AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(10): 11584–11594.
DOI:10.3934/math.2021672
Received: 05 April 2021
Accepted: 04 August 2021
Published: 10 August 2021

Research article

Double controlled quasi metric-like spaces and some topological properties of this space

A. M. Zidan[1,2,*] **and Z. Mostefaoui**[3]

[1] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut Branch 71524, Egypt
[3] Department of Mathematics, E.N.S, B.P. 92 Vieux Kouba 16050 Algiers, Algeria

* **Correspondence:** Email: zidan.math90@azhar.edu.eg; Tel: +966507940194.

Abstract: In present paper, we introduce a new extension of the double controlled metric-like spaces, so called double controlled quasi metric-like spaces “assuming that the self-distance may not be zero”. Also, if the value of the metric is zero, then it has to be “a self-distance”. After that, by using this new type of quasi metric spaces, we generalize many results in the literature and we prove fixed point theorems along with some examples illustrating.

Keywords: uasi metric-like spaces; double controlled; fixed point theory
Mathematics Subject Classification: 30L10, 37C25, 46A19

1. Introduction

Among various generalizations of concept of metric, Matthews [15] introduced a special kind of a partial metric space where the self-distance $d(x, x)$ is not necessarily zero. On the other hand, Amini-Harandi [4] redefined a dislocated metric of Hitzler and Seda [11] and introduced metric-like spaces. Combining these two concepts we get quasi-metric-like spaces. The study of partial metric spaces has wide area of application, especially in computer science [14, 18]. Therefore, we can find many fixed point results in the setting of partial metric spaces [3–6, 8–10].

The b-metric space [6, 7] and its partial versions, which extends the metric space by modifying the triangle equality metric axiom by inserting a constant multiple $s > 1$ to the right-hand side, is one of the most applied generalizations for metric spaces (see [1, 12]).

Very recently, the authors in [13] introduced a type of extended b-metric spaces by replacing the constant s by a function $\theta(x, y)$ depending on the parameters of the left-hand side of the triangle inequality.

In this paper we introduce a new type of generalized metric space, which we call as a double controlled quasi metric-Like type space. We also prove the corresponding Banach fixed point theorem on this metric space and we provide an illustrating example.

2. Preliminary assertions

In 2017 Kamran et al. [13] initiated the concept of extended b-metric spaces.

Definition 2.1. [13] Let Υ be a non empty set and $\theta : \Upsilon \times \Upsilon \to [1, \infty)$. An extended b-metric is a function $\varpi : \Upsilon \times \Upsilon \to [0, \infty)$ such that for all $v, t, r \in \Upsilon$ the following conditions hold:
(1) $\varpi(v, t) = 0 \Leftrightarrow v = t$;
(2) $\varpi(v, t) = \varpi(t, v)$;
(3) $\varpi(v, t) \leq \theta(v, t)[\varpi(v, r) + \varpi(r, t)]$,
for all $v, t, r \in \Upsilon$. The pair (Υ, ϖ) is called an extended b-metric space.

Mlaiki et al. [17] generalized the notion of b-metric spaces.

Definition 2.2. [17] Given a nonempty set Υ and $\theta : \Upsilon \times \Upsilon \to [1, \infty)$. The function $\varpi : \Upsilon \times \Upsilon \to [0, \infty)$ is called a controlled metric type if
(1) $\varpi(v, t) = 0 \Leftrightarrow v = t$;
(2) $\varpi(v, t) = \varpi(t, v)$;
(3) $\varpi(v, t) \leq \theta(v, r)\varpi(v, r) + \theta(r, t)\varpi(r, t)]$
for all $v, t, r \in \Upsilon$. The pair (Υ, ϖ) is called a controlled metric type space.

Next we present the definition of double controlled metric-type spaces.

Definition 2.3. [2] Let there be given two non-comparable functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$. Let $\varpi : \Upsilon \times \Upsilon \to [0, \infty)$ be a function satisfying
(1) $\varpi(v, t) = 0 \Leftrightarrow v = t$;
(2) $\varpi(v, t) = \varpi(t, v)$;
(3) $\varpi(v, t) \leq \beta(v, r)\varpi(v, r) + \rho(r, t)\varpi(r, t)$,
for all $v, t, r \in \Upsilon$. Then ϖ is called a double controlled metric type by β and ρ and the pair (Υ, ϖ) is a double controlled metric type space.

The following definition is a generalization of double controlled metric-type spaces to double controlled metric-like-type spaces, where the condition (1) is replaced by a weaker one.

