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Research article

Solutions for systems of complex Fermat type partial differential-difference equations with two complex variables

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Abstract: By making use of the Nevanlinna theory and difference Nevanlinna theory of several complex variables, we investigate some properties of the transcendental entire solutions for several systems of partial differential difference equations of Fermat type, and obtain some results about the existence and the forms of transcendental entire solutions of the above systems, which improve and generalize the previous results given by Cao, Gao, Liu [5, 24, 39]. Some examples are given show that there exist some significant differences in the forms of transcendental entire solutions with finite order of the systems of equations with between several complex variables and a single complex variable.

Keywords: entire solution; Nevanlinna theory; system of partial differential-difference equations; existence
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1. Introduction

As is known to all, Nevanlinna theory is an important tool in studying the value distribution of meromorphic solutions on complex differential equations [18]. In recent, with the development of difference analogues of Nevanlinna theory in $\mathbb{C}$, many scholars paid consideration attention to considering the properties on complex difference equations, by using the difference analogue of the logarithmic derivative lemma given by Chiang and Feng [3], Halburd and Korhonen [8], respectively. In particular, Liu et al. [22–24] investigated the existence of entire solutions with finite order of the

Fermat type differential-difference equations

$$f'(z)^2 + f(z+c)^2 = 1, \tag{1.1}$$
$$f'(z)^2 + [f(z+c) - f(z)]^2 = 1. \tag{1.2}$$

They proved that the transcendental entire solutions with finite order of Eq (1.1) must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = (2k+1)\pi$, k is an integer, and the transcendental entire solutions with finite order of Eq (1.2) must satisfy $f(z) = 12\sin(2z + Bi)$, where $c = (2k+1)\pi$, k is an integer, and B is a constant.

The study of complex differential-difference equations in $\mathbb{C}$ can be traced back to Naftalevich's research [28, 29]. He used operator theory and iteration method to consider the meromorphic solutions on complex differential-difference equations. But recently, by using Nevanlinna theory, a number of results on complex differential-difference equations in $\mathbb{C}$ are rapidly obtained until now, readers can refer to [25, 31, 32].

Corresponding to Eq (1.1), Gao [5] in 2016 discussed the form of solutions for a class of system of differential-difference equation

$$\begin{cases} [f_1'(z)]^2 + f_2(z+c)^2 = 1, \\ [f_2'(z)]^2 + f_1(z+c)^2 = 1, \end{cases} \tag{1.3}$$

and obtained

Theorem A (see [5, Theorem 1.1]). *Suppose that* (f_1, f_2) *is a pair of finite order transcendental entire solutions for the system of differential-difference Eq* (1.3). *Then* (f_1, f_2) *satisfies*

$$(f_1, f_2) = (\sin(z - bi), \sin(z - b_1 i)) \text{ or } (f_1(z), f_2(z)) = (\sin(z + bi), \sin(z + b_1 i)),$$

where b, b_1 *are constants, and* $c = k\pi$, k *is a integer.*

Here these conclusions are stated in several complex variables as follows. In many previous articles [13,16,19,25,26,35] about Fermat-type partial differential equations with several complex variables, G. Khavinson [16] pointed out that any entire solutions of the partial differential equations $\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = 1$ in $\mathbb{C}^2$ are necessarily linear. This partial differential equations in real variable case occur in the study of characteristic surfaces and wave propagation theory, and it is the two dimensional eiconal equation, one of the main equations of geometric optics (see [4,6]). Later, Li [20,21] further discussed a series of partial differential equations with more general forms including $\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = e^g$, $\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = p$, etc., where g, p are polynomials in $\mathbb{C}^2$, and gave a number of important and interesting results about the existence and the forms of solutions for these partial differential equations.

In 2012, Korhonen [17, Theorem 3.1] gave a logarithmic difference lemma for meromorphic functions in several variables of hyper order strictly less that 2/3. In 2016, Cao and Korhonen [2] improved it to the case for meromorphic functions with hyper order < 1 in several variables. In 2018, Xu and Cao [39] investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential-difference equations by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables [2, 17], and obtained:

Theorem B (see [39, Theorem 1.1]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then the Fermat-type partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1$$

doesn't have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Theorem C (see [39, Theorem 1.2]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any transcendental entire solutions with finite order of the partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1} = 1$, and B is a constant on $\mathbb{C}$; in the special case whenever $c_1 = 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.

Inspired by the above theorems, the authors [37] in 2020 extended the results of Theorems A, B from the complex Fermat types partial differential difference equations to the Fermat types system of partial differential-difference equations and obtained:

Theorem D (see [37, Theorem 1.1]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$, and m_j, n_j $(j = 1, 2)$ be positive integers. If the following system of Fermat-type partial differential-difference equations*

$$\begin{cases} \left(\dfrac{\partial f_1(z_1, z_2)}{\partial z_1}\right)^{n_1} + f_2(z_1 + c_1, z_2 + c_2)^{m_1} = 1, \\ \left(\dfrac{\partial f_2(z_1, z_2)}{\partial z_1}\right)^{n_2} + f_1(z_1 + c_1, z_2 + c_2)^{m_2} = 1, \end{cases} \tag{1.4}$$

satisfies one of the conditions

(*i*) $m_1 m_2 > n_1 n_2$;
(*ii*) $m_j > \frac{n_j}{n_j - 1}$ for $n_j \geq 2$, $j = 1, 2$.

Then system (1.4) *does not have any pair of transcendental entire solution with finite order.*

Theorem E (see [37, Theorem 1.3]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any pair of transcendental entire solutions with finite order for the system of Fermat-type partial differential-difference equations*

$$\begin{cases} \left(\dfrac{\partial f_1(z_1, z_2)}{\partial z_1}\right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(\dfrac{\partial f_2(z_1, z_2)}{\partial z_1}\right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1 \end{cases}$$

have the following forms

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2}, \frac{A_{21}e^{L(z)+B_1} + A_{22}e^{-(L(z)+B_1)}}{2}\right),$$

where $L(z) = a_1 z_1 + a_2 z_2$, B_1 is a constant in $\mathbb{C}$, and a_1, c, A_{21}, A_{22} satisfy one of the following cases *(i)* $A_{21} = -i$, $A_{22} = i$, *and* $a_1 = i$, $L(c) = (2k + \frac{1}{2})\pi i$, *or* $a_1 = -i$, $L(c) = (2k - \frac{1}{2})\pi i$;

(ii) $A_{21} = i$, $A_{22} = -i$, *and* $a_1 = i$, $L(c) = (2k - \frac{1}{2})\pi i$, *or* $a_1 = -i$, $L(c) = (2k + \frac{1}{2})\pi i$;

(iii) $A_{21} = 1$, $A_{22} = 1$, *and* $a_1 = i$, $L(c) = 2k\pi i$, *or* $a_1 = -i$, $L(c) = (2k + 1)\pi i$;

(iv) $A_{21} = -1$, $A_{22} = -1$, *and* $a_1 = i$, $L(c) = (2k + 1)\pi i$, *or* $a_1 = -i$, $L(c) = 2k\pi i$.

