AIMS Mathematics, 6(11): 11723–11732.
DOI: 10.3934/math.2021681
Received: 31 May 2021
Accepted: 09 August 2021
Published: 12 August 2021

http://www.aimspress.com/journal/Math

*Research article*

# On two sums related to the Lehmer problem over short intervals

**Yanbo Song***

School of Mathematics and Statistics, Xi'an Jiaotong University, Xianning West Road, Xi'an, China

* **Correspondence:** Email: syb888@stu.xjtu.edu.cn; Tel: +8618291806025.

**Abstract:** In this article, we study sums related to the Lehmer problem over short intervals, and give two asymptotic formulae for them. The original Lehmer problem is to count the numbers coprime to a prime such that the number and the its number theoretical inverse are in different parities in some intervals. The numbers which satisfy these conditions are called Lehmer numbers. It prompts a series of investigations, such as the investigation of the error term in the asymptotic formula. Many scholars investigate the generalized Lehmer problems and get a lot of results. We follow the trend of these investigations and generalize the Lehmer problem.

**Keywords:** Lehmer problem; asymptotic formula; Dirichlet character
**Mathematics Subject Classification:** 11L03, 11L05

## 1. Introduction

Let $p$ be an odd primes, $a$ be an integer coprime to $p$. So there exists a $0 < \bar{a} \le p$ such that $a\bar{a} \equiv 1 \mod p$. Let $N(p)$ denoted the number of integers $a$ $(0 < a < p)$ such that $a\bar{a} \equiv 1 \mod p$ and $2 \nmid a + \bar{a}$. D. H. Lehmer (see [1] problem F12 p. 381) asked a problem about the asymptotic formula of $N(p)$. To give an answer to this problem, Professor W. P. Zhang proved the following two theorems (see [2–4]):

**Theorem 1.1.** *Let $N(p)$ denoted the number of integers $a$ $(0 < a < p)$ such that $a\bar{a} \equiv 1 \mod p$ and $2 \nmid a + \bar{a}$. We have*

$$\begin{aligned} N(p) =& \frac{1}{2}p + \frac{1}{\pi^2}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\frac{1}{ab}(S(a,b;p) - S(a,-b;p) \\ &+ 4S(\bar{4}a,b;p) - 4S(\bar{4}a,-b;p) - 4S(\bar{2}a,b;p) \\ &+ 4S(\bar{2}a,-b;p)) + O(\log^3 p), \end{aligned}$$

*where the $S(a, b; p)$ are Kloosterman sums, defined as follows:*

$$S(m, n; p) = \sum_{a \mod p} e\left(\frac{ma + n\overline{a}}{p}\right).$$

**Theorem 1.2.** *For any odd number q, let $N(q)$ denotes the number of integers a ($0 < a < q$) which coprime to q such that $a\overline{a} \equiv 1 \mod q$ and $2 \nmid a + \overline{a}$. We have*

$$N(q) = \frac{1}{2}\phi(q) + O(q^{1/2}\tau^2(q)\log^2 q).$$

For more about Lehmer problem, one can see [5–12]. In 2015, H. Zhang and W. P. Zhang [13] investigated two new sums:

$$L(k, p) = \frac{1}{2^k} \sum_{\substack{a_1 \le p \\ a_1 \cdots a_k \equiv 1}} \cdots \sum_{\substack{a_k \le p \\ \mod p}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}}),$$

$$S(k, p) = \frac{1}{2^k} \sum_{\substack{a_1 \le p \\ a_1 \cdots a_k \equiv 1 \\ n | a_1 + \cdots + a_k}} \cdots \sum_{\substack{a_k \le p \\ \mod p}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}}),$$

which related to the Lehmer problem, and gave the asymptotic formulae for these new sums.

In this paper, we give asymptotic formulae for these new sums over short intervals.