Definition 2.4. [16] Consider a set Υ be a non empty set and non-comparable functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$. Suppose that a function $\varpi : \Upsilon \times \Upsilon \to [0, \infty)$ satisfies the following conditions for all $v, t, r \in \Upsilon$:
(1) $\varpi(v, t) = 0 \Rightarrow v = t$;
(2) $\varpi(v, t) = \varpi(t, v)$;
(3) $\varpi(v, t) \leq \beta(v, r)\varpi(v, r) + \rho(r, t)\varpi(r, t)]$.

Then the pair (Υ, ϖ) is called a double controlled metric-like space.

3. Double controlled quasi metric-Like spaces and some topological properties of this space

In this section we present our generalization of the double controlled quasi metric-like-type spaces. This concept is extension of the double controlled metric-like spaces, so called double controlled quasi metric-like spaces “assuming that the self-distance may not be zero”.

Definition 3.1. Let Υ be a non empty set and consider non-comparable functions

$$\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty).$$

Suppose that a function $\varpi : \Upsilon \times \Upsilon \to [0, \infty)$, for all $v, t, r \in \Upsilon$, satisfies the following conditions:
(ϖ_1) $\varpi(v, t) = \varpi(t, v) = 0 \Rightarrow v = t$;
(ϖ_2) $\varpi(v, t) \leq \varpi(v, r)\beta(v, r) + \varpi(r, t)\rho(r, t)$.
Then the pair (Υ, ϖ) is called a double controlled quasi metric-like space or shortly $(DCQMLS)$.
Definition 3.2. Let (Υ, ϖ) be a $DCQMLS$ and (v_n) be a sequence in Υ. Then we say

(i) (v_n) converges to $v \in \Upsilon$ if and only if

$$\lim_{n \longrightarrow +\infty} \varpi(v_n, v) = \varpi(v, v) = \lim_{n \longrightarrow +\infty} \varpi(v, v_n).$$

In this case v is called a double controlled quasi like-limit or shortly (ϖ-limit) of (v_n), and we write $\lim_{n \longrightarrow +\infty} v_n = v$.

(ii) A sequence (v_n) is a ϖ-Cauchy sequence if both $\lim_{n,m \longrightarrow +\infty} \varpi(v_n, v_m)$ and $\lim_{n,m \longrightarrow +\infty} \varpi(v_m, v_n)$ exist and are finite.

(iii) (Υ, ϖ) is ϖ-complete if for any ϖ-Cauchy sequence (v_n), there exists some $v \in \Upsilon$ such that

$$\begin{aligned} \varpi(v, v) &= \lim_{n \longrightarrow +\infty} \varpi(v_n, v) \\ &= \lim_{n \longrightarrow +\infty} \varpi(v, v_n) \\ &= \lim_{n,m \longrightarrow +\infty} \varpi(v_n, v_m) \\ &= \lim_{n,m \longrightarrow +\infty} \varpi(v_m, v_n). \end{aligned}$$

(iv) The mapping $\Xi : \Upsilon \to \Upsilon$ is said to be continuous at $v \in \Upsilon$ if for any sequence (v_n) converging to v, we have $\lim_{n \longrightarrow +\infty} \Xi v_n = \Xi v$, that is,

$$\lim_{n \longrightarrow +\infty} \varpi(\Xi v_n, \Xi v) = \varpi(\Xi v, \Xi v) = \lim_{n \longrightarrow +\infty} \varpi(\Xi v, \Xi v_n).$$

Remark 3.1.

(i) Topology of $(DCQMLS)$ is not necessarily a Hausdorff topology, so the limit of convergent sequence is not always unique.

(ii) There are convergent sequences in $(DCQMLS)$ that are not Cauchy sequences.

Example 3.1. Let $\Upsilon = \{0, 1, 2\}$ and $\varpi : \Upsilon \times \Upsilon \to [0, +\infty)$ defined with

$$\varpi(0, 0) = \varpi(0, 1) = \varpi(1, 1) = 1,$$

$$\varpi(0, 2) = \varpi(1, 0) = \varpi(1, 2) = \varpi(2, 0) = \varpi(2, 1) = \varpi(2, 2) = 2.$$

Consider the following $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$:

$$\beta(1, 1) = \beta(1, 2) = \beta(1, 0) = \beta(0, 1) = \beta(0, 0) = \beta(0, 2) = \beta(2, 0) = 1,$$

$$\beta(2,1) = \beta(2,2) = 2,$$

and

$$\rho(1,1) = \rho(0,1) = \rho(1,0) = \rho(0,0) = 1,$$

$$\rho(0,2) = \rho(2,0) = \rho(1,2) = \rho(2,1) = \rho(2,2) = 2.$$

Thus, (Υ, ϖ) is a $(DCQMLS)$.