From Theorems D and E, we can see that there only contains the partial differentiation of the first variable z_1 of the unknown functions f_1, f_2 in those systems of partial differential difference equations. Naturally, a question arises: *What will happen when the system of the partial differential-difference equations include both the difference* $f_j(z+c)$ *and* $\frac{\partial f_j(z_1,z_2)}{\partial z_1}, \frac{\partial f_j(z_1,z_2)}{\partial z_2}, (j = 1, 2)$? In the past two decades, in spite of a number of important and meaningful results about the complex difference equation of single variable and the complex Fermat difference equation were obtained (can be found in [9–11,24,31,33]), but as far as we know, there are few results concerning the complex differential and complex difference equation in several complex variables. Further more, it appears that the study of systems of this Fermat type equations in several complex variables has been less addressed in the literature before.

The main purpose of this paper is concerned with the description of the transcendental entire solutions for some Fermat-type equations systems which include both difference operator and two kinds of partial differentials by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables [2, 17]. We obtained some results about the existence and the forms of the transcendental entire solutions of some Fermat type systems of partial differential difference equations in $\mathbb{C}^2$, which improve the previous results given by Xu and Cao, Xu, Liu and Li, Gao [5, 37–40].

2. Results and examples

Here and below, let $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Now, our main results of this paper are stated below.

Theorem 2.1. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, *and* m_j, n_j $(j = 1, 2)$ *be positive integers. If the following system of Fermat-type partial differential-difference equations*

$$\begin{cases} \left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1} + \dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^{m_1} + f_2(z_1 + c_1, z_2 + c_2)^{n_1} = 1, \\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1} + \dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^{m_2} + f_1(z_1 + c_1, z_2 + c_2)^{n_2} = 1, \end{cases} \tag{2.1}$$

satisfies one of the conditions

(i) $n_1 n_2 > m_1 m_2$;
(ii) $m_j > \frac{n_j}{n_j - 1}$ *for* $n_j \geq 2$, $j = 1, 2$.

Then system (2.1) *does not admit any pair of transcendental entire solution with finite order.*

Remark 2.1. *Here,* (f, g) *is called as a pair of finite order transcendental entire solutions for system*

$$\begin{cases} f^{m_1} + g^{n_1} = 1, \\ f^{m_2} + g^{n_2} = 1, \end{cases}$$

if f, g *are transcendental entire functions and* $\rho(f, g) = \max\{\rho(f), \rho(g)\} < \infty$.

The following examples show system (2.1) admits a transcendental entire solution of finite order when $m_1 = m_2 = 2$ and $n_1 = n_2 = 1$.

Example 2.1. *Let*

$$\begin{cases} f_1(z_1,z_2) = \dfrac{5-z_1^2}{4} + \dfrac{1}{2}(z_2-z_1)(z_1-1) - \dfrac{1}{4}[(z_2-z_1)-1]^2 \\ \qquad + e^{\pi i(z_2-z_1)}(2z_1-z_2) - e^{2\pi i(z_2-z_1)}, \\ f_2(z_1,z_2) = \dfrac{5-z_1^2}{4} + \dfrac{1}{2}(z_2-z_1)(z_1-1) - \dfrac{1}{4}[(z_2-z_1)-1]^2 \\ \qquad - e^{\pi i(z_2-z_1)}(2z_1-z_2) - e^{2\pi i(z_2-z_1)}. \end{cases}$$

Then $\rho(f_1, f_2) = 1$ *and* (f_1, f_2) *satisfies system* (2.1) *with* $(c_1, c_2) = (1, 2)$*,* $m_1 = m_2 = 2$ *and* $n_1 = n_2 = 1$.

Theorem 2.2. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$*. If* (f_1, f_2) *is a pair of transcendental entire solutions with finite order for the system of Fermat-type difference equations*

$$\begin{cases} \left(\dfrac{\partial f_1(z_1,z_2)}{\partial z_1} + \dfrac{\partial f_1(z_1,z_2)}{\partial z_2}\right)^2 + f_2(z_1+c_1, z_2+c_2)^2 = 1, \\ \left(\dfrac{\partial f_2(z_1,z_2)}{\partial z_1} + \dfrac{\partial f_2(z_1,z_2)}{\partial z_2}\right)^2 + f_1(z_1+c_1, z_2+c_2)^2 = 1. \end{cases} \tag{2.2}$$

Then (f_1, f_2) *is of the following forms*

$$(f_1, f_2) = \left(\frac{e^{\phi(z)+B_0} - e^{-\phi(z)-B_0}}{2(a_1+a_2)}, \frac{A_{21}e^{\phi(z)+B_0} + A_{22}e^{-\phi(z)-B_0}}{2}\right),$$

where $\phi(z) = L(z) + H(c_2z_1 - c_1z_2)$*,* $L(z) = a_1z_1 + a_2z_2$*,* a_1, a_2, B_0 *is a constant in* $\mathbb{C}$*,* H *is a polynomial in* $\mathbb{C}$*, and*

$$(c_1 - c_2)H' \equiv 0, \quad (a_1+a_2)^2 = -1, \quad e^{2L(c)} = \pm 1,$$

and c, A_{21}, A_{22} *satisfy one of the following cases*

(*i*) $L(c) = 2k\pi i$*, here and below,* $k \in \mathbb{Z}$*,* $A_{21} = -i$ *and* $A_{22} = i$*;*

(*ii*) $L(c) = (2k+1)\pi i$*,* $A_{21} = i$ *and* $A_{22} = -i$*;*

(*iii*) $L(c) = (2k + \frac{1}{2})\pi i$*,* $A_{21} = -1$ *and* $A_{22} = -1$*;*

(*iv*) $L(c) = (2k - \frac{1}{2})\pi i$*,* $A_{21} = 1$ *and* $A_{22} = 1$.

Here, we only list the following examples to explain the existence of transcendental entire solutions with finite order for system (2.2).