**Theorem 1.3.** *Let p be an odd primes, and $k > 2$ be an integer, and $\varepsilon > 0$ is a sufficient small real number. $|A_i| \gg p^{1/2+1/k+\varepsilon}$, $A_i = \{1, 2, \cdots, t_i\}$, $[1, t_i] \in (0, p)$, $i = 1, 2, \cdots, k$. Define $N(k, p)$ as follows:*

$$N(k, p) = \frac{1}{2^k} \sum_{\substack{a_1 \in A_1 \\ a_1 \cdots a_k \equiv 1}} \cdots \sum_{\substack{a_k \in A_k \\ \mod p}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}}),$$

*then we have*

$$N(k, p) = \frac{1}{2^k(p-1)} \prod_{i=1}^{k} t_i + O\left(p^{-1/2} \log^2 p \max_{1 \le r \le k} \left\{\prod_{i \ne r} t_i\right\}\right) + O(p^{k/2+\varepsilon}),$$

*where the implied constant of the first and second big 'O' terms depends on k.*

**Theorem 1.4.** *Let p be an odd primes, $k > 4$ be an integer, and $\varepsilon > 0$ is a sufficient small real number. $p \ge n > p^{\frac{k+2}{2k-2}+\varepsilon}$ be an integer. $A_i = \{1, 2, \cdots, t_i\}$, $i = 1, 2, \cdots, k-1$, $p^{\frac{k+2}{2k-2}+\varepsilon} < t_i < n$, $A_k = \{1, 2, \cdots, n\}$. Define $M(k, p)$ as follows:*

$$M(k, p) = \frac{1}{2^k} \sum_{\substack{a_1 \in A_1 \\ a_1 \cdots a_k \equiv 1 \\ n | a_1 + \cdots + a_k}} \cdots \sum_{\substack{a_k \in A_k \\ \mod p}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}}),$$

*then we have*

$$M(k,p)=\frac{1}{2^k(p-1)}\prod_{i=1}^{k-1}t_i+O\left(p^{-1/2}\log^2 p\max_{1\le r\le k-1}\left\{\prod_{\substack{i\neq r\\1\le i\le k-1}}t_i\right\}\right)+O(p^{k/2+\varepsilon}),$$

*where the implied constant of the first and second big 'O' terms depends on k.*

*Remark* 1.5. The short interval result is usually more difficult than the whole interval result. This is why this problem is interesting and meaningful, and use some technique which are often used to obtain a short interval result such as Lemma 2.4 and formula (2.1). We use $1-e\left(\frac{a+b}{2}\right)$ instead of $(-1)^{a+b}$ to detect $2\mid a+b$.

## 2. Some Lemmas

Before we prove the theorem, we introduce some Lemmas, and through out the paper, $p$ always denotes an odd prime number, and $\chi$ and $\psi$ always denote the character $\bmod\ p$, $\chi_0$ denotes the principal character $\bmod\ p$.

We define the generalized Kloosterman sums, as follows:

$$K(m,n;q)=\sum_{a\mod q}\chi(a)e\left(\frac{ma+n\bar{a}}{q}\right),$$

where $m,n,q$ be integers and $\chi$ is a Dirichlet character $\bmod\ q$. And we define Gauss sums, as follows:

$$G(a,\chi)=\sum_{d\mod q}\chi(d)e\left(\frac{ad}{q}\right).$$

We have following estimates.

**Lemma 2.1.** *Let p be an odd primes, then we have*

$$K(m,n;p)\le p^{1/2+\varepsilon}(m,n,p)^{1/2}.$$

*Proof.* See [14]. □

**Lemma 2.2.** *Let a, b, q be natural numbers, $\chi$ runs through all the non-principal characters* mod *q, then we have*

$$\sum_{\chi\neq\chi_0}G(a,\chi)G(b,\chi)\ll q^{3/2}\tau(q)(a,q)^{1/2}(b,q)^{1/2}.$$

*Proof.* See Lemma 3 in [15]. □

Similarly, we have

**Lemma 2.3.** *Let p be an odd primes, and a, bbe natural numbers, $\psi$ runs through all the non-principal characters* mod *p, $\chi$ is a character* mod *p, then we have*

$$\sum_{\psi\neq\chi_0}G(a,\psi)G(b,\psi\chi)\ll p^{3/2+\varepsilon}(a,p)^{1/2}(b,p)^{1/2}.$$

