, and some related work is given in this setup. In Section 3, we review the notions of q-calculus and some related research in this direction. In Sections 4 and 5, we prove Hermite-Hadamard type inequalities for the convex stochastic and co-ordinated convex stochastic processes. The relationship between the findings obtained and the comparable outcomes in the current literature is also discussed. Some findings and further directions for future study are found in Section 6. We assume that the analysis initiated in this paper could provide researchers working on integral inequalities and their applications with a strong source of inspiration.

2. Convex stochastic processes

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}-$measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, .)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

(i) continuous in probability in interval I, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t, .) = X(t_0, .),$$

where $P - \lim$ denotes the limit in probability.

(ii) *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} E\left[(X(t) - X(t_0))^2\right] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t, .)$.

Indeed, mean-square continuity implies continuity in probability, but the converse implication is not true.

Definition 2.1. *Suppose we are given a sequence* $\{\Delta^m\}$ *of partitions,* $\Delta^m = \{a_{m,0}, ..., a_{m,n_m}\}$*. We say that the sequence* $\{\Delta^m\}$ *is a normal sequence of partitions if the length of the greatest interval in the* $n-th$ *partition tends to zero, i.e.,*

$$\lim_{m\to\infty} \sup_{1\le i\le n_m} \left|a_{m,i} - a_{m,i-1}\right| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties (see [55]).

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[X(t)^2\right] < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < t_2 < ... < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a, b]$, if we have

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequence of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$. Then, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)\, ds \text{ (a.e.)}.$$

For the existence of the mean-square integral, it is enough to assume the mean-square continuity of the stochastic process X.

Throughout the paper, we will frequently use the monotonicity of the mean-square integral. If $X(t, \cdot) \le Y(t, \cdot)$ (a.e.) in some interval $[a, b]$, then

$$\int_a^b X(t, \cdot)\, dt \le \int_a^b Y(t, \cdot)\, dt \text{ (a.e.)}.$$

Of course, this inequality is an immediate consequence of the definition of the mean-square integral.

Definition 2.2. *We say that a stochastic process* $X : I \times \Omega \to \mathbb{R}$ *is convex, if for all* $\lambda \in [0, 1]$ *and* $u, v \in I$ *the inequality*

$$X(\lambda u + (1-\lambda)v, \cdot) \le \lambda X(u, \cdot) + (1-\lambda) X(v, \cdot) \quad (a.e.) \tag{2.1}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$*, then the process X is Jensen-convex or* $\frac{1}{2}-$*convex. A stochastic process X is concave if* $(-X)$ *is convex. Some interesting properties of convex and Jensen-convex processes are presented in [44, 55].*

Now, we present some results proved by D. Kotrys [38] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 2.1. *If* $X : I \times \Omega \to \mathbb{R}$ *is a stochastic process of the form* $X(t, \cdot) = A(\cdot)t + B(\cdot)$*, where* $A, B : \Omega \to \mathbb{R}$ *are random variables, such that* $E\left[A^2\right] < \infty, E\left[B^2\right] < \infty$ *and* $[a, b] \subset I$*, then*

$$\int_a^b X(t, \cdot)\, dt = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b - a) \text{ (a.e.)}.$$

Proposition 2.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in intI$. Then there exists a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is*

$$X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \ (a.e.).$$

for all $t \in I$.

Theorem 2.1. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex, mean-square continuous in the interval I. Then for any $u, v \in I$ we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{v-u} \int_u^v X(t, \cdot)\, dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \ (a.e.). \tag{2.2}$$

In [39], D. Kotrys introduced the concept of strongly convex stochastic processes and investigated their properties.

Definition 2.3. *Let $C : \Omega \to \mathbb{R}$ denote a positive random variable. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called strongly convex with modulus $C(\cdot) > 0$, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality*

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) - C(\cdot)\lambda(1-\lambda)(u-v)^2 \qquad a.e.$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is strongly Jensen-convex with modulus $C(\cdot)$.

In [31], F. M. Hafiz gave the following definition of stochastic mean-square fractional integrals:

Definition 2.4. *For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals I^{α}_{u+} and I^{α}_{v+} of X of order $\alpha > 0$ is defined by*

$$I^{\alpha}_{u+}[X](t) = \frac{1}{\Gamma(\alpha)} \int_u^t (t-s)^{\alpha-1} X(x, s) ds \quad (a.e.), \quad t > u$$

and

$$I^{\alpha}_{v-}[X](t) = \frac{1}{\Gamma(\alpha)} \int_t^v (s-t)^{\alpha-1} X(x, s) ds \quad (a.e.), \quad t < v.$$

Using this concept of stochastic mean-square fractional integrals I^{α}_{a+} and I^{α}_{b+}, H. Agahi and A. Babakhani proved the following Hermite-Hadamard type inequality for convex stochastic processes:

Theorem 2.2. *[1] Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval I. Then for any $u, v \in I$, the following Hermite-Hadamard inequality*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}} \left[I^{\alpha}_{u+}[X](v) + I^{\alpha}_{v-}[X](u)\right] \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \ (a.e.) \tag{2.3}$$

holds, where $\alpha > 0$.