Example 2.2. *Let* $a = (a_1, a_2) = (2i, -i)$*,* $H = 4\pi^2(z_1 - z_2)^2$*,* $A_{21} = -i, A_{22} = i$ *and* $B_0 = 0$*. That is*

$$(f_1, f_2) = (f(z_1,z_2), f(z_1,z_2)),$$

where

$$f(z_1,z_2) = \frac{e^{i(2z_1-z_2)+4\pi^2(z_1-z_2)^2} - e^{-i(2z_1-z_2)-4\pi^2(z_1-z_2)^2}}{2i}.$$

Thus, $\rho(f_1, f_2) = 2$ *and* (f_1, f_2) *satisfies system* (2.2) *with* $(c_1, c_2) = (2\pi, 2\pi)$.

Example 2.3. *Let* $a = (a_1, a_2) = (2i, -i)$, $H = \pi^n(z_1 - z_2)^n, n \in \mathbb{Z}_+$, $A_{21} = i, A_{22} = -i$ *and* $B_0 = 0$. *That is*

$$(f_1, f_2) = (f(z_1, z_2), -f(z_1, z_2)),$$

where

$$f(z_1, z_2) = \frac{e^{i(2z_1-z_2)+\pi^n(z_1-z_2)^n} - e^{-i(2z_1-z_2)-\pi^n(z_1-z_2)^n}}{2i}.$$

Thus, $\rho(f_1, f_2) = n$ *and* (f_1, f_2) *satisfies system* (2.2) *with* $(c_1, c_2) = (\pi, \pi)$.

Remark 2.2. *From the conclusions of Theorems C and E, there only exists finite order transcendental entire solutions with growth order* $\rho(f_1, f_2) = 1(\rho(f) = 1)$. *However, in view of Theorem 2.2, we can see that there exists transcendental entire solution of system* (2.2) *with growth order* $\rho(f_1, f_2) > 1$, *for example,* $\rho(f_1, f_2) = 2$ *in Example 2.2 and* $\rho(f_1, f_2) = n, n \in \mathbb{Z}_+$ *in Example 2.3, these properties are quite different from the previous results. Hence, our results are some improvements of the previous theorems given by Xu, Liu and Li [37], Xu and Cao [39], Liu Cao and Cao [23].*

To state our last result, throughout this paper, let $s_1 = z_2 - z_1$, $G_1(s_1)$, $G_2(s_1)$ be the finite order entire period functions in s_1 with period $2s_0 = 2(c_2 - c_1)$, where $c_1 \neq c_2$, $G_1(s_1)$, $G_2(s_1)$ can not be the same at every time occurrence.

Theorem 2.3. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1 \neq c_2$. *Then any pair of transcendental entire solutions with finite order for the system of Fermat-type partial differential-difference equations*

$$\begin{cases} \left(\dfrac{\partial f_1(z_1, z_2)}{\partial z_1} + \dfrac{\partial f_1(z_1, z_2)}{\partial z_2}\right)^2 + [f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)]^2 = 1, \\ \left(\dfrac{\partial f_2(z_1, z_2)}{\partial z_1} + \dfrac{\partial f_2(z_1, z_2)}{\partial z_2}\right)^2 + [f_1(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)]^2 = 1, \end{cases} \tag{2.3}$$

are one of the following forms

(i)

$$(f_1, f_2) = (G_1(s_1) + A_0 s_1, G_2(s_1) + A_0 s_1),$$

where $A_0 = \frac{\xi_2+\xi_1}{2(c_2-c_1)}$, $G_2(s_1 + s_0) = G_1(s_1) + \frac{\xi_2-\xi_1}{2}$, *and* $\xi_1^2 = \xi_2^2 = 1$;

(ii)

$$(f_1, f_2) = (\xi_3 z_1 + A_0 s_1 + G_1(s_1), \xi_3 z_1 + A_0 s_1 + G_2(s_1)),$$

where $A_0 = \frac{\xi_1+\xi_2-2c_1\xi_3}{2(c_2-c_1)}$, $G_2(s_1 + s_0) = G_1(s_1) + \frac{\xi_1-\xi_2}{2}$, $\xi_1^2 + \xi_3^2 = 1$, *and* $\xi_1 = \pm\xi_2$;

(iii)

$$(f_1, f_2) = \left(\frac{e^{L(z)+B_1} - e^{-(L(z)+B_1)}}{-4i} + G_1(s_1), \frac{e^{L(z)+B_2} - e^{-(L(z)+B_2)}}{-4i} + G_2(s_1)\right)$$

where

$$a_1 + a_2 = -2i,\ e^{2L(c)} = 1,\ e^{B_1-B_2} = -e^{L(c)},\ G_2(s_1 + s_0) = G_1(s_1);$$

(*iv*)

$$(f_1, f_2) = \left(z_1 \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} + G_3(s_1), D_0 z_1 \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} + G_4(s_1)\right),$$

where

$$a_1 + a_2 = 0,\ e^{2L(c)} = 1,$$

and $G_3(s_1), G_4(s_1), D_0$ *satisfying one of the following cases:*

(iv_1) $L(c) = 2k\pi i$, $D_0 = 1$,

$$G_3(s_1) = G_1(s_1) - \frac{c_1 + i}{2s_0} s_1 e^{a_2 s_1 + B_1} - \frac{c_1 - i}{2s_0} s_1 e^{-a_2 s_1 - B_1},$$

and

$$G_4(s_1) = G_2(s_1) - \frac{c_1 + i}{2s_0} s_1 e^{a_2 s_1 + B_1} - \frac{c_1 - i}{2s_0} s_1 e^{-a_2 s_1 - B_1};$$

(iv_2) $L(c) = (2k + 1)\pi i$, $D_0 = -1$,

$$G_3(s_1) = G_1(s_1) - \frac{c_1 + i}{2s_0} s_1 e^{a_2 s_1 + B_1} - \frac{c_1 - i}{2s_0} s_1 e^{-a_2 s_1 - B_1},$$

and

$$G_4(s_1) = G_2(s_1) + \frac{c_1 + i}{2s_0} s_1 e^{a_2 s_1 + B_1} + \frac{c_1 - i}{2s_0} s_1 e^{-a_2 s_1 - B_1};$$

where $G_2(s + s_0) = G_1(s)$.

Some examples are listed to exhibit the existence of solutions for system (2.3).

Example 2.4. *Let* $G_1(s_1) = -G_2(s_1) = e^{-\pi i s_1}$, *and* $\xi_1 = -1$, $\xi_2 = 1$. *Then it follows that* $A_0 = 0$ *and*

$$(f_1(z_1, z_2), f_2(z_1, z_2)) = \left(e^{-\pi i(z_2 - z_1)}, -e^{-\pi i(z_2 - z_1)} - 1\right).$$

Thus, $\rho(f_1, f_2) = 1$ *and* (f_1, f_2) *satisfies the system* (2.3) *with* $(c_1, c_2) = (1, 2)$.