The constant sequence $(v_n = 1)_{n\in\mathbb{N}}$ is convergent with both 1 and 2 as limits since

$$\lim_{n\to+\infty} \varpi(v_n, 1) = \lim_{n\to+\infty} \varpi(1, v_n) = \varpi(1,1) = 1$$

$$\lim_{n\to+\infty} \varpi(v_n, 2) = \varpi(1,2) = \varpi(2,1) = \lim_{n\to+\infty} \varpi(2, v_n) = \varpi(2,2) = 2.$$

Consider the sequence $t_{2n} = 1, t_{2n-1} = 0, n \in \mathbb{N}$. Obviously, (t_n) is not a Cauchy sequence, but

$$\lim_{n\to+\infty} \varpi(t_n, 2) = \lim_{n\to+\infty} \varpi(2, t_n) = \varpi(2,2) = 2,$$

implying that $\lim_{n\to+\infty} t_n = 2$.

Definition 3.3. Let (Υ, ϖ) be a $(DCQMLS)$ with $\varepsilon > 0$, $v_0 \in \Upsilon$. The set $\delta(v_0, \varepsilon) = \{v/v \in \Upsilon, \max(\varpi(v_0, v), \varpi(v, v_0)) < \varepsilon\}$ is called ϖ-open ball of radius ε,center v_0 and $B_\varepsilon(v_0) = \{v_0\} \cup \delta(v_0, \varepsilon)$. The set $\overline{\delta}(v_0, \varepsilon) = \{v/v \in \Upsilon, \max(\varpi(v_0, v), \varpi(v, v_0)) \leq \varepsilon\}$ is called ϖ-closed ball of radius ε, center v_0 and $\overline{B}_\varepsilon(v_0) = \{v_0\} \cup \overline{\delta}(v_0, \varepsilon)$.

4. Main results

Theorem 4.1. Let (Υ, ϖ) be a complete $(DCQMLS)$ defined by functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$. Let $\Xi : \Upsilon \to \Upsilon$ be a mapping such that

$$\varpi(\Xi v, \Xi t) \leq h\varpi(v, t), \tag{4.1}$$

for all $v, t \in \Upsilon$, where $h \in (0,1)$. For $v_0 \in \Upsilon$, take $v_n = \Xi^n v_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(v_{i+1}, v_{i+2})}{\beta(v_i, v_{i+1})} \rho(v_{i+1}, v_m) < \frac{1}{h}. \tag{4.2}$$

Also assume that, for every $v \in \Upsilon$, we have

$$\lim_{n\to+\infty} \beta(v, v_n) \text{ and } \lim_{n\to+\infty} \rho(v_n, v) \text{ exist and are finite.} \tag{4.3}$$

Then Ξ has a unique fixed point.

Proof. Let $v_n = \Xi^n v_0$ in Υ be a sequence that satisfies the conditions of our theorem. By using (4.1) we get

$$\varpi(v_n, v_{n+1}) \leq h^n \varpi(v_0, v_1) \text{ for all } n \geq 0. \tag{4.4}$$

Let $n, m \in \mathbb{N}$ be such that $n < m$. Then

$$\begin{aligned}
\varpi(v_n, v_m) &\leq \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \rho(v_{n+1}, v_m)\varpi(v_{n+1}, v_m) \\
&\leq \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \rho(v_{n+1}, v_m)\beta(v_{n+1}, v_{n+2})\varpi(v_{n+1}, v_{n+2}) \\
&\quad +\rho(v_{n+1}, v_m)\rho(v_{n+2}, v_m)\varpi(v_{n+2}, v_m) \\
&\leq \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \rho(v_{n+1}, v_m)\beta(v_{n+1}, v_{n+2})\varpi(v_{n+1}, v_{n+2}) \\
&\quad +\rho(v_{n+1}, v_m)\rho(v_{n+2}, v_m)\beta(v_{n+2}, v_{n+3})\varpi(v_{n+2}, v_{n+3}) \\
&\quad +\rho(v_{n+1}, v_m)\rho(v_{n+2}, v_m)\rho(v_{n+3}, v_m)\varpi(v_{n+3}, v_m) \\
&\leq \\
&\vdots \\
&\leq \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})\varpi(v_i, v_{i+1}) \\
&\quad +\Big(\prod_{k=n+1}^{m-1}\rho(v_k, v_m)\Big)\varpi(v_{m-1}, v_m) \\
&\leq \beta(v_n, v_{n+1})h^n\varpi(v_0, v_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})h^i\varpi(v_0, v_1) \\
&\quad +\Big(\prod_{i=n+1}^{m-1}\rho(v_i, v_m)\Big)h^{m-1}\varpi(v_0, v_1) \\
&\leq \beta(v_n, v_{n+1})h^n\varpi(v_0, v_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})h^i\varpi(v_0, v_1) \\
&\quad +\Big(\prod_{i=n+1}^{m-1}\rho(v_i, v_m)\Big)h^{m-1}\beta(v_{m-1}, v_m)\varpi(v_0, v_1)
\end{aligned}$$

$$\begin{aligned}
&= \beta(v_n, v_{n+1})h^n\varpi(v_0, v_1) + \sum_{i=n+1}^{m-1}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})h^i\varpi(v_0, v_1) \\
&\leq \beta(v_n, v_{n+1})h^n\varpi(v_0, v_1) + \sum_{i=n+1}^{m-1}\Big(\prod_{j=0}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})h^i\varpi(v_0, v_1).
\end{aligned}$$