**Lemma 2.4.** *Let N be a nature number, α be a real number, then*

$$\sum_{n\le N} e(\alpha n) \ll \min\left(N, \frac{1}{2\|\alpha\|}\right),$$

*where $\|\alpha\|$ denotes the distance from α to the nearest integer.*

*Proof.* See [16]. □

**Lemma 2.5.** *Let U and γ be two positive real numbers, p be a nonzero natural number, then*

$$\sum_{k=1}^{p-1} \min\left(U, \frac{1}{\left\|\frac{k}{p}+\gamma\right\|}\right) \ll U + p\log p.$$

*Proof.* Since the inequality

$$\left\|\frac{k}{p}+\gamma\right\| < \frac{1}{p}$$

hold for at most two $k$. They contribute at most $2U$. The others contribute at most

$$2\sum_{j<p/2} p/j \ll p\log p.$$

This completes the proof of the lemma. □

**Lemma 2.6.** *Let p be an odd prime number, $A = \{1, 2, \cdots, s\}$, $[1, s] \in (0, p)$, then we have*

$$\sum_{\substack{a\in A\\ 2\nmid a+\bar{a}}} \chi(a) e\left(\frac{ta}{n}\right) \ll p^{1/2+\varepsilon},$$

*when $\chi \neq \chi_0$, and*

$$\sum_{\substack{a\in A\\ 2\nmid a+\bar{a}}} \chi(a) e\left(\frac{ta}{n}\right) = \frac{1}{2}\sum_{a\in A} e\left(\frac{ta}{n}\right) + O(p^{1/2}\log^2 p),$$

*when $\chi = \chi_0$.*

*Proof.* When $\chi = \chi_0$, we have

$$\begin{aligned}
\sum_{\substack{a\in A\\ 2\nmid a+\bar{a}}} e\left(\frac{ta}{n}\right) =& \sum_{\substack{a\in A\\ ab\equiv 1\\ 2\nmid a+b}} \sum_{\substack{b=1\\ \bmod p}}^{p-1} e\left(\frac{ta}{n}\right)\\
=&\frac{1}{2p-2}\sum_{\psi \bmod p}\sum_{a\in A}\sum_{b=1}^{p-1}\psi(ab)\left(1-e\left(\frac{a+b}{2}\right)\right)e\left(\frac{ta}{n}\right)\\
=&\frac{1}{2p-2}\sum_{a\in A}\sum_{b=1}^{p-1} e\left(\frac{ta}{n}\right) - \frac{1}{2p-2}\sum_{a\in A}\sum_{b=1}^{p-1} e\left(\frac{a+b}{2}\right)e\left(\frac{ta}{n}\right)
\end{aligned}$$

$$+\frac{1}{2p-2}\sum_{\psi\neq\chi_0}\sum_{a\in A}\sum_{b=1}^{p-1}\psi(a)\psi(b)e\left(\frac{ta}{n}\right)-\frac{1}{2p-2}\sum_{\psi\neq\chi_0}\sum_{a\in A}\sum_{b=1}^{p-1}\psi(a)\psi(b)e\left(\frac{a+b}{2}\right)e\left(\frac{ta}{n}\right)$$
$$=\frac{1}{2}\sum_{a\in A}e\left(\frac{ta}{n}\right)-\frac{1}{2p-2}\sum_{\psi\neq\chi_0}\sum_{a\in A}\sum_{b=1}^{p-1}\psi(a)\psi(b)e\left(\frac{a+b}{2}\right)e\left(\frac{ta}{n}\right).$$

Let

$$S=\frac{1}{p-1}\sum_{\psi\neq\chi_0}\sum_{a\in A}\sum_{b=1}^{p-1}\psi(a)\psi(b)e\left(\frac{a+b}{2}\right)e\left(\frac{ta}{n}\right).$$