Example 2.5. *Let* $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 1$ *and* $G_1(s_1) = G_2(s_1) = e^{2\pi i s_1}$. *Then it follows that* $A_0 = -1$ *and*

$$(f_1, f_2) = \left(2z_1 - z_2 + e^{2\pi i(z_2 - z_1)}, 2z_1 - z_2 + e^{2\pi i(z_2 - z_1)}\right).$$

Thus, $\rho(f_1, f_2) = 1$ *and* (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (1, 2)$.

Example 2.6. *Let* $L(z) = -i(z_1 + z_2)$ *and* $G_1(s_1) = -G_2(s_1) = e^{\frac{i}{4} s_1}$. *That is*

$$(f_1, f_2) = \left(\frac{e^{-L(z)} - e^{L(z)}}{4i} + e^{\frac{i}{4}(z_2 - z_1)}, -\frac{e^{-L(z)} - e^{L(z)}}{4i} - e^{\frac{i}{4}(z_2 - z_1)}\right).$$

Thus, $\rho(f) = 1$ *and* (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (-\pi, 3\pi)$.

Example 2.7. *Let* $a_1 = i$, $a_2 = -i$, $L(z) = i(z_1 - z_2)$, $G_1(s_1) = G_2(s_1) = e^{\frac{i}{4} s_1}$ *and* $B_1 = 0$. *Then, it follows that*

$$f_1(z_1, z_2) = \frac{z_1}{2}(e^{L(z)} + e^{-L(z)}) - \frac{-2\pi + i}{8\pi}(z_2 - z_1)e^{L(z)} - \frac{-2\pi - i}{8\pi}(z_2 - z_1)e^{-L(z)} + e^{\frac{i}{4}(z_2 - z_1)},$$

$$f_2(z_1, z_2) = \frac{z_1}{2}(e^{L(z)} + e^{-L(z)}) - \frac{-2\pi + i}{8\pi}(z_2 - z_1)e^{L(z)} - \frac{-2\pi - i}{8\pi}(z_2 - z_1)e^{-L(z)} - e^{\frac{i}{4}(z_2 - z_1)},$$

Thus, $\rho(f_1, f_2) = 1$ *and* (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (-2\pi, 2\pi)$.

Example 2.8. *Let* $a_1 = 1$, $a_2 = -1$, $L(z) = z_1 - z_2$, $G_1(s_1) = -G_2(s_1) = e^{s_1}$ *and* $B_1 = 0$. *Then, it follows that*

$$f_1(z_1, z_2) = \frac{z_1}{2}(e^{L(z)} + e^{-L(z)}) + \frac{\pi - 2}{4\pi}(z_2 - z_1)e^{L(z)} + \frac{\pi + 2}{4\pi}(z_2 - z_1)e^{-L(z)} + e^{z_2 - z_1},$$

$$f_2(z_1, z_2) = -\frac{z_1}{2}(e^{L(z)} + e^{-L(z)}) - \frac{\pi - 2}{4\pi}(z_2 - z_1)e^{L(z)} - \frac{\pi + 2}{4\pi}(z_2 - z_1)e^{-L(z)} - e^{z_2 - z_1}.$$

Thus, (f_1, f_2) *satisfies system* (2.3) *with* $(c_1, c_2) = (-\frac{1}{2}\pi i, \frac{1}{2}\pi i)$.

3. Conclusions and discussion

From Theorems 2.1–2.3, we can see that our results are some extension of the previous results given by Xu and Cao [39] from the equations to the systems, and some supplements of the results given by Xu, Liu and Li [37]. More importantly, Examples 2.2 and 2.3 show that system (2.2) can admit the transcendental entire solutions of any positive integer order. However, the conclusions of Theorem C and Theorem E showed that the order of the transcendental entire solutions of the equations must be equal to 1. In fact, this is a very significant difference. Finally, one can find that we only focus on the finite-order transcendental entire solutions of systems (1.4)–(2.2) in this article; thus, the following question can be raised naturally:

Question 3.1. *How should the meromorphic solutions of systems* (2.2) *and* (2.3) *be characterized?*

4. Proofs of Theorems 2.1–2.3

Similar to the argument as in the proof of Theorems 1.1 and 1.3 in Ref. [37], one can obtain the conclusions of Theorems 2.1 and 2.2 easily. Thus, we only give the proof of Theorem 2.3 as follow. However, the following lemmas play the key roles in proving Theorem 2.3.

Lemma 4.1. *([34, 36]) For an entire function* F *on* $\mathbb{C}^n$, $F(0) \neq 0$ *and put* $\rho(n_F) = \rho < \infty$. *Then there exist a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z)e^{g_F(z)}$. *For the special case* $n = 1$, f_F *is the canonical product of Weierstrass.*

Remark 4.1. *Here, denote* $\rho(n_F)$ *to be the order of the counting function of zeros of* F.

Lemma 4.2. *([30]) If* g *and* h *are entire functions on the complex plane* $\mathbb{C}$ *and* $g(h)$ *is an entire function of finite order, then there are only two possible cases: either*

(a) the internal function h *is a polynomial and the external function* g *is of finite order; or else*

(b) the internal function h *is not a polynomial but a function of finite order, and the external function* g *is of zero order.*

Lemma 4.3. *([15] or [14, Lemma 3.1]) Let* $f_j(\not\equiv 0)$, $j = 1, 2, 3$, *be meromorphic functions on* $\mathbb{C}^m$ *such that* f_1 *is not constant. If* $f_1 + f_2 + f_3 = 1$, *and if*

$$\sum_{j=1}^{3}\left\{N_2(r, \frac{1}{f_j}) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r *outside possibly a set with finite logarithmic measure, where* $\lambda < 1$ *is a positive number, then either* $f_2 = 1$ *or* $f_3 = 1$.

Remark 4.2. *Here, $N_2(r, \frac{1}{f})$ is the counting function of the zeros of f in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.*

Lemma 4.4. *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$, $c_1 \neq 0, c_2 \neq 0$ and $c_2 \neq c_1$. Let $p(z)$, $q(z)$ be two polynomial solutions of the equation*

$$\frac{\partial h}{\partial z_1} + \frac{\partial h}{\partial z_2} = \gamma_0, \tag{4.1}$$

and $q(z+c)-p(z) = \zeta_1, p(z+c)-q(z) = \zeta_2$, where $\zeta_1, \zeta_2, \gamma_0 \in \mathbb{C}$, then $p(z) = L(z)+B_1$, $q(z) = L(z)+B_2$, where $L(z) = a_1z_2 + a_2z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$.