Note that we are using the fact that $\beta(v, t) \geq 1$ and $\rho(v, t) \geq 1$. Let

$$\Phi_p = \sum_{i=0}^{p}\Big(\prod_{j=0}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})h^i.$$

Then we have

$$\varpi(v_n, v_m) \leq \varpi(v_0, v_1)[h^n\beta(v_n, v_{n+1}) + (\Phi_{m-1} - \Phi_n)]. \tag{4.5}$$

By condition (4.2), using the ratio test, we see that $\lim_{n\to+\infty} \Phi_n$ exists, and hence the real sequence (Φ_n) a Cauchy sequence. Finally, if we take the limit in inequality (4.5) as $n, m \to +\infty$, we deduce that

$$\lim_{n,m\to+\infty} \varpi(v_n, v_m) = 0. \tag{4.6}$$

Similarly proceeding we have

$$\lim_{n,m\to+\infty} \varpi(v_m, v_n) = 0.$$

Hence the sequence (v_n) is ϖ-Cauchy in (Υ, ϖ), which is a complete $(DCQMLS)$, so (v_n) converges to some $v^* \in \Upsilon$, that is,

$$\begin{aligned}
\lim_{n\to+\infty} \varpi(v_n, v^*) &= \lim_{n\to+\infty} \varpi(v^*, v_n) \\
&= \varpi(v^*, v^*) \\
&= \lim_{n,m\to+\infty} \varpi(v_n, v_m) = \lim_{n,m\to+\infty} \varpi(v_m, v_n) \\
&= 0.
\end{aligned} \tag{4.7}$$

Then $\varpi(v^*, v^*) = 0$. Next, we show that $\Xi v^* = v^*$. By the triangle inequality and (4.1) we have

$$\begin{aligned}
\varpi(v^*, \Xi v^*) &\leq \beta(v^*, v_{n+1})\varpi(v^*, v_{n+1}) + \rho(v_{n+1}, \Xi v^*)\varpi(v_{n+1}, \Xi v) \\
&= \beta(v^*, v_{n+1})\varpi(v^*, v_{n+1}) + \rho(v_{n+1}, \Xi v^*)\varpi(\Xi v_n, \Xi v^*) \\
&\leq \beta(v^*, v_{n+1})\varpi(v^*, v_{n+1}) + h\rho(v_{n+1}, \Xi v^*)\varpi(v_n, v^*).
\end{aligned}$$

and

$$\begin{aligned}
\varpi(\Xi v^*, v^*) &\leq \beta(\Xi v^*, v_{n+1})\varpi(\Xi v^*, v_{n+1}) + \rho(v_{n+1}, v^*)\varpi(v_{n+1}, v^*) \\
&= \beta(\Xi v^*, v_{n+1})\varpi(\Xi v^*, \Xi v_n) + \rho(v_{n+1}, v^*)\varpi(v_{n+1}, v^*) \\
&\leq h\beta(\Xi v^*, v_{n+1})\varpi(v^*, v_n) + \rho(v_{n+1}, v^*)\varpi(v_{n+1}, v^*).
\end{aligned}$$

Taking the limit as $n \to +\infty$, by (4.3) and (4.7) we deduce that $\varpi(v^*, \Xi v^*) = \varpi(\Xi v^*, v^*) = 0$, that is, $\Xi v^* = v^*$. Finally, assume that Ξ has two fixed points, say v and ξ. Then

$$\varpi(v, \xi) = \varpi(\Xi v, \Xi\xi) \leq h\varpi(v, \xi) < \varpi(v, \xi),$$

$$\varpi(\xi, v) = \varpi(\Xi\xi, \Xi v) \leq h\varpi(\xi, v) < \varpi(\xi, v),$$

which leads us to a contradiction. Therefore $\varpi(v, \xi) = \varpi(\xi, v) = 0$, so $v = \xi$. Hence Ξ has a unique fixed point.