3. Quantum calculus and related inequalities

In this section, we present some required definitions. Additionally, here and further we use the following notation (see [35]):

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{n=0}^{n-1} q^n, \quad q \in (0,1).$$

In [33], F. H. Jackson gave the q-Jackson integral from 0 to b for $0 < q < 1$ as follows:

$$\int_0^b F(x) \ d_q x \ = (1-q)\, b \sum_{n=0}^{\infty} q^n F(bq^n) \tag{3.1}$$

provided the sums converges absolutely. Furthermore, he gave the q-Jackson integral in a generic interval $[a,b]$ as:

$$\int_a^b F(x) \ d_q x \ = \int_0^b F(x) \ d_q x \ - \int_0^a F(x) \ d_q x \ .$$

Definition 3.1. *[56] For a continuous function $F:[a,b] \to \mathbb{R}$, the q_a-derivative of F at $x \in [a,b]$ is characterized by the expression:*

$${}_aD_q F(x) \ = \frac{F(x) - F(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{3.2}$$

If $x = a$, we define ${}_aD_q F(a) = \lim_{x \to a} \ {}_aD_q F(x)$ if it exists and it is finite.

Definition 3.2. *[56] Let $F:[a,b] \to \mathbb{R}$ be a continuous function. Then, the q_a-definite integral on $[a,b]$ is defined as:*

$$\int_a^b F(x) \ {}_a d_q x \ = (1-q)(b-a) \sum_{n=0}^{\infty} q^n F(q^n b + (1-q^n) a) = (b-a) \int_0^1 F((1-t)a + tb) \ d_q t \ .$$

In [11], N. Alp et al. proved the following q_a-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 3.1. *If $F:[a,b] \to \mathbb{R}$ is a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then , the q-Hermite-Hadamard inequalities are expressed as:*

$$F\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a} \int_a^b F(x) \ {}_a d_q x \ \leq \frac{qF(a) + F(b)}{[2]_q}. \tag{3.3}$$

On the other hand, S. Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 3.3. *[16] For a continuous function $F:[a,b]\to\mathbb{R}$, the q^b-derivative of F at $x\in[a,b]$ is characterized by the expression:*

$$^{b}D_{q}F(x) = \frac{F(qx+(1-q)b)-F(x)}{(1-q)(b-x)},\ x\neq b.$$

If $x=b$, we define $^{b}D_{q}F(b)=\lim_{x\to b}\ {}^{b}D_{q}F(x)$ if it exists and it is finite.

Definition 3.4. *[16] Let $F:[a,b]\to\mathbb{R}$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as:*

$$\int_a^b F(x)\ {}^{b}d_{q}x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n F(q^n a+(1-q^n)b) = (b-a)\int_0^1 F(ta+(1-t)b)\ d_q t\ .$$

Theorem 3.2. *[16] If $F:[a,b]\to\mathbb{R}$ is a convex differentiable function on $[a,b]$ and $0<q<1$. Then, the q-Hermite-Hadamard inequalities are expressed as:*

$$F\left(\frac{a+qb}{[2]_q}\right)\leq\frac{1}{b-a}\int_a^b F(x)\ {}^{b}d_{q}x \leq \frac{F(a)+qF(b)}{[2]_q}. \tag{3.4}$$

From Theorem 3.1 and Theorem 3.2, one can obtain the following inequalities:

Corollary 3.1. *[16] For any convex function $F:[a,b]\to\mathbb{R}$ and $0<q<1$, we have*

$$F\left(\frac{qa+b}{[2]_q}\right)+F\left(\frac{a+qb}{[2]_q}\right)\leq\frac{1}{b-a}\left\{\int_a^b F(x)\ {}_{a}d_{q}x + \int_a^b F(x)\ {}^{b}d_{q}x\right\}\leq F(a)+F(b) \tag{3.5}$$

and

$$F\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)}\left\{\int_a^b F(x)\ {}_{a}d_{q}x + \int_a^b F(x)\ {}^{b}d_{q}x\right\}\leq\frac{F(a)+F(b)}{2}. \tag{3.6}$$

In [40], M. A. Latif defined q_{ac}-integral and partial q-derivatives for two variables functions as follows:

Definition 3.5. *Suppose that $F:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous function. The definite q_{ac}-integral on $[a,b]\times[c,d]$ is defined by*

$$\begin{aligned}\int_a^x\int_c^y F(t,s)\ {}_{c}d_{q_2}s\ \ {}_{a}d_{q_1}t &= (1-q_1)(1-q_2)(x-a)(y-c)\\ &\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x+(1-q_1^n)a, q_2^m y+(1-q_2^m)c)\end{aligned}$$

for $(x,y)\in[a,b]\times[c,d]$.