Proof. The characteristic equations of the Eq (4.1) are

$$\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = 1, \quad \frac{dh}{dt} = \gamma_0,$$

Using the initial conditions: $z_1 = 0, z_2 = s_1$, and $h = h(0, s_1) := h_0(s_1)$ with a parameter. Then $z_1 = t$, $z_2 = t + s_1$, and $h = \int_0^t \gamma_0 dt + h_0(s_1) = \gamma_0 t + h_0(s)$, where $s_1 = z_2 - z_1$, $h_0(s_1)$ is a function in s_1. Since $p(z), q(z)$ are the solutions of (4.1), then it yields that

$$p(z_1, z_2) = p(t, s_1) = \gamma_0 t + h_1(s_1), \quad q(z_1, z_2) = p(t, s_1) = \gamma_0 t + h_2(s_1), \tag{4.2}$$

where $h_1(s_1), h_2(s_1)$ are two polynomials in s_1. Substituting (4.2) into $q(z+c)-p(z) = \zeta_1, q(z)-p(z+c) = \zeta_2$, it leads to

$$h_2(s_1 + s_0) - h_1(s_1) = \zeta_1 - \gamma_0 c_1, \quad h_1(s_1 + s_0) - h_2(s_1) = \zeta_2 - \gamma_0 c_1. \tag{4.3}$$

Hence, we have

$$h_1(s_1 + 2s_0) = h_1(s_1) + \varepsilon_0, \quad h_2(s_1 + 2s_0) = h_2(s_1) + \varepsilon_0, \tag{4.4}$$

where $\varepsilon_0 = \zeta_1 + \zeta_2 - 2\gamma_0 c_1$. The fact that $h_1(s_1), h_2(s_1)$ are polynomials leads to

$$h_1(s_1) = \gamma_1 s_1 + B_1, \quad h_2(s_1) = \gamma_1 s_1 + B_2, \tag{4.5}$$

where $\gamma_1 = \frac{\varepsilon_0}{2s_0}$, $B_1, B_2 \in \mathbb{C}$. By combining with (4.2) and (4.5), it follows that

$$\begin{aligned} p(z_1, z_2) &= \gamma_0 t + \gamma_1 s_1 + B_1 = \gamma_0 z_1 + \gamma_1(z_1 - z_2) + B_1 = a_1 z_1 + a_2 z_2 + B_1, \\ q(z_1, z_2) &= \gamma_0 t + \gamma_1 s_1 + B_2 = \gamma_0 z_1 + \gamma_1(z_1 - z_2) + B_2 = a_1 z_1 + a_2 z_2 + B_2, \end{aligned}$$

where $a_1 = \gamma_0 + \gamma_1$, $a_2 = -\gamma_1$.

Therefore, Lemma 4.4 is proved. □

The Proof of Theorem 2.3: Let (f_1, f_2) be a pair of transcendental entire functions with finite order satisfying system (2.3). In view of [7, 27], we know that the entire solutions of the Fermat type functional equation $f^2 + g^2 = 1$ are $f = \cos a(z), g = \sin a(z)$, where $a(z)$ is an entire function. Hence, we only consider the following cases.

(i) Suppose that $\frac{\partial f_1(z_1,z_2)}{\partial z_1} + \frac{\partial f_1(z_1,z_2)}{\partial z_2} = 0$. Then it follows from (2.3) that

$$f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2) \equiv \xi_1, \quad \xi_1^2 = 1, \tag{4.6}$$

and
$$\frac{\partial f_2(z+c)}{\partial z_1}+\frac{\partial f_2(z+c)}{\partial z_2}=\frac{\partial f_1(z_1,z_2)}{\partial z_1}+\frac{\partial f_1(z_1,z_2)}{\partial z_2}=0. \tag{4.7}$$

In view of (4.6), (4.7) and (2.3), it yields that

$$\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial z_2}=0,\quad f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)\equiv \xi_2,\quad \xi_2^2=1. \tag{4.8}$$

By solving the equations $\frac{\partial f_j}{\partial z_1}+\frac{\partial f_j}{\partial z_2}=0$ $(j=1,2)$, we have

$$f_1(z_1,z_2)=g_1(s_1):=g_1(z_2-z_1),\quad f_2(z_1,z_2)=g_2(s_1):=g_2(z_2-z_1), \tag{4.9}$$

where g_1,g_2 are transcendental entire functions of finite order. Substituting (4.9) into (4.6), (4.8), it yields that

$$\begin{aligned} f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)=&g_2(s_1+s_0)-g_1(s_1)=\xi_1,\\ f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)=&g_1(s_1+s_0)-g_2(s_1)=\xi_2. \end{aligned}$$

which implies

$$g_1(s_1+2s_0)=g_1(s_1)+\xi_1+\xi_2,\quad g_2(s_1+2s_0)=g_2(s_1)+\xi_1+\xi_2, \tag{4.10}$$
$$g_2(s_1+s_0)=g_1(s_1)+\xi_1,\ \ g_1(s_1+s_0)=g_2(s_1)+\xi_2. \tag{4.11}$$

Thus, in view of (4.10) and (4.11), we conclude that

$$g_1(s_1)=G_1(s_1)+A_0s_1,\quad g_2(s_1)=G_2(s_1)+B_0s_1,$$

where $A_0=B_0=\frac{\xi_1+\xi_2}{2(c_2-c_1)}$, $G_1(s_1),G_2(s_1)$ are the finite order entire period functions with period $2s_0$, and

$$G_2(s_1+s_0)=G_1(s_1)+\frac{\xi_1-\xi_2}{2},\ \ G_1(s_1+s_0)=G_2(s_1)+\frac{\xi_2-\xi_1}{2}.$$

Thus, this proves the conclusions of Theorem 2.3 (i).

(ii) Suppose that
$$\frac{\partial f_1(z_1,z_2)}{\partial z_1}+\frac{\partial f_1(z_1,z_2)}{\partial z_2}=\xi_3,\quad \xi_3\neq 0. \tag{4.12}$$

In view of (2.3), we conclude that

$$\frac{\partial f_2(z_1,z_2)}{\partial z_1}+\frac{\partial f_2(z_1,z_2)}{\partial z_2}=\xi_3, \tag{4.13}$$
$$f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)\equiv\xi_1,\quad \xi_1^2+\xi_3^2=1, \tag{4.14}$$
$$f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)\equiv\xi_2,\ \ \xi_2^2+\xi_3^2=1. \tag{4.15}$$

The characteristic equations of Eq (4.12) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=1,\quad \frac{df_1}{dt}=\xi_3,$$

Using the initial conditions: $z_1 = 0, z_2 = s_1$, and $f_1 = f_1(0, s_1) := g_1(s_1)$ with a parameter. Thus, we have $z_1 = t$, $z_2 = t + s_1$ and $f_1(t, s_1) = \int_0^t \xi_3 dt + g_1(s_1) = \xi_3 t + g_1(s_1)$, where $s_1 = z_2 - z_1$, $g_1(s_1)$ is a transcendental entire function of finite order. Hence, it yields that

$$f_1(z_1, z_2) = \xi_3 z_1 + g_1(z_2 - z_1). \tag{4.16}$$

Similar to the same argument for Eq (4.13), we have

$$f_2(z_1, z_2) = \xi_3 z_1 + g_2(z_2 - z_1), \tag{4.17}$$

where $g_2(s_1)$ is a transcendental entire function of finite order.