We note that for $\psi\neq\chi_0$, we have

$$\psi(a)=\frac{1}{p}\sum_{r=1}^{p-1}G(r,\psi)e\left(\frac{-ar}{p}\right). \tag{2.1}$$

$$S=\frac{1}{p-1}\sum_{\psi\neq\chi_0}\sum_{a\in A}\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)$$
$$=\frac{1}{p^2(p-1)}\sum_{r=1}^{p-1}\sum_{a\in A}e\left(\frac{a}{2}+\frac{ta}{n}-\frac{ar}{p}\right)\sum_{s=1}^{p-1}\sum_{b=1}^{p-1}e\left(\frac{b}{2}-\frac{bs}{p}\right)\sum_{\psi\neq\chi_0}G(r,\psi)G(s,\psi)$$

By Lemma 2.2 and 2.4, we have

$$S\ll\frac{1}{p^{3/2}}\sum_{r=1}^{p-1}\min\left(|A|,\left\|\frac{1}{2}+\frac{t}{n}-\frac{r}{p}\right\|^{-1}\right)\sum_{s=1}^{p-1}\min\left(p,\left\|\frac{1}{2}-\frac{s}{p}\right\|^{-1}\right).$$

By Lemma 2.5, we have

$$S\ll p^{1/2}\log^2 p.$$

So when $\chi=\chi_0$, we have

$$\sum_{\substack{a\in A\\ 2\nmid a+\bar{a}}}\chi(a)e\left(\frac{ta}{n}\right)=\frac{1}{2}\sum_{a\in A}e\left(\frac{ta}{n}\right)+O(p^{1/2}\log^2 p).$$

When $\chi\neq\chi_0$, we have

$$\sum_{\substack{a\in A\\ 2\nmid a+\overline{a}}}\chi(a)e\left(\frac{ta}{n}\right)=\sum_{\substack{a\in A\\ ab\equiv 1}}\sum_{\substack{b=1\\ \bmod p\\ 2\nmid a+b}}^{p-1}\chi(a)e\left(\frac{ta}{n}\right)$$

$$=\frac{1}{2p-2}\sum_{\psi\bmod p}\sum_{a\in A}\sum_{b=1}^{p-1}\chi(a)\psi(ab)e\left(\frac{ta}{n}\right)$$

$$-\frac{1}{2p-2}\sum_{\psi\bmod p}\sum_{a\in A}\chi(a)\psi(a)e\left(\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{a+b}{2}\right)$$

$$:=S_1-S_2.$$

We only estimate $S_2$, the estimation of $S_1$ is similar.

$$S_2=\frac{1}{2p-2}\sum_{\psi}\sum_{a\in A}\chi(a)\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)$$

$$=\frac{1}{2p-2}\sum_{\psi\neq\chi_0,\overline{\chi}}\sum_{a\in A}\chi(a)\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)$$

$$+\frac{1}{2p-2}\sum_{\psi=\chi_0}\sum_{a\in A}\chi(a)\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)$$

$$+\frac{1}{2p-2}\sum_{\psi=\overline{\chi}}\sum_{a\in A}\chi(a)\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)$$

$$:=S_{21}+S_{22}+S_{23}.$$

For $S_{21}$, by (2.1), we have

$$S_{21}=\frac{1}{2p-2}\sum_{\psi\neq\chi_0}\sum_{a\in A}\chi\psi(a)e\left(\frac{a}{2}+\frac{ta}{n}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{b}{2}\right)-S_{23}$$

So we have

$$S_{21}+S_{23}=\frac{1}{p^2(p-1)}\sum_{r=1}^{p-1}\sum_{a\in A}e\left(\frac{a}{2}+\frac{ta}{n}-\frac{ar}{p}\right)\sum_{s=1}^{p-1}\sum_{b=1}^{p-1}e\left(\frac{b}{2}-\frac{bs}{p}\right)\sum_{\psi\neq\chi_0}G(r,\chi\psi)G(s,\psi).$$

By Lemma 2.3 and 2.4, we have

$$S_{21}+S_{23}\ll\frac{1}{p^{3/2-\varepsilon}}\sum_{r=1}^{p-1}\min\left(|A|,\left\|\frac{1}{2}+\frac{t}{n}-\frac{r}{p}\right\|^{-1}\right)\sum_{s=1}^{p-1}\min\left(p,\left\|\frac{1}{2}-\frac{s}{p}\right\|^{-1}\right).$$