On the other hand, H. Budak et al. gave the following definitions of q_a^d, q_b^c and q^{bd} integrals:

Definition 3.6. *[22] Suppose that $F : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Then the following q_a^d, q_c^b and q^{bd} integrals on $[a,b] \times [c,d]$ are defined by*

$$\int_a^x \int_y^d F(t,s)\ {}^d d_{q_2} s\ \ {}_a d_{q_1} t = (1-q_1)(1-q_2)(x-a)(d-y) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + (1-q_1^n) a, q_2^m y + (1-q_2^m) d\right) \tag{3.7}$$

$$\int_x^b \int_c^y F(t,s)\ {}_c d_{q_2} s\ \ {}^b d_{q_1} t = (1-q_1)(1-q_2)(b-x)(y-c) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + (1-q_1^n) b, q_2^m y + (1-q_2^m) c\right) \tag{3.8}$$

and

$$\int_x^b \int_y^d F(t,s)\ {}^d d_{q_2} s\ \ {}^b d_{q_1} t = (1-q_1)(1-q_2)(b-x)(d-y) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + (1-q_1^n) b, q_2^m y + (1-q_2^m) d\right) \tag{3.9}$$

respectively, for $(x,y) \in [a,b] \times [c,d]$.

Recently, W. U. Haq gave the following definition of q_a-mean square integral and related results for quantum stochastic process:

Definition 3.7. *[32] For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the q_a-mean integral of X on $[a,b] \subset I$ with $E\left[(X(t))^2\right] < \infty$ for all $t \in I$, is a random variable $\Lambda : \Omega \to \mathbb{R}$ which satisfying the following equality:*

$$E\left[\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n X\left(q^n b + (1-q^n) a\right) - \Lambda\right)^2\right] = 0.$$

Hence, we can state

$$\Lambda(.) = \int_a^b X(t,.)\ {}_a d_q t \quad \text{(a.e)},$$

for the existence of the q_a-mean integral, the stochastic process must be mean square continuous.

Lemma 3.1. *[32] If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t,\cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E\left[A^2\right] < \infty$, $E\left[B^2\right] < \infty$ and $[a,b] \subset I$, then*

$$\int_a^b X(t,\cdot)\ {}^b d_q t = A(\cdot)\frac{(b-a)(b+qa)}{[2]_q} + B(\cdot)(b-a) \quad \text{(a.e.)}.$$

Theorem 3.3. *[32] If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval I, then for any $u, v \in I$ we have*

$$X\left(\frac{qu+v}{[2]_q}, \cdot\right) \leq \frac{1}{v-u} \int_u^v X(t, \cdot)\ {}_u d_q t \leq \frac{qX(u, \cdot) + X(v, \cdot)}{[2]_q} \quad \text{(a.e.)}. \tag{3.10}$$

4. Quantum Hermite-Hadamard inequality for convex stochastic processes

In this section, we introduce the notion of q^b-mean square integral for stochastic process and establish some new inequalities of Hermite-Hadamard type for convex stochastic process using the q^b-mean square integral.

Definition 4.1. *For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the q^b-mean integral of X on $[a, b] \subset I$ with $E\left[(X(t))^2\right] < \infty$ for all $t \in I$, is a random variable $\Lambda : \Omega \to \mathbb{R}$ which satisfying the following equality:*

$$E\left[\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n X(q^n a + (1-q^n) b) - \Lambda\right)^2\right] = 0.$$

Hence, we can state

$$\Lambda(.) = \int_a^b X(t, .)\ {}^b d_q t \quad \text{(a.e)},$$

for the existence of the q^b-mean integral, the stochastic process must be a mean square continuous.

Lemma 4.1. *If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, \cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E\left[A^2\right] < \infty$, $E\left[B^2\right] < \infty$ and $[a, b] \subset I$, then*

$$\int_a^b X(t, \cdot)\ {}^b d_q t = A(\cdot)\frac{(b-a)(qb+a)}{[2]_q} + B(\cdot)(b-a) \quad \text{(a.e.)}.$$

Theorem 4.1. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval I, then for any $u, v \in I$ we have*

$$X\left(\frac{u+qv}{[2]_q}, \cdot\right) \leq \frac{1}{v-u} \int_u^v X(t, \cdot)\ {}^v d_q t \leq \frac{X(u, \cdot) + qX(v, \cdot)}{[2]_q} \quad \text{(a.e.)}. \tag{4.1}$$

Proof. Since the process X is mean-square continuous, it is continuous in probability. Nikodem in [44] proved that every Jensen-convex and continuous in probability stochastic process is convex. Since X is convex, then by Proposition 2.1 it has a supporting process at any point $t_0 \in intI$. Let's take a support at $t_0 = \frac{u+qv}{[2]_q}$, then we have

$$X(t, \cdot) \geq A(\cdot)\left(t - \frac{u+qv}{[2]_q}\right) + X\left(\frac{u+qv}{[2]_q}, \cdot\right) \quad \text{(a.e.)}.$$

By q^v-integral, we have

$$\int_u^v X(t,\cdot)\ {}^v d_q t \geq A(\cdot)\frac{(v-u)(qv+u)}{[2]_q} - A(\cdot)\frac{u+qv}{[2]_q}(v-u) + X\left(\frac{u+qv}{[2]_q},\cdot\right)(v-u).$$

That is

$$\frac{1}{v-u}\int_u^v X(t,\cdot)\ {}^v d_q t \geq X\left(\frac{u+qv}{[2]_q},\cdot\right)$$

which proves the first inequality in (4.1).