Substituting (4.16), (4.17) into (4.14), (4.15), we conclude that

$$g_2(s_1 + s_0) - g_1(s_1) = -c_1\xi_3 + \xi_1, \tag{4.18}$$

$$g_1(s_1 + s_0) - g_2(s) = -c_1\xi_3 + \xi_2. \tag{4.19}$$

which lead to

$$\begin{aligned} g_2(s_1 + s_0) &= g_1(s_1) + \xi_1 - c_1\xi_3, \\ g_1(s_1 + s_0) &= g_2(s_1) + \xi_2 - c_1\xi_3, \\ g_1(s_1 + 2s_0) &= g_1(s_1) + \xi_1 + \xi_2 - 2c_1\xi_3, \\ g_2(s_1 + 2s_0) &= g_2(s_1) + \xi_1 + \xi_2 - 2c_1\xi_3. \end{aligned}$$

Since $g_1(s_1), g_2(s_1)$ are the transcendental entire functions of finite order. then we have

$$g_1(s_1) = G_1(s_1) + A_0 s_1, \quad g_2(s_1) = G_2(s_1) + B_0 s_1,$$

where $A_0 = B_0 = \frac{\xi_1+\xi_2-2c_1\xi_3}{2(c_2-c_1)}$, $G_1(s_1), G_2(s_1)$ are the finite order entire period functions with period $2s_0$, and

$$G_2(s_1 + s_0) = G_1(s_1) + \frac{\xi_1 - \xi_2}{2}.$$

Thus, this proves the conclusions of Theorem 2.3 (ii).

(iii) If $\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2}$ is transcendental, then $f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)$ is transcendental. Here, we can deduce that $f_1(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)$ and $\frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2}$ are transcendental.

Suppose that $f_1(z_1+c_1, z_2+c_2)-f_2(z_1, z_2)$ is not transcendental. Since $\frac{\partial f_1}{\partial z_1}+\frac{\partial f_1}{\partial z_2}$ is transcendental, then $\frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2}$ is transcendental. In view of (2.3), it yields that $f_1(z_1+c_1, z_2+c_2)-f_2(z_1, z_2)$ is transcendental, a contradiction.

Suppose that $\frac{\partial f_2}{\partial z_1}+\frac{\partial f_2}{\partial z_2}$ is not transcendental. In view of (2.3), it follows that $f_1(z_1+c_1, z_2+c_2)-f_2(z_1, z_2)$ is not transcendental. Thus, it yields that $\frac{\partial f_1(z+c)}{\partial z_1} + \frac{\partial f_1(z+c)}{\partial z_2}$ is not transcendental. This is a contradiction with $\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2}$ is transcendental.

Hence, if $\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2}$ is transcendental, then $f_2(z_1+c_1, z_2+c_2) - f_1(z_1, z_2)$, $f_1(z_1+c_1, z_2+c_2) - f_2(z_1, z_2)$ and $\frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2}$ are transcendental. Thus, system (2.3) can be rewritten as

$$\begin{cases} \left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2} + i[f_2(z+c) - f_1(z_1, z_2)]\right)\left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2} - i[f_2(z+c) - f_1(z_1, z_2)]\right) = 1, \\ \left(\frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + i[f_1(z+c) - f_2(z_1, z_2)]\right)\left(\frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2} - i[f_1(z+c) - f_2(z_1, z_2)]\right) = 1. \end{cases} \tag{4.20}$$

Since f_1, f_2 are transcendental entire functions with finite order, then by Lemmas 4.1 and 4.2, there exist two nonconstant polynomials $p(z), q(z)$ such that

$$\begin{cases} \frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2} + i[f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)] = e^p, \\ \frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2} - i[f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2)] = e^{-p}, \\ \frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + i[f_1(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)] = e^q, \\ \frac{\partial f_2}{\partial z_1} + \frac{\partial f_2}{\partial z_2} - i[f_1(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2)] = e^{-q}. \end{cases} \tag{4.21}$$

Thus, it follows from (4.21) that

$$\begin{cases} \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \frac{\partial f_1(z_1, z_2)}{\partial z_2} = \frac{e^{p(z_1,z_2)} + e^{-p(z_1,z_2)}}{2}, \\ f_2(z_1 + c_1, z_2 + c_2) - f_1(z_1, z_2) = \frac{e^{p(z_1,z_2)} - e^{-p(z_1,z_2)}}{2i}, \\ \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \frac{\partial f_2(z_1, z_2)}{\partial z_2} = \frac{e^{q(z_1,z_2)} + e^{-q(z_1,z_2)}}{2}, \\ f_1(z_1 + c_1, z_2 + c_2) - f_2(z_1, z_2) = \frac{e^{q(z_1,z_2)} - e^{-q(z_1,z_2)}}{2i}, \end{cases} \tag{4.22}$$

which implies

$$-i\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{p(z)+q(z+c)} - i\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{q(z+c)-p(z)} - e^{2q(z+c)} \equiv 1, \tag{4.23}$$

$$-i\left(\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} + i\right)e^{q(z)+p(z+c)} - i\left(\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} + i\right)e^{p(z+c)-q(z)} - e^{2p(z+c)} \equiv 1. \tag{4.24}$$

Obviously, $\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} \neq -i$. Otherwise, it follows that $-e^{2q(z+c)} \equiv 1$, this is impossible because $q(z)$ is a nonconstant polynomial. Similarly, $\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} \neq -i$. Thus, by Lemma 4.3, and in view of (4.23), (4.24), we conclude that

$$-i\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{q(z+c)-p(z)} \equiv 1, \ or \ -i\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{p(z)+q(z+c)} \equiv 1,$$

and

$$-i\left(\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} + i\right)e^{p(z+c)-q(z)} \equiv 1, \ or \ -i\left(\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} + i\right)e^{q(z)+p(z+c)} \equiv 1.$$

Next, we consider the following four cases.