By Lemma 2.5, we have

$$S_{21}+S_{23}\ll p^{1/2+\varepsilon}.$$

For $S_{22}$, it is simple, we have

$$S_{22} \ll p^{1/2+\varepsilon}.$$

So when $\chi \neq \chi_0$, we have

$$\sum_{\substack{a\in A \\ 2\nmid a+\overline{a}}} \chi(a) e\left(\frac{ta}{n}\right) \ll p^{1/2+\varepsilon}.$$

So we complete the proof of Lemma 2.6. □

For convenience, we record the following four special cases of Lemma 2.6.

**Corollary 2.7.** *Let p be an odd prime number,* $A = \{1, 2, \cdots, t\}$*,* $[1, t] \in (0, p)$*, then we have*

$$\sum_{\substack{a\in A \\ 2\nmid a+\overline{a}}} 1 = \frac{1}{2}|A| + O(p^{1/2}\log^2 p).$$

**Corollary 2.8.** *Let p be an odd prime number,* $A = \{1, 2, \cdots, t\}$*,* $[1, t] \in (0, p)$*,* $\chi \neq \chi_0$*, then we have*

$$\sum_{\substack{a\in A \\ 2\nmid a+\overline{a}}} \chi(a) \ll p^{1/2+\varepsilon}.$$

**Corollary 2.9.** *Let p be an odd prime number,* $A = \{1, 2, \cdots, h\}$*,* $[1, h] \in (0, p)$*, then we have*

$$\sum_{\substack{a\in A \\ 2\nmid a+\overline{a}}} e\left(\frac{ta}{n}\right) = \frac{1}{2}\sum_{a\in A} e\left(\frac{ta}{n}\right) + O(p^{1/2}\log^2 p).$$

**Corollary 2.10.** *Let p be an odd prime number,* $A = \{1, 2, \cdots, t\}$*,* $[1, t] \in (0, p)$*,* $\chi \neq \chi_0$*, then we have*

$$\sum_{\substack{a\in A \\ 2\nmid a+\overline{a}}} \chi(a) e\left(\frac{ta}{n}\right) \ll p^{1/2+\varepsilon}.$$

## 3. Proofs of the theorems

Now we prove Theorem 1.3, we give asymptotic formulae for two sums over short intervals.

$$\begin{aligned} N(k,p) &= \frac{1}{2^k} \sum_{a_1\in A_1} \cdots \sum_{\substack{a_k\in A_k \\ a_1\cdots a_k \equiv 1 \bmod p}} (1-(-1)^{a_1+\overline{a_1}})\cdots(1-(-1)^{a_k+\overline{a_k}}) \\ &= \sum_{\substack{a_1\in A_1 \\ 2\nmid a_1+\overline{a_1}}} \cdots \sum_{\substack{a_k\in A_k \\ 2\nmid a_k+\overline{a_k} \\ a_1\cdots a_k \equiv 1 \bmod p}} 1 \\ &= \frac{1}{p-1} \sum_{\chi} \sum_{\substack{a_1\in A_1 \\ 2\nmid a_1+\overline{a_1}}} \cdots \sum_{\substack{a_k\in A_k \\ 2\nmid a_k+\overline{a_k}}} \chi(a_1\cdots a_k) \end{aligned}$$

$$= \frac{1}{p-1} \sum_{\substack{a_1 \in A_1 \\ 2 \nmid a_1 + \overline{a_1}}} \cdots \sum_{\substack{a_k \in A_k \\ 2 \nmid a_k + \overline{a_k}}} 1 + \frac{1}{p-1} \sum_{\chi \neq \chi_0} \sum_{\substack{a_1 \in A_1 \\ 2 \nmid a_1 + \overline{a_1}}} \cdots \sum_{\substack{a_k \in A_k \\ 2 \nmid a_k + \overline{a_k}}} \chi(a_1 \cdots a_k)$$
$$= \frac{1}{p-1} \prod_i \sum_{\substack{a_i \in A_i \\ 2 \nmid a_i + \overline{a_i}}} 1 + \frac{1}{p-1} \sum_{\chi \neq \chi_0} \prod_i \sum_{\substack{a_i \in A_i \\ 2 \nmid a_i + \overline{a_i}}} \chi(a_i).$$