Since stochastic processes X is convex, we have

$$\begin{aligned} X(t,\cdot) &= X\left(\frac{t-u}{v-u}v + \frac{v-t}{v-u}u,\cdot\right) \\ &\leq \frac{t-u}{v-u}X(v,\cdot) + \frac{v-t}{v-u}X(u,\cdot) \\ &= \frac{X(v,\cdot)-X(u,\cdot)}{v-u}t + \frac{vX(u,\cdot)-uX(v,\cdot)}{v-u}. \end{aligned}$$

By using q^v-integral, we have

$$\begin{aligned} \int_u^v X(t,\cdot)\ {}^v d_q t &\leq \frac{X(v,\cdot)-X(u,\cdot)}{v-u}\frac{u+qv}{[2]_q}(v-u) + \frac{vX(u,\cdot)-uX(v,\cdot)}{v-u}(v-u) \\ &= \frac{X(u,\cdot)+qX(v,\cdot)}{[2]_q}(v-u). \end{aligned}$$

This completes the proof. □

Remark 4.1. *If we set $q \to 1^-$ in Theorem 4.1, then Theorem 4.1 reduces to Theorem 2.1.*

Adding the results in Theorems 3.1 and 4.1 yields the next corollary.

Corollary 4.1. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval I, then for any $u, v \in I$ we have*

$$\begin{aligned} &\frac{1}{2}\left[X\left(\frac{qu+v}{[2]_q},\cdot\right) + X\left(\frac{u+qv}{[2]_q},\cdot\right)\right] \\ \leq\ &\frac{1}{2(v-u)}\left[\int_u^v X(t,\cdot)\ {}_u d_q t + \int_u^v X(t,\cdot)\ {}^v d_q t\right] \\ \leq\ &\frac{X(u,\cdot)+X(v,\cdot)}{2} \qquad (a.e.). \end{aligned}$$

Corollary 4.2. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval I, then for any $u, v \in I$ we have*

$$\begin{aligned} & X\left(\frac{u+v}{2}, \cdot\right) \\ \leq \; & \frac{1}{2(v-u)}\left[\int_u^v X(t,\cdot)\ {}_u d_q t + \int_u^v X(t,\cdot)\ {}^v d_q t\right] \\ \leq \; & \frac{X(u,\cdot)+X(v,\cdot)}{2} \quad \text{(a.e.)}. \end{aligned}$$

Proof. Since X is convex stochastic process, we have

$$\begin{aligned} X\left(\frac{u+v}{2}, \cdot\right) &= X\left(\frac{1}{2}\frac{qu+v}{[2]_q} + \frac{1}{2}\frac{u+qv}{[2]_q}, \cdot\right) \\ &\leq \frac{1}{2}\left[X\left(\frac{qu+v}{[2]_q}, \cdot\right) + X\left(\frac{u+qv}{[2]_q}, \cdot\right)\right]. \end{aligned}$$

This completes the proof. □

Theorem 4.2. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly Jensen-convex with modulus $C(\cdot)$ and mean-square continuous in the interval I such that $E\left[C^2\right]$, then for any $u, v \in I$ we have*

$$\begin{aligned} & X\left(\frac{u+qv}{[2]_q}, \cdot\right) + C(\cdot)\left[\frac{q(u+qv)^2 + u^2 + qv^2}{[2]_q [3]_q} - \left(\frac{u+qv}{[2]_q}\right)^2\right] \\ \leq \; & \frac{1}{v-u}\int_u^v X(t,\cdot)\ {}^v d_q t \\ \leq \; & \frac{X(u,\cdot)+qX(v,\cdot)}{[2]_q} + C(\cdot)\left[\frac{q(u+qv)^2 + u^2 + qv^2}{[2]_q [3]_q} - \frac{u^2+qv^2}{[2]_q}\right] \quad \text{(a.e.)}. \end{aligned}$$

Proof. If X is strongly convex with modulus $C(\cdot)$, then the process $Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$ is convex. Therefore, if we apply the inequality (4.1) to process $Y(t,\cdot)$, then we have

$$Y\left(\frac{u+qv}{[2]_q}, \cdot\right) \leq \frac{1}{v-u}\int_u^v Y(t,\cdot)\ {}^v d_q t \leq \frac{Y(u,\cdot)+qY(v,\cdot)}{[2]_q} \quad \text{(a.e.)}.$$

That is,

$$\begin{aligned} & X\left(\frac{u+qv}{[2]_q}, \cdot\right) - C(\cdot)\left(\frac{u+qv}{[2]_q}\right)^2 \\ \leq \; & \frac{1}{v-u}\int_u^v \left[X(t,\cdot) - C(\cdot)t^2\right]\ {}^v d_q t \end{aligned}$$

$$\leq \quad \frac{X(u,\cdot) - C(\cdot)u^2 + q\left[X(v,\cdot) - C(\cdot)v^2\right]}{[2]_q} \qquad \text{(a.e.).}$$