Remark 4.1. Note that condition (4.3) in Theorem 4.1 can be changed by the assumption that Ξ and the $(DCQMLS)$ ϖ are continuous. To see this, the continuity gives us that if $v_n \to v^*$, then $\Xi v_n \to \Xi v^*$, and hence we have

$$\begin{aligned}
\lim_{n\to+\infty} \varpi(\Xi v_n, \Xi v^*) &= \lim_{n\to+\infty} \varpi(\Xi v^*, \Xi v_n) \\
&= \varpi(\Xi v^*, \Xi v^*) \\
&= \lim_{n\to+\infty} \varpi(v_{n+1}, \Xi v^*) = \lim_{n\to+\infty} \varpi(\Xi v^*, v_{n+1}) \\
&= \varpi(v^*, \Xi v^*) = \varpi(\Xi v^*, v^*),
\end{aligned}$$

then

$$\varpi(\Xi v^*, \Xi v^*) = \varpi(v^*, \Xi v^*) = \varpi(\Xi v^*, v^*) \tag{4.8}$$

Next we show that $\varpi(\Xi v^*, \Xi v^*) = 0$. In fact by (4.1) we have

$$\begin{aligned}
\varpi(\Xi v^*, \Xi v^*) &\leq \beta(\Xi v^*, v_{n+1})\varpi(\Xi v^*, v_{n+1}) + \rho(v_{n+1}, \Xi v^*)\varpi(v_{n+1}, \Xi v^*) \\
&= \beta(\Xi v^*, v_{n+1})\varpi(\Xi v^*, \Xi v_n) + \rho(v_{n+1}, \Xi v^*)\varpi(\Xi v_n, \Xi v^*) \\
&= h\beta(\Xi v^*, v_{n+1})\varpi(v^*, v_n) + h\rho(v_{n+1}, \Xi v^*)\varpi(v_n, v^*).
\end{aligned}$$

Taking the limit as $n \to +\infty$, by (4.3), (4.7) and (4.8) we deduce that $\varpi(\Xi v^*, \Xi v^*) = \varpi(v^*, \Xi v^*) = \varpi(\Xi v^*, v^*) = 0$, so $\Xi v^* = v^*$.

Definition 4.1. Let $\Xi : \Upsilon \to \Upsilon$. For some $v_0 \in \Upsilon$, consider $O(v_0) = \{v_0, \Xi v_0, \Xi^2 v_0, \cdots\}$ to be the orbit of v_0. We say that a function φ is Ξ- orbitally lower semicontinuous at $u \in \Upsilon$ if for $(v_n) \subset O(v_0)$ such that $v_n \to u$, we have $\varphi(u) \leq \lim_{n\to+\infty} \inf \varphi(v_n)$.

Corollary 4.1. Let (Υ, ϖ) be a complete $(DCQMLS)$ defined by functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$. Let $\Xi : \Upsilon \to \Upsilon$. Let $v_0 \in \Upsilon$ and $0 < h < 1$ be such that

$$\varpi(\Xi u, \Xi^2 u) \leq h\varpi(v, \Xi u) \text{ for each } u \in O(v_0). \tag{4.9}$$

Take $v_n = \Xi^n v_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(v_{i+1}, v_{i+2})}{\beta(v_i, v_{i+1})} \rho(v_{i+1}, v_m) < \frac{1}{h}. \tag{4.10}$$

Then $\lim_{n\to+\infty} v_n = u \in \Upsilon$. Moreover, $\Xi u = u \Leftrightarrow u \mapsto \varpi(u, \Xi u)$ is Ξ- orbitally lower semicontinuous at u.

Next, we present the nonlinear case.

Theorem 4.2. Let (Υ, ϖ) be a complete $(DCQMLS)$ defined by functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$ and assume that there exists a nondecreasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{n\to+\infty} \psi^n(v) = 0,\ v > 0, \quad \psi(t) < t,\ for\, all\, t > 0,$$

and

$$\varpi(\Xi v, \Xi t) \leq \psi(\Theta(v, t)),\ \Theta(v, t) = \max\{\varpi(v, t), \varpi(v, \Xi v), \varpi(t, \Xi t)\}, \tag{4.11}$$

for all $v, t \in \Upsilon$. Moreover, assume that for each $v_0 \in \Upsilon$, we have

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(v_{i+1}, v_{i+2})}{\beta(v_i, v_{i+1})} \rho(v_{i+1}, v_m) \frac{\psi^{i+1}(\varpi(v_1, v_0))}{\psi^i(\varpi(v_1, v_0))} < 1, \tag{4.12}$$

where $v_n = \Xi^n v_0, n \in \mathbb{N}$. If the $(DCQMLS)$ ϖ and Ξ are continuous, then Ξ admits a unique fixed point $v^* \in \Upsilon$ with $\Xi^n v \to v^*$ for each $v \in \Upsilon$.