Case 1.

$$\begin{cases} -i\left(\frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} + i\right)e^{q(z+c)-p(z)} \equiv 1, \\ -i\left(\frac{\partial q}{\partial z_1} + \frac{\partial q}{\partial z_2} + i\right)e^{p(z+c)-q(z)} \equiv 1. \end{cases} \tag{4.25}$$

Since $p(z), q(z)$ are polynomials, then from (4.25), we know that $\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}$ and $\frac{\partial q}{\partial z_1}+\frac{\partial q}{\partial z_2}$ are constants in $\mathbb{C}$. Otherwise, we obtain a contradiction from the fact that the right of the above equations is not transcendental, but the left is transcendental. In addition, we get that $q(z+c)-p(z)\equiv C_1$ and $p(z+c)-q(z)\equiv C_2$, that is, $p(z+2c)-p(z)\equiv C_1+C_2$ and $q(z+2c)-q(z)\equiv C_1+C_2$. In view of $c_1\neq c_2$, then by Lemma 4.4, it means that $p(z)=L(z)+B_1, q(z)=L(z)+B_2$, where L is a linear function as the form $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants.

By combining with (4.23)–(4.25), we have

$$\begin{cases} -i\,(a_1+a_2+i)\,e^{L(c)+B_2-B_1}\equiv 1, \\ -i\,(a_1+a_2+i)\,e^{L(c)+B_1-B_2}\equiv 1, \\ -i\,(a_1+a_2+i)\,e^{-L(c)+B_1-B_2}\equiv 1, \\ -i\,(a_1+a_2+i)\,e^{-L(c)-B_1+B_2}\equiv 1. \end{cases} \tag{4.26}$$

This means

$$(a_1+a_2+i)^2=-1,\quad e^{2L(c)}=1,\quad e^{B_1-B_2}=-i\,(a_1+a_2+i)\,e^{L(c)}. \tag{4.27}$$

Thus, it follows that $a_1+a_2=-2i$ or $a_1+a_2=0$.

If $a_1+a_2=-2i$. Thus, $e^{2L(c)}=1, e^{B_1-B_2}=-e^{L(c)}$. The characteristic equations of the first equation in (4.22) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=1,\quad \frac{df_1}{dt}=\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2},$$

Using the initial conditions: $z_1=0, z_2=s_1$, and $f_1=f_1(0,s_1):=g_0(s_1)$ with a parameter. Thus, it follows that $z_1=t$, $z_2=t+s_1$ and

$$\begin{aligned} f_1(t,s_1)&=\int_0^t\frac{e^{(a_1+a_2)t+a_2s_1+B_1}+e^{-[(a_1+a_2)t+a_2s_1+B_1]}}{2}dt+g_0(s_1)\\ &=\frac{e^{a_2s_1+B_1}}{2}\int_0^t e^{(a_1+a_2)t}dt+\frac{e^{-(a_2s_1+B_1)}}{2}\int_0^t e^{-(a_1+a_2)t}dt+g_0(s_1)\\ &=\frac{e^{a_2s_1+B_1}}{2(a_1+a_2)}e^{(a_1+a_2)t}-\frac{e^{-(a_2s_1+B_1)}}{2(a_1+a_2)}e^{-(a_1+a_2)t}+g_1(s_1), \end{aligned}$$

where $g_1(s_1)$ is a finite order entire function, and

$$g_1(s_1)=g_0(s_1)+\frac{e^{a_2s_1+B_1}}{2(a_1+a_2)}-\frac{e^{-(a_2s_1+B_1)}}{2(a_1+a_2)}.$$

In view of $z_1=t$, $z_2=t+s_1$, we have

$$f_1(z_1,z_2)=\frac{e^{L(z)+B_1}-e^{-L(z)-B_1}}{-4i}+g_1(s_1). \tag{4.28}$$

Similar to the same argument for the third equation in (4.22), we have

$$f_2(z_1,z_2)=\frac{e^{L(z)+B_2}-e^{-L(z)-B_2}}{-4i}+g_2(s_1), \tag{4.29}$$

where $g_2(s_1)$ is a finite order entire function.

Substituting (4.28), (4.29) into (4.22), and applying (4.27), it yields that

$$g_2(s_1+s_0)-g_1(s_1)=0,\quad g_1(s_1+s_0)-g_2(s_1)=0. \tag{4.30}$$

Thus, from (4.30), we can deduce that

$$g_1(s_1)=G_1(s_1),\quad g_2(s_1)=G_2(s_1),$$

where $G_1(s_1), G_2(s_1)$ are the finite order entire period functions with period $2s_0$, and

$$G_2(s_1+s_0)=G_1(s_1).$$

If $a_1+a_2=0$, then it follows that $e^{2L(c)}=1, e^{B_1-B_2}=e^{L(c)}$. In view of $z_1=t$, $z_2=t+s_1$, it leads to $L(z)=a_1z_1+a_2z_2=a_2s_1$. The characteristic equations of the first equation in (4.22) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=1,\quad \frac{df_1}{dt}=\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2},$$

Using the initial conditions: $z_1=0, z_2=s_1$, and $f_1=f_1(0,s):=G_3(s)$ with a parameter. This leads to

$$\begin{aligned} f_1(t,s_1) &= \int_0^t \frac{e^{a_2s_1+B_1}+e^{-(a_2s_1+B_1)}}{2}dt+G_3(s_1)\\ &= t\left(\frac{e^{a_2s_1+B_1}}{2}+\frac{e^{-(a_2s_1+B_1)}}{2}\right)+G_3(s_1)\\ &= t\frac{e^{a_2s_1+B_1}+e^{-(a_2s_1+B_1)}}{2}+G_3(s_1), \end{aligned}$$

that is

$$f_1(z_1,z_2)=z_1\frac{e^{L(z)+B_1}+e^{-(L(z)+B_1)}}{2}+G_3(s_1), \tag{4.31}$$

where $G_3(s_1)$ is an entire function of finite order. Similarly, we have

$$f_2(z_1,z_2)=z_1\frac{e^{L(z)+B_2}+e^{-L(z)-B_2}}{2}+G_4(s_1),$$

where $G_4(s_1)$ is an entire function of finite order.