By calculating the integrals, we have

$$X\left(\frac{u+qv}{[2]_q},\cdot\right) + C(\cdot)\left[\frac{q(u+qv)^2 + u^2 + qv^2}{[2]_q[3]_q} - \left(\frac{u+qv}{[2]_q}\right)^2\right]$$

$$\leq \quad \frac{1}{v-u}\int_u^v X(t,\cdot)\ {}^{v}d_q t$$

$$\leq \quad \frac{X(u,\cdot) + qX(v,\cdot)}{[2]_q} + C(\cdot)\left[\frac{q(u+qv)^2 + u^2 + qv^2}{[2]_q[3]_q} - \frac{u^2 + qv^2}{[2]_q}\right] \qquad \text{(a.e.)}$$

which gives the required result. □

Remark 4.2. *If we set $q \to 1^-$ in Theorem 4.2, then Theorem 4.2 reduces to [39, Theorem 11].*

5. Quantum Hermite-Hadamard inequality for co-ordinated convex stochastic processes

In this section, we review the definition of co-ordinated convex stochastic process and introduce the notion of q_1q_2-mean square integrals for co-ordinated stochastic processes. Moreover, we prove Hermite-Hadamard inequalities for co-ordinated convex stochastic process using the q_1q_2-mean square integrals which is the main motivation of this section.

Definition 5.1. *[53] Let $\Lambda := T_1 \times T_2$, $T_1, T_2 \subset \mathbb{R}$. A stochastic process $X : \Lambda \times \Omega \to \mathbb{R}$ is called co-ordinated convex on Λ if the following inequality holds for all $\alpha, \beta \in [0,1]$ and $(t_1, s_1), (t_2, s_2) \in \Lambda$:*

$$X((\alpha t_1 + (1-\alpha)t_2, \beta s_1 + (1-\beta)s_2), \cdot)$$

$$\leq \quad \alpha\beta X((t_1,s_1),\cdot) + \alpha(1-\beta)X((t_1,s_2),\cdot)$$

$$+ (1-\alpha)\beta X((t_2,s_1),\cdot) + (1-\alpha)(1-\beta)X((t_2,s_2),\cdot).$$

We can give the definitions of q_1q_2-mean square integrals as follows:

Definition 5.2. *Let $X : \Lambda \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[X(t,s)^2\right] < \infty$ for all $(t,s) \in \Lambda$.*

i) A random variable $\Pi_1 : \Omega \to \mathbb{R}$ is called the q_{ac}-mean-square integral of the process X on $[a,b] \times [c,d] \subset \Lambda$, if we have

$$E\left[\left((1-q_1)(1-q_2)(b-a)(d-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m X(q_1^n b + (1-q_1^n)a, q_2^m d + (1-q_2^m)c) - \Pi_1\right)^2\right] = 0.$$

Then, we can state

$$\Pi_1(.) = \int_a^b \int_c^d X((t,s),\cdot)\ {}_c d_{q_2} s\ \ {}_a d_{q_1} t \quad \textit{(a.e).}$$

ii) A random variable $\Pi_2 : \Omega \to \mathbb{R}$ is called the q_a^d-mean-square integral of the process X on $[a,b] \times [c,d] \subset \Lambda$, if we have

$$E\left[\left((1-q_1)(1-q_2)(b-a)(d-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m X\left(q_1^n b + (1-q_1^n)a, q_2^m c + (1-q_2^m)d\right) - \Pi_2\right)^2\right] = 0.$$

Then, we can define

$$\Pi_2(.) = \int_a^b \int_c^d X((t,s),\cdot)\ {}^d d_{q_2} s\ \ {}_a d_{q_1} t \quad (a.e).$$

iii) A random variable $\Pi_3 : \Omega \to \mathbb{R}$ is called the q_c^b-mean-square integral of the process X on $[a,b] \times [c,d] \subset \Lambda$, if we have

$$E\left[\left((1-q_1)(1-q_2)(b-a)(d-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m X\left(q_1^n a + (1-q_1^n)b, q_2^m d + (1-q_2^m)c\right) - \Pi_3\right)^2\right] = 0.$$

Then, we can define

$$\Pi_3(.) = \int_a^b \int_c^d X((t,s),\cdot)\ {}_c d_{q_2} s\ \ {}^b d_{q_1} t \quad (a.e).$$

iv) A random variable $\Pi_4 : \Omega \to \mathbb{R}$ is called the q^{bd}-mean-square integral of the process X on $[a,b] \times [c,d] \subset \Lambda$, if we have

$$E\left[\left((1-q_1)(1-q_2)(b-a)(d-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m X\left(q_1^n a + (1-q_1^n)b, q_2^m c + (1-q_2^m)d\right) - \Pi_4\right)^2\right] = 0.$$

Then, we can state

$$\Pi_4(.) = \int_a^b \int_c^d X((t,s),\cdot)\ {}^d d_{q_2} s\ \ {}^b d_{q_1} t \quad (a.e).$$

For the existence of the $q_1 q_2$-mean-square integrals, it is enough to assume that the stochastic process X is the mean-square continuous.