Proof. Assume that there exists $k \in \mathbb{N}$ such that $v_k = v_{k+1} = \Xi v_k$, which implies that v_k is a fixed point. So we may assume that $v_{n+1} \neq v_n$ for each n. From condition (4.11) we have

$$\varpi(v_n, v_{n+1}) = \varpi(\Xi v_n, \Xi v_{n-1}) \leq \psi(\Theta(v_{n-1}, v_n)), \tag{4.13}$$

where $\Theta(v_{n-1}, v_n) = \max\{\varpi(v_{n-1}, v_n), \varpi(v_n, v_{n+1})\}$. If for some n we accept that $\Theta(v_{n-1}, v_n) = \varpi(v_n, v_{n+1})$, then by (4.13) and the assumption $\psi(t) < t$ for all $t > 0$, we deduce that

$$0 < \varpi(v_n, v_{n+1}) \leq \psi(\varpi(v_n, v_{n+1})) < \varpi(v_n, v_{n+1}), \tag{4.14}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$, we obtain $\Theta(v_{n-1}, v_n) = \varpi(v_{n-1}, v_n)$. It follows that $0 < \varpi(v_n, v_{n+1}) \leq \psi(\varpi(v_{n-1}, v_n))$. By using induction we easily see that for all $n \geq 0$,

$$0 < \varpi(v_n, v_{n+1}) \leq \psi^n(\varpi(v_0, v_1)).$$

By the properties of ψ we can easily deduce that

$$\lim_{n\to+\infty} \varpi(v_n, v_{n+1}) = 0.$$

Using the argument in the proof of Theorem 4.1, for $n, m \in \mathbb{N}$ such that $n < m$, we can easily deduce that

$$\varpi(v_n, v_m) \le \beta(v_n, v_{n+1})\psi^n(\varpi(v_0, v_1)) + \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \rho(v_j, v_m))\beta(v_i, v_{i+1})\psi^i(\varpi(v_0, v_1)). \quad (4.15)$$

and

$$\varpi(v_m, v_n) \le \beta(v_m, v_{m+1})\psi^m(\varpi(v_0, v_1)) + \sum_{i=m+1}^{n-1} (\prod_{j=0}^{i} \rho(v_j, v_n))\beta(v_i, v_{i+1})\psi^i(\varpi(v_0, v_1)).$$

By condition (4.12), using the ratio test, we can easily deduce that the sequence (v_n) is ϖ-Cauchy. Since (Υ, ϖ) is a complete $(DCQMLS)$, if $v_n \to r$ as $n \to +\infty$, then $\lim_{n\to+\infty} \varpi(v_n, r) = \lim_{n\to+\infty} \varpi(r, v_n) = 0$. Hence by Remark 4.1 we conclude that $\Xi r = r$. Finally, assume that r and t are two fixed points of Ξ such that $r \neq t$. From assumption (4.11) we have

$$\varpi(r, t) = \varpi(\Xi r, \Xi t) \le \psi(\Theta(r, t)) = \psi(\varpi(r, t)) < \varpi(r, t),$$

and

$$\varpi(t, r) = \varpi(\Xi t, \Xi r) \le \psi(\Theta(t, r)) = \psi(\varpi(t, r)) < \varpi(t, r),$$

which leads to a contradiction. Therefore $r = t$, as desired.

Remark 4.2. Note that if $\psi(v) = \alpha v,\ 0 < \alpha < 1$, then condition (4.11) in Theorem 4.2 becomes

$$\varpi(\Xi v, \Xi t) \le \alpha \max\{\varpi(v, t), \varpi(v, \Xi v), \varpi(t, \Xi t)\}. \quad (4.16)$$

Next, we prove the following result for mappings satisfying Kannan-type contraction.

Theorem 4.3. Let (Υ, ϖ) be a complete $(DCQMLS)$ defined by functions $\beta, \rho : \Upsilon \times \Upsilon \to [1, \infty)$. Let $\Xi : \Upsilon \to \Upsilon$ be a Kannan mapping defined as follows:

$$\varpi(\Xi v, \Xi t) \le \delta\{\varpi(v, \Xi v) + \varpi(t, \Xi t)\} \quad (4.17)$$

for $v, t \in \Upsilon$, where $\delta \in (0, \frac{1}{2})$. For $v_0 \in \Upsilon$, take $v_n = \Xi^n v_0$. Suppose that

$$\sup_{m\ge1} \lim_{i\to+\infty} \frac{\beta(v_{i+1}, v_{i+2})}{\beta(v_i, v_{i+1})}\rho(v_{i+1}, v_m) < \frac{1-a}{a}. \quad (4.18)$$

Also, assume that for every $v \in \Upsilon$, we have

$$\lim_{n\to+\infty} \beta(v, v_n) < \frac{1}{\delta} \textit{ and } \lim_{n\to+\infty} \rho(v_n, v) < \frac{1}{\delta} \quad (4.19)$$

Then Ξ has a fixed point. Moreover, if for every fixed point z, we have $\varpi(z, z) = 0$, then the fixed point is unique.