By Corollary 2.7, we have

$$\sum_{\substack{a_i \in A_i \\ 2 \nmid a_i + \overline{a_i}}} 1 = \frac{1}{2}|A_i| + O(p^{1/2} \log^2 p).$$

By Corollary 2.8, we have

$$\sum_{\substack{a_i \in A_i \\ 2 \nmid a_i + \overline{a_i}}} \chi(a_i) \ll p^{1/2+\varepsilon}.$$

So we have

$$N(k,p) = \frac{1}{2^k} \sum_{a_1 \in A_1} \cdots \sum_{\substack{a_k \in A_k \\ a_1 \cdots a_k \equiv 1 \bmod p}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}})$$
$$= \frac{1}{2^k (p-1)} \prod_{i=1}^{k} (t_i + O(p^{1/2} \log^2 p)) + O(p^{k/2+\varepsilon})$$
$$= \frac{1}{2^k (p-1)} \prod_{i=1}^{k} t_i + O\left( p^{-1/2} \log^2 p \max_{1 \le r \le k} \left\{ \prod_{i \neq r} t_i \right\} \right) + O(p^{k/2+\varepsilon}).$$

Note that the implied constant of the first big 'O' term is dependent on $k$. This completes the proof of Theorem 1.3.

Now we prove Theorem 1.4, we have

$$M(k,p) = \frac{1}{2^k} \sum_{a_1 \in A_1} \cdots \sum_{\substack{a_k \in A_k \\ a_1 \cdots a_k \equiv 1 \bmod p \\ n | a_1 + \cdots + a_k}} (1 - (-1)^{a_1 + \overline{a_1}}) \cdots (1 - (-1)^{a_k + \overline{a_k}})$$
$$= \sum_{\substack{a_1 \in A_1 \\ 2 \nmid a_1 + \overline{a_1}}} \cdots \sum_{\substack{a_k \in A_k \\ 2 \nmid a_k + \overline{a_k} \\ a_1 \cdots a_k \equiv 1 \bmod p \\ n | a_1 + \cdots + a_k}} 1$$
$$= \frac{1}{n(p-1)} \sum_{t=1}^{n} \sum_{\chi} \sum_{\substack{a_1 \in A_1 \\ 2 \nmid a_1 + \overline{a_1}}} \cdots \sum_{\substack{a_k \in A_k \\ 2 \nmid a_k + \overline{a_k}}} \chi(a_1 \cdots a_k) e\left( \frac{t(a_1 + \cdots + a_k)}{n} \right)$$
$$= \frac{1}{n(p-1)} \sum_{t=1}^{n} \sum_{\substack{a_1 \in A_1 \\ 2 \nmid a_1 + \overline{a_1}}} \cdots \sum_{\substack{a_k \in A_k \\ 2 \nmid a_k + \overline{a_k}}} e\left( \frac{t(a_1 + \cdots + a_k)}{n} \right)$$

$$+\frac{1}{n(p-1)}\sum_{t=1}^{n}\sum_{\chi\neq\chi_0}\sum_{\substack{a_1\in A_1\\ 2\nmid a_1+\overline{a_1}}}\cdots\sum_{\substack{a_k\in A_k\\ 2\nmid a_k+\overline{a_k}}}\chi(a_1\cdots a_k)e\left(\frac{t(a_1+\cdots+a_k)}{n}\right)$$
$$=\frac{1}{n(p-1)}\sum_{t=1}^{n}\prod_{i=1}^{k}\sum_{\substack{a_i\in A_i\\ 2\nmid a_i+\overline{a_i}}}e\left(\frac{ta_i}{n}\right)+\frac{1}{n(p-1)}\sum_{t=1}^{n}\sum_{\chi\neq\chi_0}\prod_{i=1}^{k}\sum_{\substack{a_i\in A_i\\ 2\nmid a_i+\overline{a_i}}}\chi(a_i)e\left(\frac{ta_i}{n}\right).$$