If $e^{L(c)}=1$, that is, $L(c)=2k\pi i$, which leads to $e^{B_1-B_2}=1$. Thus, it follows that

$$f_2(z_1,z_2)=z_1\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2}+G_4(s_1). \tag{4.32}$$

Substituting (4.31), (4.32) into the second and fourth equations in (4.22), we have

$$\begin{aligned} G_4(s_1+s_0)-G_3(s_1) &= \frac{e^{a_2s_1+B_1}-e^{-a_2s_1-B_1}}{2i}-c_1\frac{e^{a_2s_1+B_1}+e^{-a_2s_1-B_1}}{2},\\ G_3(s_1+s_0)-G_4(s_1) &= \frac{e^{a_2s_1+B_1}-e^{-a_2s_1-B_1}}{2i}-c_1\frac{e^{a_2s_1+B_1}+e^{-a_2s_1-B_1}}{2}. \end{aligned}$$

that is,

$$G_4(s_1+2s_0)-G_4(s_1)=-(c_1+i)e^{a_2s_1+B_1}-(c_1-i)e^{-a_2s_1-B_1},$$
$$G_3(s_1+2s_0)-G_3(s_1)=-(c_1+i)e^{a_2s_1+B_1}-(c_1-i)e^{-a_2s_1-B_1}.$$

Thus, we can deduce that

$$G_3(s_1)=G_1(s_1)-\frac{c_1+i}{2s_0}s_1e^{a_2s_1+B_1}-\frac{c_1-i}{2s_0}s_1e^{-a_2s_1-B_1},$$
$$G_4(s_1)=G_2(s_1)-\frac{c_1+i}{2s_0}s_1e^{a_2s_1+B_1}-\frac{c_1-i}{2s_0}s_1e^{-a_2s_1-B_1},$$

where $G_1(s_1), G_2(s_1)$ are the finite order entire period functions with period $2s_0$, and $G_2(s_1+s_0)=G_1(s_1)$.

If $e^{L(c)}=-1$, that is $L(c)=(2k+1)\pi i$, then it follows that $e^{B_1-B_2}=-1$, which means

$$f_2(z_1,z_2)=-z_1\frac{e^{L(z)+B_1}+e^{-L(z)-B_1}}{2}+G_4(s_1). \tag{4.33}$$

Substituting (4.31), (4.33) into the second and fourth equations in (4.22), we have

$$G_4(s_1+s_0)-G_3(s_1)=\frac{e^{a_2s_1+B_1}-e^{-a_2s_1-B_1}}{2i}-c_1\frac{e^{a_2s_1+B_1}+e^{-a_2s_1-B_1}}{2},$$
$$G_3(s_1+s_0)-G_4(s_1)=-\frac{e^{a_2s_1+B_1}-e^{-a_2s_1-B_1}}{2i}+c_1\frac{e^{a_2s_1+B_1}+e^{-a_2s_1-B_1}}{2}.$$

Thus, it yields

$$G_4(s_1+2s_0)-G_4(s_1)=(c_1+i)e^{a_2s_1+B_1}+(c_1-i)e^{-a_2s_1-B_1},$$
$$G_3(s_1+2s_0)-G_3(s_1)=-(c_1+i)e^{a_2s_1+B_1}-(c_1-i)e^{-a_2s_1-B_1}.$$

This leads to

$$G_3(s_1)=G_1(s_1)-\frac{c_1+i}{2s_0}s_1e^{a_2s_1+B_1}-\frac{c_1-i}{2s_0}s_1e^{-a_2s_1-B_1},$$
$$G_4(s_1)=G_2(s_1)+\frac{c_1+i}{2s_0}s_1e^{a_2s_1+B_1}+\frac{c_1-i}{2s_0}s_1e^{-a_2s_1-B_1},$$

where $G_1(s_1), G_2(s_1)$ are the finite order entire period functions with period $2s_0$, and $G_2(s+s_0)=G_1(s)$.

Case 2.

$$\begin{cases} -i\left(\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}+i\right)e^{q(z+c)-p(z)}\equiv 1,\\ -i\left(\frac{\partial q}{\partial z_1}+\frac{\partial q}{\partial z_2}+i\right)e^{q(z)+p(z+c)}\equiv 1. \end{cases} \tag{4.34}$$

In view of (4.34), the fact that $p(z), q(z)$ are polynomials leads to $q(z+c)-p(z)\equiv C_1$ and $q(z)+p(z+c)\equiv C_2$. This means $q(z+2c)+q(z)\equiv C_1+C_2$, this is a contradiction with the assumption of $q(z)$ being a nonconstant polynomial.

Case 3.

$$\begin{cases} -i(\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}+i)e^{p(z)+q(z+c)}\equiv 1, \\ -i(\frac{\partial q}{\partial z_1}+\frac{\partial q}{\partial z_2}+i)e^{p(z+c)-q(z)}\equiv 1. \end{cases} \tag{4.35}$$

In view of (4.35), the fact that $p(z), q(z)$ are polynomials leads to $p(z)+q(z+c)\equiv C_1$ and $p(z+c)-q(z)\equiv C_2$. This means $p(z+2c)+p(z)\equiv C_1+C_2$, this is a contradiction with the assumption of $p_2(z)$ being a nonconstant polynomial.

Case 4.

$$\begin{cases} -i\left(\frac{\partial p}{\partial z_1}+\frac{\partial p}{\partial z_2}+i\right)e^{p(z)+q(z+c)}\equiv 1, \\ -i\left(\frac{\partial q}{\partial z_1}+\frac{\partial q}{\partial z_2}+i\right)e^{q(z)+p(z+c)}\equiv 1. \end{cases} \tag{4.36}$$

In view of (4.36), the fact that $p_1(z), p_2(z)$ are polynomials leads to $p(z)+q(z+c)\equiv C_1$ and $q(z)+p(z+c)\equiv C_2$, that is, $p(z+2c)-p(z)\equiv C_1+C_2$ and $q(z+2c)-q(z)\equiv C_2+C_1$. Similar to the same argument in Case 1 of Theorem 2.3, we obtain that $p(z)=L(z)+B_1, q(z)=-L(z)+B_2$, where L is a linear function as the form $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants. In view of (4.23), (4.24) and (4.36), we have

$$\begin{cases} -i(a_1+a_2+i)e^{-L(c)+B_1+B_2}\equiv 1, \\ -i(a_1+a_2+i)e^{L(c)-B_1-B_2}\equiv 1, \\ i(a_1+a_2-i)e^{L(c)+B_1+B_2}\equiv 1, \\ i(a_1+a_2-i)e^{-L(c)-B_1-B_2}\equiv 1, \end{cases} \tag{4.37}$$

which implies $(a_1+a_2+i)^2=(a_1+a_2-i)^2$, that is, $a_1+a_2=0$. Thus, we conclude from (4.37) that

$$a_1+a_2=0,\quad e^{2L(c)}=1,\quad e^{B_1+B_2}=e^{L(c)}. \tag{4.38}$$

Similar to the argument as in Case 1 of Theorem 2.3, we get the conclusions of Theorem 2.3 (iv).

Therefore, from Cases 1–4, we complete the proof of Theorem 2.3.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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