Theorem 5.1. *If $X : \Lambda \times \Omega \to \mathbb{R}$ is a co-ordinated convex stochastic process on Λ, then we have*

$$X\left(\left(\frac{u_1 q_1 + v_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}}\right), \cdot\right) \tag{5.1}$$
$$\leq \frac{1}{2}\left[\frac{1}{v_1 - u_1}\int_{u_1}^{v_1} X\left(\left(t, \frac{u_2 q_2 + v_2}{[2]_{q_2}}\right), \cdot\right)\ {}_{u_1} d_{q_1} t\right.$$
$$\left. + \frac{1}{v_2 - u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1 q_1 + v_1}{[2]_{q_1}}, s\right), \cdot\right)\ {}_{u_2} d_{q_2} s\ \right]$$

$$\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ {}_{u_2}d_{q_2}s\ \ {}_{u_1}d_{q_1}t$$
$$\leq \frac{q_1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((u_1,s),\cdot)\ {}_{u_2}d_{q_2}s + \frac{1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((v_1,s),\cdot)\ {}_{u_2}d_{q_2}s$$
$$+ \frac{q_2}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,u_2),\cdot)\ {}_{u_1}d_{q_1}t + \frac{1}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,v_2),\cdot)\ {}_{u_1}d_{q_1}t$$
$$\leq \frac{q_1q_2X((u_1,u_2),\cdot)+q_1X((u_1,v_2),\cdot)+q_2X((v_1,u_2),\cdot)+X((v_1,v_2),\cdot)}{[2]_{q_1}[2]_{q_2}}$$

for $0<q_1,q_2<1$.

Proof. Since $X:\Lambda\times\Omega\to\mathbb{R}$ is a co-ordinated convex stochastic process on Λ, then $Y:T_2\times\Omega\to\mathbb{R}$, $Y(s,\cdot)=X((t,s),\cdot)$ is convex stochastic process on T_2 for all $t\in T_1$. Then by applying Theorem 3.3 to $Y(s,\cdot)$, we obtain

$$Y\left(\frac{q_2u_2+v_2}{[2]_{q_2}},\cdot\right)\leq \frac{1}{v_2-u_2}\int_{u_2}^{v_2} Y(t,\cdot)\ \ {}_{u_2}d_{q_2}s\ \leq \frac{q_2Y(u_2,\cdot)+Y(v_2,\cdot)}{[2]_{q_2}}\qquad \text{(a.e.)}.$$

i.e.

$$X\left(\left(t,\frac{q_2u_2+v_2}{[2]_{q_2}}\right),\cdot\right)\ \leq\ \frac{1}{v_2-u_2}\int_{u_2}^{v_2} X((t,s),\cdot)\ \ {}_{u_2}d_{q_2}s \tag{5.2}$$
$$\leq\ \frac{q_2X((t,u_2),\cdot)+X((t,v_2),\cdot)}{[2]_{q_2}}\qquad \text{(a.e.)}.$$

By the q_u-integral we have,

$$\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{q_2u_2+v_2}{[2]_{q_2}}\right),\cdot\right)\ {}_{u_1}d_{q_1}t \tag{5.3}$$
$$\leq\ \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ \ {}_{u_2}d_{q_2}s\ \ {}_{u_1}d_{q_1}t$$
$$\leq\ \frac{1}{v_1-u_1}\int_{u_1}^{v_1}\frac{q_2X((t,u_2),\cdot)+X((t,v_2),\cdot)}{[2]_{q_2}}\ {}_{u_1}d_{q_1}t\qquad \text{(a.e.)}.$$

By similar argument applied for mapping $Z:T_1\times\Omega\to\mathbb{R}$, $Z(t,\cdot)=X((t,s),\cdot)$, we get

$$\frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{q_1u_1+v_1}{[2]_{q_1}},s\right),\cdot\right)\ {}_{u_2}d_{q_2}s \tag{5.4}$$

$$\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ {}_{u_2}d_{q_2}s\ {}_{u_1}d_{q_1}t$$

$$\leq \frac{1}{v_2-u_2}\int_{u_2}^{v_2} \frac{q_2X((u_1,s),\cdot)+X((v1,s),\cdot)}{[2]_{q_2}}\ {}_{u_2}d_{q_2}s \qquad \text{(a.e.)}.$$

By adding the inequalities (5.3) and (5.4), we obtain the second and third inequalities in (5.1).

By the first inequality in (3.10), we get

$$X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right) \leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right)\ {}_{u_1}d_{q_1}t \tag{5.5}$$

and

$$X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right) \leq \frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},s\right),\cdot\right)\ {}_{u_2}d_{q_2}s\ . \tag{5.6}$$

This gives, by addition the inequalities (5.3) and (5.4), the first inequality in (5.1).