Proof. Consider the sequence ($v_n = \Xi v_{n-1}$) in Υ satisfying hypotheses (4.18) and (4.19). From (4.17) we obtain

$$\begin{aligned}\varpi(v_n, v_{n+1}) &= \varpi(\Xi v_{n-1}, \Xi v_n)\\ &\leq \delta\{\varpi(v_{n-1}, \Xi v_{n-1}) + \varpi(v_n, \Xi v_n)\}\\ &= \delta\{\varpi(v_{n-1}, v_n) + \varpi(v_n, v_{n+1})\}.\end{aligned}$$

Then $\varpi(v_n, v_{n+1}) \leq \frac{\delta}{1-\delta}\varpi(v_{n-1}, v_n)$. By induction we get

$$\varpi(v_n, v_{n+1}) \leq (\frac{\delta}{1-\delta})^n \varpi(v_0, v_1), \ \ \forall n \geq 0. \tag{4.20}$$

Next we show that (v_n) is a ϖ-Cauchy sequence. For two natural numbers $n < m$, we have

$$\varpi(v_n, v_m) \leq \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \rho(v_{n+1}, v_m)\varpi(v_{n+1}, v_m).$$

Similarly to the proof of Theorem 4.1, we get

$$\begin{aligned}\varpi(v_n, v_m) \leq\ & \beta(v_n, v_{n+1})\varpi(v_n, v_{n+1}) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})\varpi(v_i, v_{i+1})\\ & +\Big(\prod_{k=n+1}^{m-1}\rho(v_k, v_m)\Big)\varpi(v_{m-1}, v_m)\\ \leq\ & \beta(v_n, v_{n+1})(\tfrac{\delta}{1-\delta})^n\varpi(v_0, v_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\rho(v_j, v_m)\Big)\beta(v_i, v_{i+1})(\frac{\delta}{1-\delta})^i\varpi(v_0, v_1)\\ & +\Big(\prod_{i=n+1}^{m-1}\rho(v_i, v_m)\Big)(\frac{\delta}{1-\delta})^{m-1}\beta(v_{m-1}, v_m)\varpi(v_0, v_1)\end{aligned}$$

Similarly proceeding we have

$$\begin{aligned}\varpi(v_m, v_n) \leq\ & \beta(v_m, v_{m+1})\varpi(v_m, v_{m+1}) + \sum_{i=m+1}^{n-2}\Big(\prod_{j=m+1}^{i}\rho(v_j, v_n)\Big)\beta(v_i, v_{i+1})\varpi(v_i, v_{i+1})\\ & +\Big(\prod_{k=m+1}^{n-1}\rho(v_k, v_n)\Big)\varpi(v_{n-1}, v_n)\\ \leq\ & \beta(v_m, v_{m+1})(\tfrac{\delta}{1-\delta})^m\varpi(v_0, v_1) + \sum_{i=m+1}^{n-2}\Big(\prod_{j=m+1}^{i}\rho(v_j, v_n)\Big)\beta(v_i, v_{i+1})(\frac{\delta}{1-\delta})^i\varpi(v_0, v_1)\\ & +\Big(\prod_{i=m+1}^{n-1}\rho(v_i, v_n)\Big)(\frac{\delta}{1-\delta})^{n-1}\beta(v_{n-1}, v_n)\varpi(v_0, v_1)\end{aligned}$$

Since $0 \leq \delta < \frac{1}{2}$, we have $0 < \frac{\delta}{1-\delta} < 1$, and similarly to the argument in the proof of Theorem 4.1, we obtain that (v_n) is a ϖ-Cauchy sequence in the complete $(DCQMLS)$ (Υ, ϖ). Thus (v_n) converges to some $z \in \Upsilon$. Suppose that $\Xi z \neq z$. Then

$$\begin{aligned}0 < \varpi(z, \Xi z) &\leq \beta(z, v_{n+1})\varpi(z, v_{n+1}) + \rho(v_{n+1}, \Xi z)\varpi(v_{n+1}, \Xi z)\\ &\leq \beta(z, v_{n+1})\varpi(z, v_{n+1}) + \rho(v_{n+1}, \Xi z)\{\delta\varpi(v_n, v_{n+1}) + \delta\varpi(z, \Xi z)\}\ ,\end{aligned}$$

.

and

$$\begin{aligned} 0 < \varpi(\Xi z, z) &\leq \beta(\Xi z, v_{n+1})\varpi(\Xi z, v_{n+1}) + \rho(v_{n+1}, z)\varpi(v_{n+1}, z) \\ &\leq \beta(\Xi z, v_{n+1})\{\delta\varpi(z, \Xi z) + \delta\varpi(v_n, v_{n+1})\} + \rho(v_{n+1}, z)\varpi(v_{n+1}, z). \end{aligned}$$