By Corollary 2.9 and 2.10, we have

$$\sum_{\substack{a_i\in A\\ 2\nmid a_i+\overline{a_i}}}e\left(\frac{ta_i}{n}\right)=\frac{1}{2}\sum_{a_i\in A}e\left(\frac{ta_i}{n}\right)+O(p^{1/2}\log^2 p),$$
$$\sum_{\substack{a_i\in A\\ 2\nmid a_i+\overline{a_i}}}\chi(a_i)e\left(\frac{ta_i}{n}\right)\ll p^{1/2+\varepsilon}.$$

By the following formula

$$\sum_{a=0}^{p-1}e\left(\frac{ma}{p}\right)=\begin{cases}p, & \text{if } p\mid m;\\ 0, & \text{otherwise,}\end{cases}$$

and the definition of interval $A_k$, we have

$$M(k,p)=\frac{1}{2^k(p-1)}\prod_{i=1}^{k-1}t_i+O\left(p^{-1/2}\log^2 p\max_{1\le r\le k-1}\left\{\prod_{\substack{i\neq r\\ 1\le i\le k-1}}t_i\right\}\right)+O(p^{k/2+\varepsilon}).$$

Note that the implied constant of the first big 'O' term is dependent on $k$. This completes the proof of Theorem 1.4.

## 4. Conclusions

In this paper, we investigated two problems which related to the Lehmer problem, and gave the asymptotic formulae for these new sums over short intervals.


## Acknowledgements

The author would like to express his heartfelt thanks to the reviewers for their valuable comments and suggestions.


## Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

## References


1. R. K. Guy, *Unsolved problems in number theory*, New York: Springer-Verlag, 2004.
2. W. P. Zhang, On D. H. Lehmer problem, *Chin. Sci. Bull.*, **21** (1992), 1765–1769.
3. W. P. Zhang, On a problem of D. H. Lehmer and its generalization, *Compos. Math.*, **86** (1993), 307–316.
4. W. P. Zhang, A problem of D. H. Lehmer and its generalization (II), *Compos. Math.*, **91** (1994), 47–56.
5. Y. Yi, W. P. Zhang, On the generalization of a problem of D. H. Lehmer, *Kyushu J. Math.*, **56** (2002), 235–241.
6. I. E. Shparlinski, On a generalised Lehmer problem for arbitrary powers, *East-West J. Math.*, 2008.
7. I. E. Shparlinski, On a generalisation of a Lehmer problem, *Math. Z.*, **263** (2009), 619–631.
8. Y. Lu, Y. Yi, On the generalization of the D. H. Lehmer problem, *Acta Math. Sinica, Engl. Ser.*, **25** (2009), 1269–1274.
9. Z. F. Xu, W. P. Zhang, On a problem of D. H. Lehmer over short intervals, *J. Math. Anal. Appl.*, **320** (2006), 756–770.
10. H. N. Liu, W. P. Zhang, On a problem of D. H. Lehmer, *Acta Math. Sinica, Engl. Ser.*, **22** (2006), 61–68.
11. Z. Y. Zheng, On the ordinary problem of D. H. Lehmer, *Chin. Sci. Bull.*, **38** (1993), 1060–1065.
12. P. Xi, Y. Yi, Kloosterman sums and a problem of D. H. Lehmer, *Chin. Ann. Math. Ser. B*, **41** (2020), 361–370.
13. H. Zhang, W. P. Zhang, Some new sums related to D. H. Lehmer problem, *Czech. Math. J.*, **65** (2015), 915–922.
14. S. Chowla, On Kloosterman's Sum, *Norske Vid. Selsk. Forh.*, **40** (1967), 70–72.
15. P. Xi, Y. Yi, Generalized D. H. Lehmer problem over short intervals, *Glasgow Math. J.*, **53** (2011), 293–299.
16. C. D. Pan, C. B. Pan, *Goldbach conjecture*, Beijing: Science Press, 1981.