Finally, by second inequality in (3.10), we have

$$\frac{1}{v_2-u_2}\int_{u_2}^{v_2} X((u_1,s),\cdot)\ {}_{u_2}d_{q_2}s\ \leq \frac{q_2X((u_1,u_2),\cdot)+X((u_1,v_2),\cdot)}{[2]_{q_2}}, \tag{5.7}$$

$$\frac{1}{v_2-u_2}\int_{u_2}^{v_2} X((v_1,s),\cdot)\ {}_{u_2}d_{q_2}s\ \leq \frac{q_2X((v_1,u_2),\cdot)+X((v_1,v_2),\cdot)}{[2]_{q_2}}, \tag{5.8}$$

$$\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X((t,u_2),\cdot)\ {}_{u_1}d_{q_1}t\ \leq \frac{q_1X((u_1,u_2),\cdot)+X((v_1,u_2),\cdot)}{[2]_{q_1}} \tag{5.9}$$

and

$$\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X((t,v_2),\cdot)\ {}_{u_1}d_{q_1}t\ \leq \frac{q_1X((u_1,v_2),\cdot)+X((v_1,v_2),\cdot)}{[2]_{q_1}}. \tag{5.10}$$

This completes the proof. □

Theorem 5.2. *If $X:\Lambda\times\Omega\to\mathbb{R}$ is a co-ordinated convex stochastic process on Λ, then we have*

$$X\left(\left(\frac{u_1+v_1q_1}{[2]_{q_1}},\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right) \tag{5.11}$$

$$\leq \frac{1}{2}\left[\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{q_2u_2+v_2}{[2]_{q_2}}\right),\cdot\right)\ {}^{v_1}d_{q_1}t\right.$$

$$
\left. + \frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1+q_1v_1}{[2]_{q_1}}, s\right), \cdot\right) \ {}_{u_2}d_{q_2}s \ \right]
$$

$$
\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot) \ {}_{u_2}d_{q_2}s \ {}^{v_1}d_{q_1}t
$$

$$
\leq \frac{1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((u_1,s),\cdot) \ {}_{u_2}d_{q_2}s + \frac{q_1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((v_1,s),\cdot) \ {}_{u_2}d_{q_2}s
$$

$$
+ \frac{q_2}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,u_2),\cdot) \ {}^{v_1}d_{q_1}t + \frac{1}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,v_2),\cdot) \ {}^{v_1}d_{q_1}t
$$

$$
\leq \frac{q_2X((u_1,u_2),\cdot) + X((u_1,v_2),\cdot) + q_1q_2X((v_1,u_2),\cdot) + q_1X((v_1,v_2),\cdot)}{[2]_{q_1}[2]_{q_2}}
$$

for $0 < q_1, q_2 < 1$.

Proof. By using q^v-integration in (5.2), we have

$$
\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t, \frac{q_2u_2+v_2}{[2]_{q_2}}\right), \cdot\right) \ {}^{v_1}d_{q_1}t \tag{5.12}
$$

$$
\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot) \ {}_{u_2}d_{q_2}s \ {}^{v_1}d_{q_1}t
$$

$$
\leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1} \frac{q_2X((t,u_2),\cdot) + X((t,v_2),\cdot)}{[2]_{q_2}} \ {}^{v_1}d_{q_1}t \qquad \text{(a.e.)}.
$$

By applying Theorem 4.1 to mapping $Z : T_1 \times \Omega \to \mathbb{R}$, $Z(t,\cdot) = X((t,s),\cdot)$, we have

$$
Z\left(\frac{u_1+q_1v_1}{[2]_{q_1}}, \cdot\right) \leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1} Z(t,\cdot) \ {}^{v_1}d_{q_1}t \leq \frac{Z(u_1,\cdot) + q_1Z(v_1,\cdot)}{[2]_{q_1}} \qquad \text{(a.e.)}.
$$

That is,

$$
X\left(\left(\frac{u_1+q_1v_1}{[2]_{q_1}}, s\right), \cdot\right) \leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1} X((t,s),\cdot) \ {}^{v_1}d_{q_1}t \leq \frac{X((u_1,s),\cdot) + q_1X((v_1,s),\cdot)}{[2]_{q_1}} \qquad \text{(a.e.)}.
$$

Then it follows that

$$
\frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1+q_1v_1}{[2]_{q_1}}, s\right), \cdot\right) \ {}_{u_2}d_{q_2}s \tag{5.13}
$$

$$\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ {}_{u_2}d_{q_2}s\ {}^{v_1}d_{q_1}t$$

$$\leq \frac{1}{v_2-u_2}\int_{u_2}^{v_2}\frac{X((u_1,s),\cdot)+q_1X((v_1,s),\cdot)}{[2]_{q_1}}\ {}_{u_2}d_{q_2}s \qquad \text{(a.e.).}$$

Adding the inequalities (5.12) and (5.13), then we obtain the second and third inequalities in (5.11).

By the first inequality in (3.10), we get

$$X\left(\left(\frac{u_1+v_1q_1}{[2]_{q_1}},\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right)\leq \frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},s\right),\cdot\right)\ {}_{u_2}d_{q_2}s \tag{5.14}$$

and by the first inequality in (4.1), we have

$$X\left(\left(\frac{u_1+v_1q_1}{[2]_{q_1}},\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right)\leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{u_2q_2+v_2}{[2]_{q_2}}\right),\cdot\right)\ {}^{v_1}d_{q_1}t\ . \tag{5.15}$$

Then, by adding the inequalities (5.14) and (5.15), we have the first inequality in (5.11).