Taking the limit in both sides of these inequalities and using (4.19), we deduce that $0 < \varpi(z, \Xi z) < \varpi(z, \Xi z)$ and $0 < \varpi(\Xi z, z) < \varpi(\Xi z, z)$, which is a contradiction. Hence $\Xi z = z$. Now assume that for every fixed point w, we have $\varpi(z, z) = 0$ and suppose that Ξ has more than one fixed point, say z and η. Then

$$\begin{aligned} \varpi(z, \eta) = \varpi(\Xi z, \Xi\eta) &\leq \delta\{\varpi(z, \Xi z) + \varpi(\eta, \Xi\eta)\} \\ &= \delta\{\varpi(z, z) + \varpi(\eta, \eta)\} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \varpi(\eta, z) = \varpi(\Xi\eta, \Xi z) &\leq \delta\{\varpi(\eta, \Xi\eta) + \varpi(z, \Xi z)\} \\ &= \delta\{\varpi(\eta, \eta) + \varpi(z, z)\} \\ &= 0. \end{aligned}$$

Thereby $z = \eta$, as required.

Remark 4.3. It will be interesting to find more applications to our current paper in other fields see [19–23].

Acknowledgments

The first author (A. M. Zidan) extends his appreciation to the Deanship of Scientific Research at King Khalid University, Abha 61413, Saudi Arabia for funding this work through research groups program under grant number R.G.P-2/142/42.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. Abdeljawad, K. Abodayeh, N. Mlaiki, On fixed point generalizations to partial b-metric spaces, *J. Comput. Anal. Appl.*, **19** (2015), 883–891.
2. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6** (2018), 320.
3. T. Abedeljawad, E. Karapinar, K. Tas, Existence and uniqueness of common fixed point on partial metric spaces, *Appl. Math. Lett.*, **24** (2011), 1900–1904.
4. A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory Appl.*, **2012** (2012), 204.
5. H. Aydi, E. Karapinar, C. Vetro, On Ekeland's variational principle in partial metric spaces, *Appl. Math. Inf. Sci.*, **9** (2015), 257–262.
6. I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal.,* **30** (1989), 26–37.

7. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11.
8. V. Berinde, Generalized contractions in quasimetric spaces, *Seminar Fixed Point Theory*, **3** (1993), 3–9.
9. V. Berinde, Sequences of operators and fixed points in quasimetric spaces, *Stud. Univ. Babes-Bolyai Math.*, **16** (1996), 23–27.
10. M. Bousselsal, Z. Mostefaoui, Some fixed point results in partial metric spaces for generalized rational type contraction mappings, *NFAA*, **20** (2015), 43–54.
11. P. Hitzler, A. K. Seda, Dislocated topologies, *J. Electr. Eng.*, **51** (2000), 3–7.
12. E. Karapinar, S. Czerwik, H. Aydi, (α, ψ)- Meir-Keeler contraction mappings in generalized b-metric spaces, *J. Funct. Spaces*, **2018** (2018), 3264620.
13. T. Kamran, M. Samreen, Q. U. Ain, A Generalization of b-metric space and some fixed point theorems, *Mathematics,* **5** (2017), 19.
14. S. G. Matthews, *Metric domains for completeness*, PhD thesis, University of Warwick, Academic Press, 1986.
15. S. G. Matthews, Partial metric spaces, *Annals of the New York Academy of Sciences-Paper Edition*, **728** (1994), 183–197.
16. N. Mlaiki, Double controlled metric-like spaces, *J. Inequal. Appl.*, **2020** (2020), 189.
17. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6** (2018), 194.
18. J. J. M. M. Rutten, Elements of generalized ultrametric domain theory, *Theor. Comput. Sci.*, **170** (1996), 349–381.
19. A. H. Soliman, A. M. Zidan, A new coupled fixed point result in extended metric spaces with an application to study the stability of set-valued functional equations, *J. Funct. Spaces*, **2019** (2019), 4146328.
20. A. H. Soliman, T. Nabil, A. M. Zidan, On quasi-partial generalized type of metric spaces and an application to complexity analysis of computer algorithms, *Alex. Eng. J.*, **59** (2020), 1233–1238.
21. A. H. Soliman, A. M. Zidan, Existential examination of the coupled fixed point in generalized b-metric spaces and an application, *J. Intell. Fuzzy Syst.*, **38** (2020), 2801–2807.
22. A. M. Zidan, A. H Soliman, T. Nabil, M. A. Barakat, An investigation of new quicker implicit iterations in hyperbolic spaces, *Therm. Sci.*, **24** (2020), 199–207.
23. A. M. Zidan, A. Al Rwaily, On new type of F-contractive mapping for quasipartial b-metric spaces and some results of fixed-point theorem and application, *J. Math.*, **2020** (2020), 8825805.