Finally, by using second inequality in (4.1), we get

$$\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X((t,u_2),\cdot)\ {}^{v_1}d_{q_1}t\ \leq \frac{X((u_1,u_2),\cdot)+q_1X((v_1,u_2),\cdot)}{[2]_{q_1}} \tag{5.16}$$

and

$$\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X((t,v_2),\cdot)\ {}^{v_1}d_{q_1}t\ \leq \frac{X((u_1,v_2),\cdot)+q_1X((v_1,v_2),\cdot)}{[2]_{q_1}}. \tag{5.17}$$

By the inequalities (5.7), (5.8), (5.16) and (5.17), then one can obtain the last inequality in (5.11). Thus, the proof is completed. □

Theorem 5.3. *If $X:\Lambda\times\Omega\to\mathbb{R}$ is a co-ordinated convex stochastic process on Λ, then we have*

$$\begin{aligned}
&X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},\frac{u_2+v_2q_2}{[2]_{q_2}}\right),\cdot\right)\\
&\leq \frac{1}{2}\left[\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{u_2+v_2q_2}{[2]_{q_2}}\right),\cdot\right)\ {}_{u_1}d_{q_1}t\right.\\
&\left.+\frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1q_1+v_1}{[2]_{q_1}},s\right),\cdot\right)\ {}_{u_2}d_{q_2}s\ \right]
\end{aligned}$$

$$\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ {}^{v_2}d_{q_2}s\ \ {}_{u_1}d_{q_1}t$$

$$\leq \frac{q_1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((u_1,s),\cdot)\ {}^{v_2}d_{q_2}s + \frac{1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((v_1,s),\cdot)\ {}^{v_2}d_{q_2}s$$

$$+\frac{1}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,u_2),\cdot)\ {}_{u_1}d_{q_1}t + \frac{q_2}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,v_2),\cdot)\ {}_{u_1}d_{q_1}t$$

$$\leq \frac{q_1X((u_1,u_2),\cdot)+q_1q_2X((u_1,v_2),\cdot)+X((v_1,u_2),\cdot)+q_2X((v_1,v_2),\cdot)}{[2]_{q_1}[2]_{q_2}}$$

and

$$X\left(\left(\frac{u_1+v_1q_1}{[2]_{q_1}},\frac{u_2+v_2q_2}{[2]_{q_2}}\right),\cdot\right) \tag{5.18}$$

$$\leq \frac{1}{2}\left[\frac{1}{v_1-u_1}\int_{u_1}^{v_1} X\left(\left(t,\frac{u_2+v_2q_2}{[2]_{q_2}}\right),\cdot\right)\ {}^{v_1}d_{q_1}t + \frac{1}{v_2-u_2}\int_{u_2}^{v_2} X\left(\left(\frac{u_1+v_1q_1}{[2]_{q_1}},s\right),\cdot\right)\ {}_{u_2}d_{q_2}s\right]$$

$$\leq \frac{1}{(v_1-u_1)(v_2-u_2)}\int_{u_1}^{v_1}\int_{u_2}^{v_2} X((t,s),\cdot)\ {}^{v_2}d_{q_2}s\ \ {}^{v_1}d_{q_1}t$$

$$\leq \frac{1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((u_1,s),\cdot)\ {}^{v_2}d_{q_2}s + \frac{q_1}{2[2]_{q_1}(v_2-u_2)}\int_{u_2}^{v_2} X((v_1,s),\cdot)\ {}^{v_2}d_{q_2}s$$

$$+\frac{1}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,u_2),\cdot)\ {}^{v_1}d_{q_1}t + \frac{q_2}{2[2]_{q_2}(v_1-u_1)}\int_{u_1}^{v_1} X((t,v_2),\cdot)\ {}^{v_1}d_{q_1}t$$

$$\leq \frac{X((u_1,u_2),\cdot)+q_2X((u_1,v_2),\cdot)+q_1X((v_1,u_2),\cdot)+q_1q_2X((v_1,v_2),\cdot)}{[2]_{q_1}[2]_{q_2}}$$

for $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorems 5.1 and 5.2 by using Theorems 4.1 and 5.1. □

Remark 5.1. *By taking the limit* $q\to 1^-$ *, then Theorem 5.1, Theorem 5.2 and Theorem 5.3 reduce to [53, Theorem 2.5].*

6. Conclusions

In this investigation, we have introduced the notions for q-mean square stochastic processes. We have derived some new quantum inequalities of Hermite-Hadamard type for convex stochastic process and co-ordinated stochastic processes using the newly defined integrals. Moreover, we have proved that the results offered in this research are the strong generalization of several known results inside the literature. It is an interesting and new problem that the upcoming mathematicians can offer the similar inequalities for different kinds of stochastic convexities in their future research.

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Conflict of interest

The authors declare no conflict of interest.

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